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# Optimal quadrature formulas with derivatives in the Sobolev space

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**Abstract.** In the present paper we construct the optimal quadrature formulas with derivatives in the Sobolev space. We give a new method of construction of such quadrature formulas using the discrete analogue of the differential operator $d^2/dx^2$. Finally we get the explicit forms of the optimal coefficients.

**Keywords:** quadrature formula, error functional, extremal function, optimal coefficients.

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## 1 Introduction. Statement of the problem

The results of many experiments in science and technology are usually given in the form of tabular data. These data are considered as values of some function and its derivatives. It is required to calculate with possible high exactness the values of a definite integral using this data. In this connection we consider the following quadrature formula

$$\int_0^1 \varphi(x)\,\mathrm{d}x \cong \sum_{\alpha=0}^{n}\sum_{\beta=0}^{N} C_\alpha[\beta]\varphi^{(\alpha)}(x_\beta) \tag{1.1}$$

with the error functional

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\alpha=0}^{n}\sum_{\beta=0}^{N}(-1)^\alpha C_\alpha[\beta]\delta^{(\alpha)}(x-x_\beta), \tag{1.2}$$

where $0 < t < 1$, $C_\alpha[\beta]$ are the coefficients, $x_\beta$ ($\in [0,1]$) are the nodes, $N$ is a natural number, $n = \overline{0,3}$, $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$, $\delta$ is the Dirac delta-function, $\varphi$ is an element of the space $L_2^{(m)}(0,1)$. Here $L_2^{(m)}(0,1)$ is the Sobolev space of functions with a square integrable $m$th generalized derivative and equipped with the norm

$$\|\varphi|L_2^{(m)}(0,1)\| = \left\{\int_0^1 (\varphi^{(m)}(x))^2\,\mathrm{d}x\right\}^{1/2}.$$

Since the functional $\ell$ of the form (1.2) is defined on the space $L_2^{(m)}(0,1)$ it is necessary to impose the following conditions (see [1, 2])

$$(\ell, x^\alpha) = 0, \quad \alpha = 0, 1, 2, ..., m-1. \tag{1.3}$$

Hence it is clear that for existence of the quadrature formulas of the form (1.1) the condition $N \geq m-1$ has to be met.

The difference

$$(\ell, \varphi) = \int\limits_{-\infty}^{\infty} \ell(x)\varphi(x)\,\mathrm{d}x = \int\limits_{0}^{1} \varphi(x)\,\mathrm{d}x - \sum_{\alpha=0}^{n}\sum_{\beta=0}^{N} C_\alpha[\beta]\varphi^{(\alpha)}(x_\beta) \tag{1.4}$$

is called *the error* of the formula (1.1).

By the Cauchy-Schwarz inequality

$$|(\ell, \varphi)| \leq \left\|\varphi|L_2^{(m)}\right\| \cdot \left\|\ell|L_2^{(m)*}\right\|$$

estimation of the error (1.4) on functions of the space $L_2^{(m)}(0,1)$ is reduced to finding the norm of the error functional $\ell$ in the conjugate space $L_2^{(m)*}(0,1)$.

It is well known [1, 2] that for any functional $\ell$ in $L_2^{(m)*}$ the equality

$$\begin{aligned}\|\ell|L_2^{(m)*}\|^2 &= (\ell, \psi_\ell) = (\ell(x), (-1)^m(\ell * G_m)(x)) \\ &= \int_{-\infty}^{\infty} \ell(x)\left((-1)^m \int_{-\infty}^{\infty} \ell(y)G_m(x-y)\,\mathrm{d}y\right)\mathrm{d}x\end{aligned}$$

holds. Here

$$\psi_\ell(x) = (-1)^m(\ell * G_m)(x) + P_{m-1}(x), \tag{1.5}$$

where

$$G_m(x) = \frac{|x|^{2m-1}}{2 \cdot (2m-1)!}, \tag{1.6}$$

$P_{m-1}(x)$ is a polynomial of degree $m-1$, and $*$ is the operation of convolution, i.e.

