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«Gipergeometrik funksiyalar»

mavzusidagi

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KIRISH

O’zbekiston Respublikasi Prezidenti I.A.Karimov Oliy Majlisning XIV sessiyasida so’zlagan nutqida kadrlar tayyorlashning ahamiyatiga izoh berib shunday degan edi:

«Biz oldimizga qanday vazifa qo’ymaylik, qanday muammoni yechish zaruriyati tug’ilmasin, gap oxir oqibat, baribir kadrlarga borib qadalaveradi. Mubolag’asiz aytish mumkinki, bizning kelajagimiz, mamlakatimiz kelajagi, o’rnimizga kim kelishiga yoki boshqacharoq qilib aytganda, qanday kadrlar tayyorlashimizga bog’liq.

…Mamlakatimiz kelajagi uchun Oliy Majlisning IX sessiyasida qabul qilingan «Kadrlar tayyorlash bo’yicha milliy dasturi»ning amalga oshirilishi juda ham muhim ahamiyatga ega.

…Yuqori malakali pedagog kadrlar tayyorlash va qayta tayyorlashga alohida e’tibor berish lozim. Kadrlar tayyorlashning sifati, erkin fikrlovchi shaxs - fuqaroni kamol toptirishiga ertaga sinfxonalar va auditoriyalarda kimlar dars va saboq berishiga bog’liq».

Darhaqiqat, barkamol inson shaxsining shakllanishi bevosita uzluksiz ta’lim jarayonida amalga oshadi. Shunday ekan, har jabhada muvaffaqiyatga erishish, jumladan yuqori malakali kadrlar tayyorlashda milliy dasturni o’rni va ahamiyati beqiyosdir.

Kadrlar tayyorlash milliy dasturida oliy ta’limning asosiy maqsadi bozor iqtisodiyoti sharoitida mustaqil ishlashga qodir, raqobatbardosh, yuqori malakali mutaxassislar tayyorlashdan iborat. Bu maqsadga erishish uchun, shuningdek Respublikamiz Prezidenti aytgani kabi «mamlakatimizning boy ilmiy - texnikaviy salohiyatidan keng foydalangan holda, yuksak texnologiya va fan yutuqlariga

asoslangan ishlab chiqarish sohalari - avtomobilsozlik, samolyotsozlik, mikrobiologiya, elektrotexnika va elektronika sanoatlarini telekommunikatsiya va zamonaviy axborot texnologiya vositalarini tez suratlarda rivojlantirish» uchun saboq olayotgan har bir shaxs o'zi o'rgangan ta'lim mazmunini chuqur anglashi, qayerda va qanday tatbiq qilishni bilishi, hayotda esa o'zi amaliyotga tatbiq qila olishi kerak.

Respublikamiz prezidenti I.A.Karimov tashabbusi bilan qabul qilingan "Kadrlar tayyorlash Milliy dasturi" mamlakatda bozor iqtisodiyotiga asoslangan jamiyatda raqobatga bardosh bera oladigan mutaxassis kadrlar tayyorlash vazifasini o‘z oldiga maqsad qilib qo‘ygan. Mamlakatimiz mustaqillikka erishgan kunlaridanoq oldinga surilgan bu maqsadning to‘g‘ri qo‘yilganligi hozirgi kunda ayniqsa, jahon moliyaviy iqtisodiy krizisi u yoki bu darajada mamlakat iqtisodiga ta`sir etishi mumkin bo‘lgan bir sharoitda, o‘z tasdig‘ini yana bir bor topmoqda. Chunki "Milliy dastur"da tayyorlanishi ko‘zda tutilgan mutaxassis zaruriyati tug‘ilgan sharoitda o‘z mutaxassisligi bo‘yicha ustunlikka ega bo‘lishi zarur, yoki aks holda o‘z mutaxassisligini juda tezlikda boshqa mutaxassislikka yo‘naltira olish malakasiga ega bo‘lishi kerak. Buning uchun esa bugungi kunda tayyorlanayotgan har bir mutaxassis nafaqat belgilangan reja va dastur bo‘yicha bilimlarni o‘zlashtirish, balki chuqur va atroflicha egallashi, unga yondosh ilm va bilimlar majmuasi va ko‘nikmalariga ham ega bo‘lishi zarur.

Maxsus funksiyalar nazariyasi keng tarmoqli bo‘lib, gipergeometrik funksiyalar uning markazida turadi. Ular bir tomondan ma'lum differensial tenglamaning yechimi sifatida ahamiyatli bo‘lsa, ikkinchi tomondan ma'lum ko‘inishdagi qatorning yig'indisi sifatida ahamiyatlidir.

