OLIY VA O’RTA MAXSUS TA’LIM VAZIRLIGI

TOSHKENT ARXITEKTURA QURILISH INSTITUTI

MATEMATIKA VA TABIIY FANLAR KAFEDRASI

REFERAT

MAVZU. MATRITSA USTIDA ALMASHTIRISHLAR

TOSHKENT 2016

MATRITSA USTIDA ALMASHTIRISHLAR

Reja:

1. Teskari matritsa
2. Teskari matritsani topish usullari
3. Matritsaning rangi

Matritsa ustida almashtirishlar chiziqli algebrada muhim ro‘l o‘ynaydi. Jumladan, chiziqli algebraik tenglamalar sistemasining umumiy yechimini topishda, teskari matritsani aniqlashda, matritsaning rangini hisoblashda matritsa ustidagi almashtirishlardan keng foydalaniladi [1].

Matritsa satri (ustuni) ustida elementar almashtirishlar uch tipda bo‘ladi [2]:

I. ikkita satrning (ustunning) o‘rnini almashtirish;

II. satrni (ustunni) noldan farqli songa ko‘paytirish;

III. satrga (ustunga) noldan farqli songa ko‘paytirilgan boshqa satrni (ustunni) qo‘shish.

Biri ikkinchisidan elementar almashtirishlar natijasida hosil qilingan A va B matritsalarga *ekvivalent matritsalar* deyiladi va $A \sim B$ ko‘rinishda yoziladi.

3.1. Teskari matritsa

Asosiy ushunchalar

Matritsalarni qo‘shish, ayirish va ko‘paytirish sonlar ustida bajariladigan mos amallarga monand (hamohang) amallar hisoblanadi. Ushbu bandda matritsalar uchun sonlarni bo‘lish amaliga monand amal bilan tanishamiz.

Ma’lumki, agar k soni nolga teng bo‘lmasa, u holda har qanday m soni uchun $kx = m$ tenglama yagona $x = \frac{m}{k} = k^{-1}m$ yechimga ega bo‘ladi, bu yerda k^{-1} soni k soniga teskari son deb ataladi.

---

[1] E.Kreyszig. Advancet engineering Matematics. Copyright. 2011, pp. 267-268

[2] Lay, David C. Linear algebra and is applications. Copyright. 2012, pp.162-169

Sonlar uchun keltirilgan bu tasdiq matritsali tenglamalarni sonli tenglamalarga monand yechishda muhim ro‘l o‘ynaydi. Xususan, sonli tenglamalar uchun $kk^{-1}=1$ va $k^{-1}k=1$ shartlarining bajarilishi hal qiluvchi hisoblansa, matritsali tenglamalar uchun $AA^{-1}=I$ va $A^{-1}A=I$ shartlarning bajarilishi muhim hisoblanadi, bu yerda $A, I-$ bir xil o‘lchamli kvadrat matritsalar [3].

Agar A va A^{-1} kvadrat matritsalar uchun $AA^{-1}=A^{-1}A=I$ tenglik bajarilsa, A^{-1} matritsa A matritsaga *teskari matritsa* deyiladi.

Sonlarda, k^{-1} mavjud bo‘lishi uchun $k\neq 0$ bo‘lishi talab etilgani kabi, matritsalarda, A^{-1} mavjud bo‘lishi uchun $\det A\neq 0$ bo‘lishi talab qilinadi.

Agar $\det A=0$ bo‘lsa, A matritsaga *singular matritsa* deyiladi. Bunda singular so‘ziga sinonim sifatida «*xos*» yoki «*maxsus*» terminlaridan ham foydalaniladi. Agar $\det A\neq 0$ bo‘lsa, A matritsa *nosingular* (yoki *xosmas* yoki *maxsusmas*) *matritsa* deb ataladi.

Agar A matritsada avval elementlarni mos algebraik to‘ldiruvchilar bilan almashtirilsa va keyin transponirlansa, hosil bo‘lgan matritsa A matritsaga *biriktirilgan matritsa* deyiladi va $\text{adj}A$ bilan belgilanadi [4]:

$$\text{adj}A=\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}.$$

Teskari matritsa haqida teoremalar

1- teorema. Xos matritsa teskari matritsaga ega bo‘lmaydi.

Isboti. A matritsa uchun A^{-1} mavjud bo‘lsin deb faraz qilaylik. U holda $AA^{-1}=I$ bo‘ladi. Bundan $\det(AA^{-1})=\det I$ yoki $\det A\cdot\det A^{-1}=\det I$ kelib chiqadi. Bunda $\det A=0$ va $\det I=1$ ekanini hisobga olsak, $0=1$ ziddiyat hosil bo‘ladi. Bu ziddiyat qilingan faraz noto‘g‘ri ekanini ko‘rsatadi, ya’ni teoremani isbotlaydi.

[3] Lay, David C. Linear algebra and is applications. Copyright. 2012, pp.162-169

[4] Kenneth L. Kuttler-Elementary Linear Algebra [Lecture notes] (2015). pp. 96-99

2- teorema. Har qanday xosmas A matritsa uchun teskari matritsa mavjud va yagona bo‘ladi.