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} f(y)g(x-y)\,\mathrm{d}y.$$

Applying this equality to the error functional (1.2) and taking into account (1.5) we obtain the following

$$\|\ell\|^2 =$$
$$= (-1)^m \Bigg[ \sum_{k=0}^{n}\sum_{\alpha=0}^{n}\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N}(-1)^k C_k[\gamma]C_\alpha[\beta]\frac{(h\beta-h\gamma)^{2m-1-\alpha-k}\mathrm{sgn}(h\beta-h\gamma)}{2(2m-1-\alpha-k)!}$$
$$-2\sum_{\alpha=0}^{n}\sum_{\beta=0}^{N}(-1)^\alpha C_\alpha[\beta]\int_0^1\frac{(x-h\beta)^{2m-1-\alpha}\mathrm{sgn}(x-h\beta)}{2(2m-1-\alpha)!}\,\mathrm{d}x \tag{1.7}$$
$$+\int_0^1\int_0^1\frac{(x-y)^{2m-1}\mathrm{sgn}(x-y)}{2(2m-1)!}\,\mathrm{d}x\,\mathrm{d}y\Bigg].$$

Now we consider the minimization problem of the norm (1.7) of the error functional $\ell$ under the conditions (1.3).

It should be noted that minimization of $\|\ell\|^2$ by $C_\alpha[\beta]$, $\alpha=\overline{0,n}$, $\beta=\overline{0,N}$ is very hard. Here we suggest successive minimization of $\|\ell\|^2$ by $C_\alpha[\beta]$, i.e. first we consider the case $m=1$ and the expression (1.7) of $\|\ell\|^2$ we minimize by $C_0[\beta]$. Further we consider the case $m=2$, and using the obtained values for $C_0[\beta]$, the expression (1.7) of $\|\ell\|^2$ we minimize by $C_1[\beta]$. After that in the case $m=3$, using the obtained values of $C_0[\beta]$ and $C_1[\beta]$, the expression (1.7) for $\|\ell\|^2$ we minimize by $C_2[\beta]$ and so on.

## 2 Minimization of the norm of the error functional

Next we realize this successive minimization for the cases $m=1,2,3$ and $m=4$. Here we use the Lagrang method. We consider the function

$$\Phi(\mathbf{C},\lambda_{m-1}) = \|\ell\|^2 - 2(-1)^m\lambda_{m-1}(\ell, x^{m-1}),$$

where $\|\ell\|^2$ is defined by (1.7) and

$$\begin{aligned}\mathbf{C} \;=\; &(C_0[0],C_0[1],...,C_0[N],C_1[0],C_1[1],...,C_1[N],\\ &...,C_{m-1}[0],C_{m-1}[1],...,C_{m-1}[N]).\end{aligned}$$

We consider *the case* $m = 1$ then the quadrature formula (1.1) has the form

$$\int_0^1 \varphi(x)\,\mathrm{d}x \cong \sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta) \tag{2.1}$$

and $\|\ell\|^2$ depends only on $C_0[\beta]$ $(\beta = \overline{0, N})$.

Equating to zero partial derivatives of $\Phi(\mathbf{C}, \lambda_0)$ by $C_0[\beta]$ and $\lambda_0$ we get the following system of linear equations

$$\sum_{\gamma=0}^{N} C_0[\gamma]\frac{(h\beta - h\gamma)\ \mathrm{sgn}(h\beta - h\gamma)}{2} + \lambda_0 = F_0(h\beta), \tag{2.2}$$

$$\beta = 0, 1, ..., N,$$

$$\sum_{\gamma=0}^{N} C_0[\gamma] = g_0, \tag{2.3}$$

where

$$\begin{aligned} F_0(h\beta) &= \int_0^1 \frac{(x - h\beta)\ \mathrm{sgn}(x - h\beta)}{2}\,\mathrm{d}x = \frac{1}{2}\left[(h\beta)^2 - h\beta + \frac{1}{2}\right], \\ g_0 &= \int_0^1 \mathrm{d}x = 1. \end{aligned}$$

Further we consider *the case* $m = 2$. In this case the quadrature formula (1.1) takes the form

$$\int_0^1 \varphi(x)\,\mathrm{d}x \cong \sum_{\beta=0}^{N} \Big(C_0[\beta]\varphi(h\beta) + C_1[\beta]\varphi'(h\beta)\Big) \tag{2.4}$$

and expression (1.7) of $\|\ell\|^2$ depends on $C_0[\beta]$ and $C_1[\beta]$. Then using the solution $C_0[\beta]$ and $\lambda_0$ of the system (2.2)-(2.3), equating to zero parial derivatives of the function $\Phi(\mathbf{C},\lambda_1)$ by $C_1[\beta]$ and $\lambda_1$ we get

$$\sum_{\gamma=0}^{N} C_1[\gamma]\frac{(h\beta-h\gamma)\ \mathrm{sgn}(h\beta-h\gamma)}{2}-\lambda_1=F_1(h\beta), \tag{2.5}$$