Ana shu nuqtai nazardan qaraganda mazkur bitiruv malakaviy ishi mavzusi dolzarb bo‘lib, u kirish, asosiy qism, xulosa va foydalanilgan adabiyotlar ro‘yxatidan iborat.

Asosiy qism ikki bobdan iborat. Birinchi bobda gipergeometrik funksiyaning ta'rifi va xossalari keltirilib, uning differensiallash, qo‘shish (ayirish) va analitik

davom ettirish formulalari berilgan va isbotlangan.

Ikkinchi bobda esa gipergeometrik funksiyani turli yo‘nalishlarda umumlashtiruvchi maxsus funksiyalar va ularning turli xossalari keltirilgan.

Xulosa qismida mazkur mavzuga oid yakunlovchi fikrlar berilib, mavzuning ahamiyati ochib berilgan.

I BOB. GIPERGEOMETRIK FUNKSIYA

1-§. Asosiy ta’riflar.

Ushbu

$$x(1-x)y'' + \left[c - (a+b+1)x\right]y' - aby = 0 \qquad (1)$$

gipergeometrik tenglama yoki Gauss tenglamasi deb ataluvchi tenglamani tekshiramiz. Bu yerda a,b,c - uchta ixtiyoriy parametrlar bo‘lib, haqiqiy yoki kompleks qiymatlarni qabul qilishi mumkin. Bulardan ikkitasi: a va b tenglamada simmetrik ishtirok etadi.

$x = 0$ va $x = 1$ bo‘lganda tenglamaning tartibi buzilib, birinchi tartibli tenglama hosil bo‘ladi:

$$cy' - aby = 0$$

$$(c - a - b - 1)y' - aby = 0$$

$x = \infty$ da esa (1) tenglama umuman mahnosini yo‘qotadi. Shuning uchun bu nuqtalar maxsus nuqtalar hisoblanadi.

(1) tenglamaning $x = 0$ maxsus nuqta atrofidagi yechimini

$$y = \sum_{n=0}^{\infty} A_n x^n \qquad (2)$$

darajali qator ko‘rinishida izlaymiz. Bundan birinchi tartibli hosilani olsak

$$y' = \sum_{n=1}^{\infty} nA_n x^{n-1}$$

yoki

$$y' = \sum_{n=0}^{\infty} (n+1)A_{n+1} x^n,$$

ikkinchi tartibli hosilasini olib quyidagiga ega bo'lamiz:

$$y'' = \sum_{n=1}^{\infty} n(n+1)A_{n+1}x^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2)A_{n+2}x^n$$

Bu hosilalarning qiymatini va y ni (1) tenglamaga ko'yamiz. U holda

$$\sum_{n=0}^{\infty} x(1-x)(n+1)(n+2)A_{n+2}x^n +$$

$$+\sum_{n=0}^{\infty}\left[c-(a+b+1)x\right](n+1)A_{n+1}x^n - \sum_{n=0}^{\infty} abA_n x^n = 0.$$

Noma'lum $A_1,...,A_n,...$ o'zgarmaslarni topish uchun aniqmas koeffitsientlar usulidan foydalanamiz, bunga asosan x ning bir xil darajalari oldidagi koeffitsientlarni yig'indisini nolga tenglash kerak.Buning uchun yuqoridagi tenglikni quyidagicha yozib olamiz:.

$$\sum_{n=0}^{\infty}\{[(n+1)(n+2)A_{n+2}]x^{n+1} - [(n+1)(n+2)A_{n+2}]x^{n+2} +$$

x^k oldidagi koeffitsientlarni yig'indisini nolga tenglab, quyidagi tenglikni hosil qilamiz

$$\sum_{n=0}^{\infty}\{(x-x^2)(n+1)(n+2)A_{n+2}x^n + [(a+b+1)x(n+1)A_{n+1}x^n -$$

$$- abA_n x^n\} = 0,$$

$$[k(k+1)A_{k+1}]x^k - k(k-1)A_k x^k +$$

$$+ckA_k x^k - k(a+b+1)A_k x^k - abA_k x^k = 0,$$

$$-k(k-1)A_k + k(k+1)A_{k+1} - k(a+b+1)A_k +$$

$$+c(k+1)A_{k+1} - abA_k = 0,$$

Oxirgi tenglikdan

$$(k+(k+1)+c(k+1))A_{k+1} = (k(a+b+1)+k(k-1)+ab)A_k$$

$$(k+1)(k+c)A_{k+1} = (k^2 + ka + kb + ab)A_k =$$

$$= k(k+a) + b(k+a)A_k = (k+a)(b+k)A_k$$

tenglikni hosil qilamiz. Bundan

$$A_{k+1} = \frac{(k+a)(k+b)}{(k+1)(c+k)} A_k$$

rekurrent formulaga ega bo‘lamiz.