Isboti. A matritsa xosmas, ya’ni $\det A \neq 0$ bo‘lsin. Avval A^{-1} mavjud bo‘lishini ko‘rsatamiz. Buning uchun A matritsani $\frac{1}{\det A}\text{adj}A$ matritsaga ko‘paytiramiz va ko‘paytmaga determinantning 9- va 10- xossalarini qo‘llaymiz:

$$A\cdot\left(\frac{1}{\det A}adjA\right)=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}\cdot\begin{pmatrix} \frac{A_{11}}{\det A} & \frac{A_{21}}{\det A} & \dots & \frac{A_{n1}}{\det A} \\ \frac{A_{12}}{\det A} & \frac{A_{22}}{\det A} & \dots & \frac{A_{n2}}{\det A} \\ \dots & \dots & \dots & \dots \\ \frac{A_{1n}}{\det A} & \frac{A_{2n}}{\det A} & \dots & \frac{A_{nn}}{\det A} \end{pmatrix}=$$

$$=\begin{pmatrix} \frac{a_{11}A_{11}+a_{12}A_{12}+\ldots+a_{1n}A_{1n}}{\det A} & \frac{a_{11}A_{21}+a_{12}A_{22}+\ldots+a_{1n}A_{2n}}{\det A} & \ldots & \frac{a_{11}A_{n1}+a_{12}A_{n2}+\ldots+a_{1n}A_{nn}}{\det A} \\ \frac{a_{21}A_{11}+a_{22}A_{12}+\ldots+a_{2n}A_{1n}}{\det A} & \frac{a_{21}A_{21}+a_{22}A_{22}+\ldots+a_{2n}A_{2n}}{\det A} & \ldots & \frac{a_{21}A_{n1}+a_{22}A_{n2}+\ldots+a_{2n}A_{nn}}{\det A} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{a_{n1}A_{11}+a_{n2}A_{12}+\ldots+a_{nn}A_{1n}}{\det A} & \frac{a_{n1}A_{21}+a_{n2}A_{22}+\ldots+a_{nn}A_{2n}}{\det A} & \ldots & \frac{a_{n1}A_{n1}+a_{n2}A_{n2}+\ldots+a_{nn}A_{nn}}{\det A} \end{pmatrix}=$$

$$=\begin{pmatrix} \frac{\det A}{\det A} & 0 & \dots & 0 \\ 0 & \frac{\det A}{\det A} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{\det A}{\det A} \end{pmatrix}=\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}=I=AA^{-1}.$$

Demak, A matritsaga teskari matritsa mavjud va bu matritsa

$$A^{-1}=\frac{1}{\det A}adjA \tag{1.3.1}$$

formula bilan topiladi. Bunda $AA^{-1}=I$ tenglik bajariladi.

$A^{-1}A=I$ tenglikning bajarilishi shu kabi ko‘rsatiladi.

Endi A^{-1} yagona ekanini ko‘rsatamiz. Buning uchun A^{-1}dan boshqa A matritsaga teskari C matritsa mavjud bo‘lsin deb faraz qilamiz. U holda ta’rifga ko‘ra $AC=I$ bo‘ladi. Bu tenglikning har ikkala tamonini A^{-1} ga chapdan ko‘paytiramiz:

$$A^{-1}AC=A^{-1}I.$$

$A^{-1}A = I$ bo‘lgani uchun $IC = A^{-1}I$ bo‘ladi. Endi $IC = C$ va $A^{-1}I = A^{-1}$ ekanini hisobga olsak, $C = A^{-1}$ kelib chiqadi. Teorema to‘liq isbot qilindi.

3- teorema. Teskari matritsa uchun ushbu *xossalar* o‘rinli bo‘ladi [5]:

1^o. A matritsa A^{-1} teskari matritsaga ega bo‘lsa, $\det A^{-1} = \dfrac{1}{\det A}$ bo‘ladi;

2^o. A matritsa A^{-1} teskari matritsaga ega bo‘lsa, $(A^{-1})^{-1} = A$ bo‘ladi;

3^o. $n \times n$ o‘lchamli A va B matritsalar A^{-1} va B^{-1} teskari matritsalarga ega bo‘lsa, $(AB)^{-1} = B^{-1}A^{-1}$ bo‘ladi;

4^o. A matritsa A^{-1} teskari matritsaga ega bo‘lsa, $(A^T)^{-1} = (A^{-1})^T$ bo‘ladi.

Isboti. 1) A matritsa uchun A^{-1} mavjud bo‘lsin. U holda $AA^{-1} = I$ yoki $\det(AA^{-1}) = \det I$ bo‘ladi. Bundan $\det A \cdot \det A^{-1} = 1$ yoki $\det A^{-1} = \dfrac{1}{\det A}$ kelib chiqadi.

2) A matritsa uchun A^{-1} mavjud bo‘lsin. U holda $AA^{-1} = I = A^{-1}A$ tengliklarga ko‘ra A^{-1} matritsa uchun teskari matritsa mavjud va u A dan iborat, ya’ni $(A^{-1})^{-1} = A$ bo‘ladi.