$\beta=0,1,...,N,$

$$\sum_{\gamma=0}^{N} C_1[\gamma]=g_1-\sum_{\gamma=0}^{N} C_0[\gamma](h\gamma), \tag{2.6}$$

where

$$F_1(h\beta) = -f_1(h\beta)+\sum_{\gamma=0}^{N} C_0[\gamma]\frac{(h\beta-h\gamma)^2\ \mathrm{sgn}(h\beta-h\gamma)}{4}, \tag{2.7}$$

$$f_1(h\beta) = -\int_0^1 \frac{(x-h\beta)^2\ \mathrm{sgn}(x-h\beta)}{4}\,\mathrm{d}x=-\frac{1}{12}\left[(1-h\beta)^3-(h\beta)^3\right],$$

$$g_1 = \int_0^1 x\,\mathrm{d}x=\frac{1}{2}. \tag{2.8}$$

In *the case* $m=3$ the quadrature formula (1.1) has the form

$$\int_0^1 \varphi(x)\,\mathrm{d}x\cong\sum_{\beta=0}^{N}\Big(C_0[\beta]\varphi(h\beta)+C_1[\beta]\varphi'(h\beta)+C_2[\beta]\varphi''(h\beta)\Big) \tag{2.9}$$

and $\|\ell\|^2$, defined by equality (1.7), depends on $C_0[\beta]$, $C_1[\beta]$ and $C_2[\beta]$. Then using solutions $C_0[\beta]$ and $\lambda_0$ of system (2.2)-(2.3) and $C_1[\beta]$, $\lambda_1$ of system (2.5)-(2.6), equating to zero partial derivatives of $\Phi(\mathbf{C},\lambda_2)$ by $C_2[\beta]$ and $\lambda_2$ we have the following system of linear equations

$$\sum_{\gamma=0}^{N} C_2[\gamma]\frac{(h\beta-h\gamma)\ \mathrm{sgn}(h\beta-h\gamma)}{2}+\lambda_2=F_2(h\beta), \tag{2.10}$$

$\beta=0,1,...,N,$

$$\sum_{\gamma=0}^{N} C_2[\gamma]=\frac{g_2}{2!}-\sum_{i=0}^{1}\sum_{\gamma=0}^{N} C_i[\gamma]\frac{(h\gamma)^{2-i}}{(2-i)!}, \tag{2.11}$$

where

$$
\begin{aligned}
F_2(h\beta) &= f_2(h\beta) - \sum_{l=0}^{1}\sum_{\gamma=0}^{N}(-1)^l C_l[\gamma]\frac{(h\beta-h\gamma)^{3-l}\ \mathrm{sgn}(h\beta-h\gamma)}{2(3-l)!},\\
f_2(h\beta) &= \int_0^1 \frac{(x-h\beta)^3\ \mathrm{sgn}(x-h\beta)}{12}\,\mathrm{d}x = \frac{1}{48}\Big[(h\beta)^4+(1-h\beta)^4\Big],\\
g_2 &= \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}.
\end{aligned}
$$

In *the case* $m=4$ the quadrature formula (1.1) has the form

$$
\int_0^1 \varphi(x)\,\mathrm{d}x \cong \sum_{\beta=0}^{N}\Big(C_0[\beta]\varphi(h\beta)+C_1[\beta]\varphi'(h\beta)+C_2[\beta]\varphi''(h\beta)+C_3[\beta]\varphi'''(h\beta)\Big) \tag{2.12}
$$

and $\|\ell\|^2$, defined by equality (1.7), depends on $C_0[\beta]$, $C_1[\beta]$, $C_2[\beta]$ and $C_3[\beta]$. Then using solutions $C_0[\beta]$ and $\lambda_0$ of system (2.2)-(2.3), $C_1[\beta]$ and $\lambda_1$ of system (2.5)-(2.6) and $C_2[\beta]$, $\lambda_2$ of system (2.10)-(2.11), equating to zero partial derivatives of $\Phi(\mathbf{C},\lambda_3)$ by $C_3[\beta]$ and $\lambda_3$ we have the following system of linear equations

$$
\sum_{\gamma=0}^{N} C_3[\gamma]\frac{(h\beta-h\gamma)\ \mathrm{sgn}(h\beta-h\gamma)}{2}+\lambda_3 = F_3(h\beta), \tag{2.13}
$$