Bu yerda $A_0 = 1$ va $c \neq 0, -1, -2, ..., -n, ...$ deb hisoblaymiz. (1) gipergeometrik tenglamaning birinchi xususiy yechimi y_1 ni $F(a,b,c;x)$ orqali belgilab, A_n koeffitsientlarning topilgan qiymatlarini (2) qatorga qo‘yamiz. U holda

$$y_1 = F(a,b,c;x) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n . \qquad (3)$$

Bu yerda

$$a_n = a(a+1)...(a+n-1) = \Gamma(a+n) / \Gamma(a),$$

xususiy holda, $(a)_0 = 1$ va $\forall n \in N$ uchun $(1)_n = n!$.

(3) qator gipergeometrik qator, bu qatorning yig’indisi bo‘lgan $F(a,b,c;x)$ funksiya esa gipergeometrik funksiya deyiladi.

Dalamber printsipiga asosan,

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(a+n)(b+n)}{(n+1)(c+n)} x \right| = |x|.$$

Demak, (3) qator $|x| < 1$ da absolut yaqinlashuvchi, $|x| > 1$ da uzoqlashuvchi bo‘ladi. Raabe belgisidan foydalanib ko‘rsatish mumkinki, $x = 1$ bo‘lganda, agar $c - a - b > 0$ bo‘lsa, (3) qator absolut yaqinlashuvchi, agar $c - a - b \leq 0$ bo‘lsa, uzoqlashuvchi; $x = -1$ bo‘lganda esa, agar $c - a - b > 0$ bo‘lsa, absolut yaqinlashuvchi, agar $-1 < c - a - b \leq 0$ bo‘lsa, shartli yaqinlashuvchi, agar $c - a - b \leq -1$ bo‘lsa uzoqlashuvchi bo‘ladi.

Agar (3) formulada $b = c$ bo‘lsa,

$$(a)_n = (-1)^n(-a)(-a-1)...(-a-n+1) = (-1)^n\binom{-a}{n}n!$$

tenglikka asosan

$$F(a,b,c;x) = 1+\sum_{n=1}^{\infty}(-1)^n\binom{-a}{n}x^n = (1-x)^{-a}$$

binomial qator hosil bo‘ladi.

Agarda $a = 1,\ b = c$ bo‘lsa, (3) formula ushbu

$$F(1,b,b;x) = 1+\sum_{n=1}^{\infty}x^n = \frac{1}{1-x}$$

ko‘rinishga ega bo‘ladi, ya’ni $a = 1,\ b = c$ bo‘lgan holda gipergeometrik qator geometrik progressiyaga aylanadi, shuning uchun ham u gipergeometrik qator deb atalgan.

(1) tenglamaning ikkinchi xususiy, umuman aytganda, (3) ga chiziqli bog’liq bo‘lmagan yechimini topish uchun (1) tenglamada

$$y = x^{\rho}\eta$$

almashtirishni bajaramiz.

Bajarilgan almashtirishimizda y dan 1- va 2- tartibli hosilalarni olib, (1) tenglamaga qo‘yamiz

$$y' = r\,x^{r-1}h + x^r h',$$

$$y'' = r(r-1)x^{r-2}h + r\,x^{r-1}h' + r\,x^{r-1}h' + x^r h'' =$$

$$= r(r-1)x^{r-2}h + 2r\,x^{r-1}h' + x^r h''$$

$$x(1-x)\left[r(r-1)x^{r-2}h + 2r\,x^{r-1}h' + x^r h''\right] +$$

$$+\left[c-(a+b+1)x\right]\left(r\,x^{r-1}h + x^r h'\right) - abx^r h = 0,$$

$$r(r-1)x^{r-1}h + 2rx^{r}h' + x^{r+1}h'' -$$

$$- r(r-1)x^{r}h - 2rx^{r+1}h' - x^{r-2}h'' +$$

$$+ crx^{r-2}h + cx^{r}h' - (a+b+1)rx^{r}h -$$

$$- (a+b+1)x^{r+1}h' - abx^{r}h = 0.$$

U holda (1) quyidagi ko‘rinishda yoziladi:

$$x(1-x)\eta'' + \left[(c+2\rho) - (a+b+1+2\rho)x\right]\eta' -$$

$$-\left[ab + \rho(a+b+\rho) - \frac{\rho(\rho+c-1)}{x}\right]\eta = 0.$$

Bu tenglama (1) tenglama tipiga tegishli bo‘lishi uchun $\rho = 1-c$ (yoki $\rho = 0,$ bu hol bizni qiziqtirmaydi) bo‘lishi kerak. U holda