3) $n \times n$ o‘lchamli A va B matritsalar A^{-1} va B^{-1} teskari matritsalarga ega bo‘lsin. U holda AB va $B^{-1}A^{-1}$ matritsalar uchun

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

bo‘ladi. Demak, AB uchun teskari matritsa mavjud va $(AB)^{-1} = B^{-1}A^{-1}$ bo‘ladi.

4) A matritsa uchun A^{-1} mavjud bo‘lsin. U holda A^T va $(A^{-1})^T$ matritsalar uchun

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I,$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

bo‘ladi. Demak, A^T uchun teskari matritsa mavjud va $(A^T)^{-1} = (A^{-1})^T$ bo‘ladi.

[5] Lay, David C. Linear algebra and is applications. Copyright. 2012, pp.162-169

1-izoh. 3^o xossani k ta $n\times n$ o‘lchamli va teskari matritsalarga ega bo‘lgan matritsalar uchun quyidagicha umumlashtirish mumkin:

$$\left(A_1A_2\cdot\ldots\cdot A_{k-1}A_k\right)^{-1}=A_k^{-1}A_{k-1}^{-1}\cdot\ldots\cdot A_2^{-1}A_1^{-1}.$$

Bu formula matematik induksiya metodi bilan isbotlanadi [6].

3.1-misol. $A=\begin{pmatrix}3&4\\1&2\end{pmatrix}$ matritsaga teskari matritsani toping va natijani tekshiring.

Yechish. Berilgan matritsaning determinantini hisoblaymiz:

$$\det A=\begin{vmatrix}3&4\\1&2\end{vmatrix}=6-4=2.$$

$\det A\neq 0$ va A matritsa uchun teskari matritsa mavjud.

Matritsa elementlarining algebraik to‘ldiruvchilarini topamiz:

$$A_{11}=(-1)^{1+1}2=2,\qquad A_{12}=(-1)^{1+2}1=-1,$$
$$A_{21}=(-1)^{2+1}4=-4,\qquad A_{22}=(-1)^{2+2}3=3.$$

A matritsaga biriktirilgan matritsani topamiz:

$$adjA=\begin{pmatrix}A_{11}&A_{21}\\A_{12}&A_{22}\end{pmatrix}=\begin{pmatrix}2&-4\\-1&3\end{pmatrix}.$$

Shunday qilib,

$$A^{-1}=\frac{1}{\det A}\begin{pmatrix}2&-4\\-1&3\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2&-4\\-1&3\end{pmatrix}=\begin{pmatrix}1&-2\\-\frac{1}{2}&\frac{3}{2}\end{pmatrix}.$$

Tekshirish:

$$AA^{-1}=\begin{pmatrix}3&4\\1&2\end{pmatrix}\cdot\begin{pmatrix}1&-2\\-\frac{1}{2}&\frac{3}{2}\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=I.$$

3.2-misol. $A=\begin{pmatrix}1&-2&1\\2&0&-1\\-2&1&1\end{pmatrix}$ matritsaga teskari matritsani toping .

Yechish. Bu matritsa uchun:

$$\det A=\begin{vmatrix}1&-2&1\\2&0&-1\\-2&1&1\end{vmatrix}=0-4+2-0+4+1=3\neq 0.$$

---

[6] Lay, David C. Linear algebra and is applications. Copyright. 2012, pp.162-169

Matritsa elementlarining algebraik to‘ldiruvchilarini topamiz:

$$A_{11} = \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = 1, \qquad A_{21} = -\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = 3, \qquad A_{31} = -\begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix} = 2,$$

$$A_{12} = -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} = 0, \qquad A_{22} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3, \qquad A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3,$$

$$A_{13} = \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = 2, \qquad A_{23} = -\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 3, \qquad A_{33} = \begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} = 4.$$

A matritsaga biriktirilgan matritsani topamiz:

$$\text{adj}A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 3 & 3 \\ 2 & 3 & 4 \end{pmatrix}.$$

Demak,

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} 1 & 3 & 2 \\ 0 & 3 & 3 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 1 & \frac{2}{3} \\ 0 & 1 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} \end{pmatrix}.$$

Teskari matritsani topishning Gauss-Jordan usuli

A xosmas matritsaning A^{-1} teskari matritsasini topishning qulay usullaridan biri matritsa satrlari ustida elementar almashtirishlarga asoslangan *Gauss-Jordan usuli* hisoblanadi.

A^{-1} matritsani topishning Gauss-Jordan usuli ushbu tartibda amalga oshiriladi [7].

Gauss-Jordan usulining algoritmi

1^o. A va I matritsalarni yonma-yon yozib, $(A|I)$ kengaytirilgan matritsa tuziladi;

2^o. Elementar almashtirishlar yordamida $(A|I)$ matritsa $(I|B)$ ko‘rinishga keltiriladi. Bunda B matritsa A matritsa uchun teskari matritsa bo‘ladi.