$\beta = 0,1,...,N,$

$$
\sum_{\gamma=0}^{N} C_3[\gamma] = \frac{g_3}{3!} - \sum_{i=0}^{2}\sum_{\gamma=0}^{N} C_i[\gamma]\frac{(h\gamma)^{3-i}}{(3-i)!}, \tag{2.14}
$$

where

$$\begin{aligned}
F_3(h\beta) &= -f_3(h\beta) + \sum_{l=0}^{2}\sum_{\gamma=0}^{N}(-1)^l C_l[\gamma]\frac{(h\beta-h\gamma)^{4-l}\ \mathrm{sgn}(h\beta-h\gamma)}{2(4-l)!},\\
f_3(h\beta) &= -\int_0^1 \frac{(x-h\beta)^4\ \mathrm{sgn}(x-h\beta)}{48}\,\mathrm{d}x = -\frac{1}{240}\Big[(1-h\beta)^5-(h\beta)^5\Big],\\
g_3 &= \int_0^1 x^3\,\mathrm{d}x = \frac{1}{4}.
\end{aligned}$$

Suppose, continuing by this way, for the cases $m = 1, 2, ..., k-1$ we found $C_0[\beta], C_1[\beta], ..., C_{k-2}[\beta]$ and $\lambda_0, \lambda_1, ..., \lambda_{k-2}$. We consider *the case* $m = k$. Then square of the norm (1.7) of the error functional $\ell$ of quadrature formulas (1.1) depends on $C_0[\beta], C_1[\beta], ..., C_{k-2}[\beta]$ and $C_{k-1}[\beta]$. Further using the obtained solutions $C_0[\beta], C_1[\beta], ..., C_{k-2}[\beta]$ and $\lambda_0, \lambda_1, ..., \lambda_{k-2}$ of corresponding systems, equating to zero partial derivatives of the function $\Phi(\mathbf{C}, \lambda_{k-1})$ by $C_{k-1}[\beta]$ and $\lambda_{k-1}$ we arrive to the system of linear equations

$$\begin{aligned}
&\sum_{\gamma=0}^{N}(-1)^{k-1}C_{k-1}[\gamma]\tfrac{(h\beta-h\gamma)\ \mathrm{sgn}(h\beta-h\gamma)}{2} + (k-1)!\lambda_{k-1} = F_{k-1}(h\beta),\\
&\beta = 0, 1, ..., N,\\
&\sum_{\gamma=0}^{N} C_{k-1}[\gamma] = \tfrac{g_{k-1}}{(k-1)!} - \sum_{i=0}^{k-2}\sum_{\gamma=0}^{N} C_i[\gamma]\tfrac{(h\gamma)^{k-1-i}}{(k-1-i)!},
\end{aligned}$$

where

$$\begin{aligned}
F_{k-1}(h\beta) &= f_{k-1}(h\beta) - \sum_{l=0}^{k-2}\sum_{\gamma=0}^{N}(-1)^l C_l[\gamma]\frac{(h\beta-h\gamma)^{k-l}\ \mathrm{sgn}(h\beta-h\gamma)}{2(k-l)!},\\
f_{k-1}(h\beta) &= \int_0^1 \frac{(-1)^{k-1}(x-h\beta)^k\ \mathrm{sgn}(x-h\beta)}{2k!}\,\mathrm{d}x,\\
g_{k-1} &= \int_0^1 x^{k-1}\,\mathrm{d}x.
\end{aligned}$$

Further, in the next section we solve systems (2.2)-(2.3), (2.5)-(2.6), (2.10)-(2.11) and (2.13)-(2.14), i.e. we find optimal coefficients of quadrature formulas of the forms (2.1), (2.4), (2.9) and (2.12).

# 3 The main results

Here we use the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given in [1, 2]. For completeness we give some definitions.

Assume that the nodes $x_\beta$ are equally spaced, i.e., $x_\beta = h\beta$, $h = \frac{1}{N}$, $N = 1, 2, ...$, functions $\varphi$ and $\psi$ are real-valued and defined on the real line $\mathbb{R}$.