[7] Kenneth L. Kuttler-Elementary Linear Algebra [Lecture notes] (2015). pp. 96-99

3.3-misol. $A=\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}$ matritsaga teskari marritsani Gauss-Jardon usuli bilan toping va natijani tekshiring.

Yechish.

$$(A\,|\,I)=\left(\begin{array}{cc|cc}3 & 1 & 1 & 0\\ 1 & 2 & 0 & 1\end{array}\right)r_1\to r_1+(-2)r_2 \sim$$

$$\sim\left(\begin{array}{cc|cc}1 & -3 & 1 & -2\\ 1 & 2 & 0 & 1\end{array}\right)r_2\to r_2+(-1)r_1 \sim\left(\begin{array}{cc|cc}1 & -3 & 1 & -2\\ 0 & 5 & -1 & 3\end{array}\right)r_2\to r_2:5 \sim$$

$$\sim\left(\begin{array}{cc|cc}1 & -3 & 1 & -2\\ 0 & 1 & -\frac{1}{5} & \frac{3}{5}\end{array}\right)r_2\to r_2:5 \sim\left(\begin{array}{cc|cc}1 & 0 & \frac{2}{5} & -\frac{1}{5}\\ 0 & 1 & -\frac{1}{5} & \frac{3}{5}\end{array}\right)=(I\,|\,A^{-1}).$$

Yuqorida keltirilgan $r_i\to r_i+\lambda r_k$ belgilash i-satr bu satrga λ songa ko‘paytirilgan k- satrni qo‘shish natijasida hosil qilinganini, $r_i\to r_i:\lambda$ belgi esa i- satr bu satrni λ songa bo‘lish natijasida hosil qilinganini bildiradi.

Demak, $A^{-1}=\begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ -\frac{1}{5} & \frac{3}{5}\end{pmatrix}=\frac{1}{5}\begin{pmatrix}2 & -1\\ -1 & 3\end{pmatrix}.$

Tekshirish:

$$AA^{-1}=\begin{pmatrix}3 & 1\\ 1 & 2\end{pmatrix}\cdot\frac{1}{5}\begin{pmatrix}2 & -1\\ -1 & 3\end{pmatrix}=\frac{1}{5}\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}=I.$$

3.4-misol. $A=\begin{pmatrix}1 & -1 & 2\\ -1 & 3 & 0\\ 2 & 1 & 4\end{pmatrix}$ matritsaga teskari marritsani Gauss-Jardon usuli bilan toping.

Yechish.

$$(A \mid I)=\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right)\begin{matrix} r_2 \to r_2 + r_1 \\ r_3 \to r_3 + (2)r_1 \end{matrix} \sim$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 3 & 0 & -2 & 0 & 1 \end{array}\right) r_2 \to r_2 : 2 \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 3 & 0 & -2 & 0 & 1 \end{array}\right)\begin{matrix} r_1 \to r_1 + r_2 \\ r_3 \to r_3 + (-3)r_2 \end{matrix} \sim$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -3 & -\frac{7}{2} & -\frac{3}{2} & 1 \end{array}\right) r_3 \to r_3 : (-3) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{7}{6} & \frac{1}{2} & -\frac{1}{3} \end{array}\right)\begin{matrix} r_1 = r_1 + (-3)r_3 \\ r_2 = r_2 + (-1)r_3 \end{matrix} \sim$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{1}{2} & -\frac{1}{3} \end{array}\right) = (I \mid A^{-1}).$$

Demak,

$$A^{-1} = \begin{pmatrix} -2 & -1 & 1 \\ -\frac{2}{3} & 0 & \frac{1}{3} \\ \frac{7}{6} & \frac{1}{2} & -\frac{1}{3} \end{pmatrix}.$$

3.2. Matritsani LU yoyish

Chiziqli algebrada matritsalarning turli yoyilmalari keng qo‘llaniladi.

Matrisani yoyish deb, uni biror xossaga (masalan, ortogonallik, simmetriklik, diagonallik xossasiga) ega bo‘lgan ikki va ikkidan ortiq martitsalar ko‘paytmasi shaklida ifodalashga aytiladi. Bunday yoyishlardan biri *matritsani LU yoyish* hisoblanadi.

Matritsani LU yoyishda $m \times n$ o‘lchamli A matritsa $A = LU$ shaklda ifodalanadi, bu yerda $L -$diagonal elementlari birlardan iborat bo‘lgan $m \times m$ o‘lchamli quyi uchburchak (Lower-triangular) matritsa; $U - m \times n$ o‘lchamli yuqori uchburchak ($m \neq n$ da trapetsiya) (Upper-triangular) matritsa [8].

Masalan,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix} \cdot \begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Matritsaning LU yoyilmasi yana *matritsaning LU faktorizasiyasi* deb ataladi. Matritsaning LU yoyilmasidan chiziqli algebraik tenglamalar sistemasini yechishda va teskari matritsani topishda foydalaniladi.

$m \times n$ o‘lchamli A matritsa $A = LU$ shaklga keltirish (LU yoyish) umuman olganda A matritsaning satrlariga noldan farqli songa ko‘paytirilgan boshqa satrni qo‘shish orqali quyidagi tartibda amalga oshiriladi.

A matritsani LU yoyish algoritmi

1^o. A matritsa satrlarida ketma-ket elementar almashtirishlar bajariladi va U shaklga keltiriladi;

2^o. Satrlarda bajarilgan elementar almashtirishlar ketma-ketligi asosida L yozuv hosil qilinadi va bu yozuvda barcha diagonal elementlar ustunlarni bo‘lish orqali birlarga aylantiriladi.

3.5-misol. $A = \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ 6 & 0 & 7 & -3 & 1 \end{pmatrix}$ matritsani LU yoying.

Yechish. Matritsa satrlarida ketma-ket elementar almashtirishlar bajaramiz:

$$A = \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 4 & -5 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} = U.$$

[8] Lay, David C. Linear algebra and is applications. Copyright. 2012, pp.162-169

A matritsa 4 ta satrdan tashkil topgani sababli L matritsa 4×4 o‘lchamli bo‘ladi. Birinchi qadamda belgilangan yozuvlar L matritsa yozuvining ustunlarini tashkil qiladi. Bu yozuvda barcha diagonal elementlarni birlarga aylantiramiz:

$$\begin{pmatrix} 2 & & & \\ -4 & 3 & & \\ 2 & -9 & 2 & \\ -6 & 12 & 4 & 5 \end{pmatrix}$$
$$\begin{matrix} :2 & :3 & :2 & :5 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{matrix}$$
$$\begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{pmatrix}. \quad \text{Bundan } L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}.$$

Demak,

$$\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ 6 & 0 & 7 & -3 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}\cdot\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

n- tartibli kvadrat matritsa berilgan bo‘lsin. Bunda A matritsani LU yoyish turli algoritmlar bilan amalga oshirilishi mumkin[9]. Shunday algoritmlardan biri bilan tanishamiz.

A matritsa xosmas bo‘lsin. U holda ta’rifga ko‘ra

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ l_{21} & 1 & 0 & \dots & 0 \\ l_{31} & l_{32} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & l_{n3} & \dots & 1 \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & u_{nn} \end{pmatrix}}_{U}=\underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}}_{A}.$$

Bundan

---

[9] Kenneth L. Kuttler-Elementary Linear Algebra [Lecture notes] (2015). pp. 96-99

$$a_{ij}=\sum_{k=1}^{n} l_{ik}u_{kj}=\sum_{k=1}^{\min(i,j)} l_{ik}u_{kj/}\ .$$

Bu yig‘indidagi oxirgi qo‘shiluvchilarni ajratib, topamiz:

$$u_{ij}=a_{ij}-\sum_{k=1}^{i-1} l_{ik}u_{kj},\ \text{agar}\ \ i\le j\ \text{bo‘lsa;} \qquad (1.3.2)$$

$$l_{ij}=\frac{1}{u_{jj}}\left(a_{ij}-\sum_{k=1}^{j-1} l_{ik}u_{kj}\right),\ \text{agar}\ \ i>j\ \text{bo‘lsa.} \qquad (1.3.3)$$

Shunday qilib, L va U matritsalarning noma’lum elementlari a_{ij} va topilgan l_{ik}, u_{kj} lar orqali ketma-ket ifodalanadi.

2-izoh. (1.3.2) va (1.3.3) formulalar shunday tartiblanganki, bunda avval barcha u_{ij} larni va keyin barcha l_{ij} larni hisoblab bo‘lmaydi, va aksincha. Bu formulalar orqali hisoblashlar quyidagi tartibda bajariladi:

$$u_{1j}=a_{1j}, \qquad j=1,2,\ldots,n;$$

$$l_{i1}=\frac{a_{i1}}{u_{11}}, \qquad i=2,3,\ldots,n;$$

$$u_{2j}=a_{2j}-l_{21}u_{1j}, \qquad j=2,3,\ldots,n;$$

$$l_{i2}=\frac{a_{i2}-l_{i1}u_{12}}{u_{22}}, \qquad i=3,4,\ldots,n;$$

va hokazo, ya’ni U matritsaning satrlari va L matritsaning ustunlari almashlab hisoblanadi.

3.5-misol. $A=\begin{pmatrix}8&2&9\\4&9&4\\6&7&9\end{pmatrix}$ matritsani LU yoying.

Yechish. Berilgan matritsa xosmas, chunki $\det A=166\neq 0$.