**Definition 3.1.** The function $\varphi(h\beta)$ is a *function of discrete argument* if it is given on some set of integer values of $\beta$.

**Definition 3.2.** *The inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

**Definition 3.3.** *The convolution of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product*

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

It should be noted that the discrete analog of the differential operator $\mathrm{d}^{2m}/\mathrm{d}x^{2m}$ was firstly introduced and investigated by S.L. Sobolev [1]. In the work [3] the discrete analogue of the differential operator $\mathrm{d}^{2m}/\mathrm{d}x^{2m}$ was constructed. In particular, when $m = 1$ from the results of the work [3] we get the discrete analogue $D_1(h\beta)$ of the differential operator $\mathrm{d}^2/\mathrm{d}x^2$ which has the form

$$D_1(h\beta) = \begin{cases} 0, & |\beta| \geq 2, \\ h^{-2}, & |\beta| = 1, \\ -2h^{-2}, & \beta = 0 \end{cases} \tag{3.1}$$

and the following properties of the operator $D_1(h\beta)$

$$D_1(h\beta) * 1 = 0, \qquad D_1(h\beta) * (h\beta) = 0, \tag{3.2}$$

$$hD_1(h\beta) * \frac{|h\beta|}{2} = \delta(h\beta). \tag{3.3}$$

where $\delta(h\beta)$ is the discrete delta function.

In this section we solve systems (2.2)-(2.3), (2.5)-(2.6), (2.10)-(2.11) and (2.13)-(2.14).

It should be noted that in the process of solution of these systems only the discrete analogue $D_1(h\beta)$ of the differential operator $\mathrm{d}^2/\mathrm{d}x^2$ is used and each of these systems is reduced to the system of two linear equations with two unknowns as shown below in the proof of Theorem 3.5.

The system (2.2)-(2.3) was solved in [**?**] and the following was proved

**Theorem 3.4.** *In the space $L_2^{(1)}(0,1)$ when $\beta = 0,1,...,N$, the coefficients of optimal quadrature formulas of the form (2.1) are defined as follows*

$$\begin{array}{rcl} C_0[0] &=& \frac{h}{2}, \\ C_0[\beta] &=& h \\ C_0[N] &=& \frac{h}{2}. \end{array}$$

Now, using Theorem 3.4, we solve system (2.5)-(2.6).

The following holds

**Theorem 3.5.** *In the space $L_2^{(2)}(0,1)$ when $\beta = 0,1,...,N$, the coefficients of optimal quadrature formulas of the form (2.4) have the form*

$$\begin{array}{rcl} C_1[0] &=& \frac{h^2}{12}, \\ C_1[\beta] &=& 0 \\ C_1[N] &=& -\frac{h^2}{12}, \end{array}$$

*where the optimal coefficients $C_0[\beta]$, $\beta = 0,1,...,N$ are defined in Theorem 3.4.*

**Proof.** Suppose $C_1[\beta] = 0$ when $\beta < 0$ and $\beta > N$. Then, using Definition 3.3, we can rewrite the system (2.5)-(2.6) in the following convolution form

$$C_1[\beta] * \frac{|h\beta|}{2} - \lambda_1 = F_1(h\beta), \quad \beta = 0,1,...,N, \tag{3.4}$$

$$\sum_{\gamma=0}^{N} C_1[\gamma] = g_1 - \sum_{\gamma=0}^{N} C_0[\gamma](h\gamma), \tag{3.5}$$

Denoting by

$$u_1(h\beta) = C_1[\beta] * \frac{|h\beta|}{2} - \lambda_1, \tag{3.6}$$

using (3.1), (3.2) and (3.3), we obtain

$$C_1[\beta] = hD_1(h\beta) * u_1(h\beta). \tag{3.7}$$

In order to calculate the convolution (3.7) we need to determine the function $u_1(h\beta)$ for all integer values of $\beta$. From equality (3.4) we get that $u_1(h\beta) = F_1(h\beta)$ when $\beta = 0, 1, 2, ..., N$. Now we need to find representation of the function $u_1(h\beta)$ when $\beta < 0$ and $\beta > N$.