$LU=A$ yoyilmani tuzamiz:

$$\begin{pmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{pmatrix}\cdot\begin{pmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{pmatrix}=\begin{pmatrix}8&2&9\\4&9&4\\6&7&9\end{pmatrix}.$$

L va U matritsalarning noma'lum elementlarini (1.3.2) va (1.3.3) formulalar bilan aniqlaymiz:

$$u_{11}=a_{11}=8,\quad u_{12}=a_{12}=2,\quad u_{13}=a_{13}=9,\quad l_{21}=\frac{1}{u_{11}}a_{21}=\frac{4}{8}=\frac{1}{2},$$

$$u_{22}=a_{22}-l_{21}u_{12}=9-\frac{1}{2}\cdot 2=8,\qquad u_{23}=a_{23}-l_{21}u_{13}=4-\frac{1}{2}\cdot 9=-\frac{1}{2},$$

$$l_{31}=\frac{1}{u_{11}}a_{31}=\frac{6}{8}=\frac{3}{4},\qquad l_{32}=\frac{1}{u_{22}}(a_{32}-l_{31}u_{12})=\frac{1}{8}\left(7-\frac{3}{4}\cdot 2\right)=\frac{11}{16},$$

$$u_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}=9-\frac{3}{4}\cdot 9-\frac{11}{16}\cdot\left(-\frac{1}{2}\right)=\frac{83}{32}.$$

Demak,

$$\begin{pmatrix}8&2&9\\4&9&4\\6&7&9\end{pmatrix}=\begin{pmatrix}1&0&0\\\frac{1}{2}&1&0\\\frac{3}{4}&\frac{11}{16}&1\end{pmatrix}\cdot\begin{pmatrix}8&2&9\\0&8&-\frac{1}{2}\\0&0&\frac{83}{32}\end{pmatrix}.$$

3.3. Matritsaning rangi

$m\times n$ o'lchamli A matritsa berilgan bo'lsin. Bu matritsadan biror k $(k\le\min(m;n))$ ta satr va k ta ustunni ajratamiz. Ajratilgan satr va ustunlarning kesishishida joylashgan elementlardan k - tartibli kvadrat matritsani tuzamiz. Bu matritsaning determinantiga A *matritsaning* k *- tartibli minori* deyiladi.

A matritsa noldan farqli minorlari tartibining eng kattasiga A *matritsaning rangi* deyiladi va $r(A)$ (yoki $rangA$) kabi belgilanadi.

Tartibi $r(A)$ ga teng bo'lgan minorga A *matritsaning bazis minori* deyiladi. Matritsa bir nechta bazis minorga ega bo'lishi mumkin.

Matritsa ranginining ta'rifidan quyidagi tasdiqlar kelib chiqadi.

1. Matritsaning rangi 0 bilan m,n sonlarining kichigi orasidagi butun son orqali ifodalanadi, ya'ni $0\le r(A)\le\min(m;n)$.

2. Faqat $A=O$ matritsa uchun $r(A)=0$ bo‘ladi.

3. n- tartibli kvadrat matritsa nosingular bo‘lganidagina $r(A)=n$ bo‘ladi.

Matritsanng rangi ushbu xossalarga bo‘ysunadi [10].

1^o. Transponiplash natijasida matritsaning rangi o‘zgarmaydi;

2^o. Elementar almashtirishlar natijasida matritsaning rangi o‘zgarmaydi.

Isboti. Bilamizki:

a) transponirlash natijasida determinantnig qiymati o‘zgarmaydi;

b) ikkita satrning (ustunning) o‘rni almashtirilsa, determinantning ishorasi o‘zgaradi;

c) satrni (ustunni) noldan farqli songa ko‘paytirilsa, determinant shu songa ko‘payadi.

d) datrga (ustunga) noldan farqli songa ko‘paytirilgan boshqa satrni (ustunni) qo‘shilsa determinant o‘zgarmaydi.

Demak, transponirlash va elementar almashtirishlar natijasida xos matritsa xosligicha va xosmas matritsa xosmasligicha qoladi, ya’ni uning rangi o‘zgarmaydi.

$r(A)$ni ta’rif asosida topish usuli *minorlar ajratish usuli* deb ataladi. Bu usulda matritsaning rangi quyidagicha topiladi: agar barcha birinchi tartibli minorlar (matritsa elementlari) nolga teng bo‘lsa, $r(A)=0$ bo‘ladi; agar birinchi tartibli minorlardan hech bo‘lmaganda bittasi noldan farqli va barcha ikkinchi tartibli minorlar nolga teng bo‘lsa, $r(A)=1$ bo‘ladi; agar ikkinchi tartibli noldan farqli minor mavjud bo‘lsa, uchinchi tartibli minorlar tekshiriladi; bu jarayon yoki barcha k- tartibli minorlar nolga teng bo‘lishi aniq bo‘lquncha yoki k- tartibli minorlar mavjud bo‘lmaguncha davom ettiriladi, bunda $r(A)=k-1$ bo‘ladi.

3.6-misol. $A=\begin{pmatrix} 2 & -1 & 3 & -2 \\ 4 & -2 & 5 & 1 \\ 2 & -1 & 1 & 8 \end{pmatrix}$ matritsaning rangini minorlar ajratish usuli bilan toping.

---

[10] Lay, David C. Linear algebra and is applications. Copyright. 2012, pp.162-169

Yechish. Ravshanki, $1 \le r(A) \le \min(3;5) = 3.$

Ikkinchi tartibli minorlardan biri

$$\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix} = -5 + 6 = 1 \ne 0.$$

Uchinchi tartibli minorlarni hisoblaymiz (ularning soni $C_3^3 \cdot C_4^3 = 4$ ta):

$$M_1^{(3)} = \begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & 5 \\ 2 & -1 & 1 \end{vmatrix} = 0; \qquad M_2^{(3)} = \begin{vmatrix} 2 & -1 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 8 \end{vmatrix} = 0;$$

$$M_3^{(3)} = \begin{vmatrix} 2 & 3 & -2 \\ 4 & 5 & 1 \\ 2 & 1 & 8 \end{vmatrix} = 0; \qquad M_4^{(3)} = \begin{vmatrix} -1 & 3 & -2 \\ -2 & 5 & 1 \\ -1 & 1 & 8 \end{vmatrix} = 0.$$

Barcha uchinchi tartibli minorlar nolga teng. Demak $r(A) = 2.$

$r(A)$ ni topishning minorlar ajratish usuli hamma vaqt ham qulay bo‘lavermaydi, chunki ayrim hollarda bir qancha hisoblashlar bajarishga to‘g‘ri keladi.

Elementar almashtirishlar orqali har qanday matritsani bosh diagonalning birinchi bir nechta elementlari birlardan va qolgan elementlari nollardan iborat bo‘lgan matritsa ko‘rinishiga keltirish mumkin, masalan,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Bunday matritsaga *kanonik matritsa* deyiladi. Kanonik matritsaning rangi uning bosh diagonalida joylashgan birlar soniga teng bo‘ladi.

$r(A)$ ni kanonik matritsaga keltirib topish usuli matritsani *kanonik ko‘rinishga keltirish* usuli deb ataladi.

3.7-misol. $A=\begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 2 & 0 & 1 & -1 & 2 \\ -1 & 3 & -5 & -10 & 5 \end{pmatrix}$ matritsaning rangini uni kanonik ko‘rinishga keltirish usuli bilan toping.

Yechish. $A=\begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 2 & 0 & 1 & -1 & 2 \\ -1 & 3 & -5 & -10 & 5 \end{pmatrix}\begin{matrix} \\ r_2 \to r_2+(-2)r_1 \\ r_3 \to r_3 + r_1 \end{matrix} \sim$

$$\sim\begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 0 & 2 & -3 & -7 & 4 \\ 0 & 2 & -3 & -7 & 4 \end{pmatrix} r_3 \to r_3+(1)r_2 \quad \sim\begin{pmatrix} 1 & -1 & 2 & 3 & -1 \\ 0 & 2 & -3 & -7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} A \to A^T \sim$$

$$\sim\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -3 & 0 \\ 3 & -7 & 0 \\ -1 & 4 & 0 \end{pmatrix}\begin{matrix} \\ r_2 \to r_2 + r_1 \\ r_3 \to r_3+(-2)r_1 \\ r_4 \to r_4+(-3)r_1 \\ r_5 \to r_5 + r_1 \end{matrix} \sim\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & -7 & 0 \\ 0 & 4 & 0 \end{pmatrix}\begin{matrix} \\ r_2 \to r_2 : 2 \\ r_3 \to r_3 : 3 \\ r_4 \to r_4 : 7 \\ r_5 \to r_5 : (4) \end{matrix} \sim$$

$$\sim\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{matrix} \\ \\ r_3 \to r_3 + r_2 \\ r_4 \to r_4 + r_2 \\ r_5 \to r_5 + r_2 \end{matrix} \sim\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Demak, $r(A)=2.$

3. Mashqlar

3.1. Agar A matritsa nosingular va simmetrik bo‘lsa, A^{-1} matritsa ham nosingular va simmetrik bo‘lishini ko‘rsating,

3.2. Agar A kvadrat matritsa va $(I-A)$ nosingular matritsa bo‘lsa, $A(I-A)^{-1}=(I-A)^{-1}A$ tenglik bajarilishini ko‘rsating.

3.3. $A=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ matritsaning teskari matritsaga ega bo‘lishi shartini toping.

3.4. $A=\begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}$ va $C=\begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}$ bo‘lsin. $C=A^{-1}$ ekanini ko‘rsating.

3.5. $B=\begin{pmatrix} 4 & 0 & -5 \\ -18 & 1 & 24 \\ -3 & 0 & 4 \end{pmatrix}$ matritsa $A=\begin{pmatrix} 4 & 0 & 5 \\ 0 & 1 & -6 \\ 3 & 0 & 4 \end{pmatrix}$ matritsaning teskari matritsasi bo‘lishini ko‘rsating.

3.6. $B=\dfrac{1}{36}\begin{pmatrix} 11 & -3 & 5 \\ -17 & 21 & -11 \\ -10 & 6 & 2 \end{pmatrix}$ matritsa $A=\begin{pmatrix} 3 & 1 & -2 \\ 4 & 2 & 1 \\ 3 & -1 & 5 \end{pmatrix}$ matritsaning teskari matritsasi bo‘lishini ko‘rsating.

3.7. Berilgan matritsalardan qaysi birlari uchun teskari matritsa mavjud bo‘ladi?

1) $A=\begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix}$; 2) $B=\begin{pmatrix} 0 & 5 \\ 7 & 2 \end{pmatrix}$; 3) $C=\begin{pmatrix} 1 & -2 & 0 \\ 2 & 2 & 6 \\ 3 & 5 & 11 \end{pmatrix}$; $D=\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 0 & 3 & 10 \end{pmatrix}$.