Taking into account that $C_1[\beta] = 0$ when $\beta < 0$ and $\beta > N$ for $u_1(h\beta)$ we get the following

$$u_1(h\beta) = \begin{cases} a_1^-, & \beta \leq 0, \\ F_1(h\beta), & 0 \leq \beta \leq N, \\ a_1^+, & \beta \geq N, \end{cases} \tag{3.8}$$

and

$$a_1^- = \mu_1 - \lambda_1, \quad a_1^+ = \mu_1 + \lambda_1,$$

here $\mu_1 = \frac{1}{2} \sum\limits_{\gamma=0}^{N} C_1[\gamma](h\gamma)$ and $\lambda_1$ are unknowns. If we find unknowns $a_1^-$ and $a_1^+$ then from the last system of equations we have

$$\mu_1 = \frac{1}{2}(a_1^+ + a_1^-), \quad \lambda_1 = \frac{1}{2}(a_1^+ - a_1^-). \tag{3.9}$$

Now from (3.8) for $a_1^+$ и $a_1^-$ when $\beta = 0$ and $\beta = N$ we get the following

$$a_1^+ = F_1(1), \quad a_1^- = F_1(0),$$

where $F_1(0)$ and $F_1(1)$ are obtained from (2.7) putting $\beta = 0$ and $\beta = N$, respectively, and $g_1$ is defined by (2.8). This means that we obtained the explicit form of the function $u_1(h\beta)$.

Further, using (3.1) and (3.8) from (3.7) calculating the convolution $hD_1(h\beta) * u_1(h\beta)$ for $\beta = \overline{0, N}$ we get

$$\begin{aligned} C_1[\beta] &= hD_1(h\beta) * u_1(h\beta) = h \sum_{\gamma=-\infty}^{\infty} D_1(h\beta - h\gamma) u_1(h\gamma) \\ &= h \Bigg[ \sum_{\gamma=0}^{N} D_1(h\beta - h\gamma) F_1(h\gamma) + \sum_{\gamma=1}^{\infty} D_1(h\beta + h\gamma) a_1^- \\ &\quad + \sum_{\gamma=1}^{\infty} D_1(h(N+\gamma) - h\beta) a_1^+ \Bigg]. \end{aligned} \tag{3.10}$$

From (3.10) for $\beta = 0$ we get

$$C_1[0] = \frac{1}{h}\Big[F_1(1) - F_1(0)\Big], \tag{3.11}$$

for $\beta = 1, ..., N-1$ we have

$$C_1[\beta] = \frac{1}{h}\Big[F_1(h(\beta-1)) - 2F_1(h\beta) + F_1(h(\beta+1))\Big], \tag{3.12}$$

and for $\beta = N$ we obtain

$$C_1[N] = \frac{1}{h}\Big[F_1(1-h) - F_1(1)\Big]. \tag{3.13}$$

Using equalities (2.7) and (2.8) from (3.11), (3.12) and (3.13), after some simplifications, we get explicit formulas for coefficients $C_1[\beta]$, $\beta = 0, 1, ..., N$, which are given in the statement of Theorem 3.5. Theorem 3.5 is proved.

Now using Theorems 3.4 and 3.5, we get the following result for the coefficients of the quadrature formula (2.9), i.e. we get the solution of the system (2.10)-(2.11).

**Theorem 3.6.** *In the space $L_2^{(3)}(0,1)$ when $\beta = 0, 1, 2, ..., N$, the coefficients of the optimal quadrature formulas of the form (2.9) have the form $C_2[\beta] = 0$, where the optimal coefficients $C_0[\beta]$ and $C_1[\beta]$ are defined by Theorems 3.4 and 3.5, respectively.*

Finally, using Theorems 3.4, 3.5 and 3.6 we arrive to the following result.

**Theorem 3.7.** *In the space $L_2^{(4)}(0,1)$ when $\beta = 0, 1, 2, ..., N$, the coefficients of the optimal quadrature formulas of the form (2.12) have the form*

$$\begin{aligned} C_3[0] &= \tfrac{h^4}{720}, \\ C_3[\beta] &= 0 \\ C_3[N] &= -\tfrac{h^4}{720}, \end{aligned}$$

*where the optimal coefficients $C_0[\beta]$, $C_1[\beta]$ and $C_2[\beta]$ are defined by Theorems 3.4, 3.5 and 3.6, respectively.*

Theorem 3.6 and 3.7 are proved similarly as Theorem 3.5.

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