3.8. $A=\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ bo‘lsin. $A^2=A^{-1}$ va $A^3=I$ bo‘lishini ko‘rsating.

3.9. Berilgan matritsalardan qaysi birlari o‘zaro teskari matritsalar bo‘ladi?

1) $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ va $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$; 2) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ va $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$;

3) $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$ va $\dfrac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$; 4) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}$ va $\begin{pmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{pmatrix}$.

3.10. $A=\begin{pmatrix} -3 & 6 \\ 2 & -5 \end{pmatrix}$ matritsa berilgan. A^{-1} matritsani toping.

3.11. $A=\begin{pmatrix} 5 & 2 \\ -1 & 4 \end{pmatrix}$. matritsa berilgan. A^{-1} matritsani toping.

3.12. Berilgan shartlarni qanoatlantiruvchi A matritsani toping:

1) $(3A)^{-1}=\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$; 2) $(2A)^T=\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}^{-1}$;

3) $(A^T-2I)^{-1}=\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$; 4) $A^{-1}=\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$.

3.13. $ABC=\begin{pmatrix}1&0&0\\-5&1&0\\0&0&1\end{pmatrix}$ bo‘lsin. $C^{-1}B^{-1}A^{-1}$ ni toping.

3.14. $A=\begin{pmatrix}-3&2&3\\4&1&6\\7&5&-1\end{pmatrix}$ matritsa berilgan. $C=A\cdot \mathrm{adj}A$ ko‘paytmaning barcha nodiagonal elementlarini toping.

3.15. $A=\begin{pmatrix}1&2&3\\0&2&4\\1&3&0\end{pmatrix}$ matritsa berilgan. $C=A\cdot \mathrm{adj}A$ ko‘paytmaning barcha diagonal elementlarini toping.

A matritsa berilgan. A^{-1} matritsani toping:

3.16. $A=\begin{pmatrix}1&1&1\\1&2&-1\\2&2&4\end{pmatrix}$.

3.17. $A=\begin{pmatrix}1&2&3\\2&6&4\\3&10&8\end{pmatrix}$.

A matritsa berilgan. A^{-1} matritsani Jordan-Gauss usuli bilan toping:

3.18. $A=\begin{pmatrix}1&0&1&2\\2&1&0&1\\1&1&2&1\\-1&1&2&1\end{pmatrix}$.

3.19. $A=\begin{pmatrix}1&-1&0&1\\-1&0&1&0\\2&-1&1&2\\0&1&2&0\end{pmatrix}$.

3.20. A matritsa berilgan. Matritsaning LU yoyilmasini toping:

1) $A=\begin{pmatrix}2&1\\8&7\end{pmatrix}$;

2) $A=\begin{pmatrix}6&4\\12&5\end{pmatrix}$;

3) $A=\begin{pmatrix}3&1&2\\-9&0&-4\\9&9&14\end{pmatrix}$;

4) $A=\begin{pmatrix}2&3&2\\4&13&9\\-6&5&4\end{pmatrix}$;

5) $A=\begin{pmatrix}2&0&5&2\\-6&3&-13&-3\\4&6&16&17\end{pmatrix}$;

6) $A=\begin{pmatrix}2&-3&4\\-4&8&-7\\6&-5&14\\-6&9&-12\\8&-6&10\end{pmatrix}$.

A matritsa berilgan. $r(A)$ ni minorlar ajratish usuli bilan toping:

3.21. $A=\begin{pmatrix} 1 & -1 & 2 & 3 \\ -1 & 3 & 0 & 1 \\ 3 & 4 & 1 & 1 \end{pmatrix}.$ **3.22.** $A=\begin{pmatrix} 1 & -2 & 3 \\ -1 & 4 & -2 \\ 2 & -2 & 7 \end{pmatrix}.$

A matritsa berilgan. $r(A)$ ni elementar almashtirishlar usuli bilan toping:

3.23. $A=\begin{pmatrix} 1 & -3 & 2 & -1 \\ 2 & -1 & 4 & -6 \\ -3 & -1 & -6 & 11 \end{pmatrix}.$ **3.24.** $A=\begin{pmatrix} 1 & -1 & 3 & 4 \\ 2 & -1 & 3 & -2 \\ 1 & -4 & 3 & 1 \\ 1 & -3 & 0 & -9 \end{pmatrix}.$

Adabiyotlar

1. Yo.U.Soatov. Oliy matematika 1-tom., T, “O’qituvchi” 1992
2. Yo.U.Soatov. Oliy matematika 2-tom., T, “O’qituvchi” 1992
3. Lay, David C. Linear algebra and is applications. Copyright. 2012, pp.162-169.
4. Kenneth L. Kuttler-Elementary Linear Algebra [Lecture notes] (2015). pp. 96-99.
5. Sh.R.Xurramov ”Matematika” Toshkent- 2016.