$$x(1-x)\eta'' + \left\{(2-c) - \left[(a-c+1)+(b-c+1)+1\right]x\right\}\eta' -$$

$$-(a-c+1)(b-c+1)\eta = 0$$

tenglamaga ega bo‘lamiz. Shunday qilib, $\rho = 1-c$ bo‘lganda $y = x^{\rho}\eta$ almashtirish (1) tenglamani xuddi shu ko‘rinishdagi tenglamaga o‘tkazadi, bunda faqat a, b, c larni mos ravishda $a-c+1,$ $b-c+1,$ $2-c$ larga almashtirish zarur. Demak, berilgan (1) tenglama y_1 ga chiziqli bog’liq bo‘lmagan

$$y_2 = x^{1-c}F(a-c+1, b-c+1, 2-c; x)$$

yechimga ega bo‘ladi. Shu bilan birga, y_2 funksiya

$$2-c \neq 0, -1, -2, ..., -n, ...$$

bo‘lgandagina ma’noga ega bo‘ladi. Shunday qilib, (1) tenglamaning umumiy yechimini quyidagi ko‘rinishda yozish mumkin:

$$y = c_1F(a,b,c;x) + c_2x^{1-c}F(a-c+1, b-c+1, 2-c; x),$$

bu yerda c_1 va c_2 - ixtiyoriy o‘zgarmaslar.

Agar gipergeometrik funksiyaga simmetrik bo‘lib kirgan a va b parametrlardan bittasi manfiy butun son $(-n)$ ga teng bo‘lsa, (3) gipergeometrik qator uzilib qoladi va u n - darajali ko‘phadga aylanadi.

Agarda $a=-n_1,\quad b=-n_2$ bo‘lib, bunda $n_1,\quad n_2\in N$ bo‘lsa, u holda gipergeometrik qator ko‘phadga aylanib, uning darajasi n_1 va n_2 sonlarning kichigiga teng bo‘ladi.

(1) tenglamaning $x=1$ maxsus nuqta atrofidagi yechimlarini topish uchun $t=1-x$ almashtirish qilamiz. Natijada (1) tenglamadan yana o‘ziga o‘xshash tenglama hosil bo‘lib, yangi tenglama $a_1=a,\ b_1=b,\ c_1=a+b-c+1$ parametrlarga ega bo‘ladi. Buni e’tiborga olib, (1) tenglamaning $x=1$ maxsus nuqta atrofidagi chiziqli bog’liq bo‘lmagan yechimlarini yozish mumkin:

$$y_3=F\left(a,b,a+b-c+1;1-x\right),$$

$$y_4=\left(1-x\right)^{c-a-b}F\left(c-a,c-b,c+1-a-b;1-x\right),$$

bu yerda $c-a-b\notin Z$ va $\left|1-x\right|<1$.

Agar (1) tenglamada $t=x^{-1},\ \omega=t^{-a}y$ almashtirish bajarsak, $\omega\left(t\right)$ funksiyaga nisbatan, parametrlari $a_1=a,\ b_1=1+a-c,\ c_1=1+a-b,$ bo‘lgan tenglamaga ega bo‘lamiz. Bunda (1) tenglamaning $x=\infty$ maxsus nuqtasi yangi tenglamaning $t=0$ maxsus nuqtasiga almashadi. Bularni e’tiborga olib, (1) tenglamaning $x=\infty$ maxsus nuqtasi atrofidagi chiziqli bog’liq bo‘lmagan ikki yechimlarini topamiz:

$$y_5=x^{-a}F\left(a,1+a-c,1+a-b;1/x\right),$$

$$y_6=x^{-b}F\left(b,1+b-c,1+b-a;1/x\right),$$

bu yerda $a-b\notin Z$ va $\left|x\right|>1.$

Shunday qilib, (1) tenglamning parametrlari $c,\ c-a-b,\ a-b\notin Z$ shartlarni qanoatlantirganda uning oltita asosiy yechimlari gipergeometrik funksiya orqali yozilishini topdik.

2-§. Asosiy ta’riflardan kelib chiquvchi formulalar.

(3) qatorni hadlab differensiallash natijasida darhol ushbu

$$\frac{d}{dx}F(a,b,c;x)=\frac{ab}{c}F(a+1,b+1,c+1;x)$$

formulani hosil qilamiz.

Isbot: Gipergeometrik funksiyaning qator ko‘rinishini yozib olamiz:

$$F(a,b,c;x)=1+\frac{a\cdot b}{c\cdot 1!}x+\frac{a(a+1)b(b+1)}{c(c+1)2!}x^2+$$

$$+\ldots+\frac{a(a+1)\ldots(a+n-1)a(b+1)\ldots(a+n-1)}{c(c+1)\ldots(c+n-1)(n-1)!}x^{n-1}=$$

$$=1+\frac{ab}{c}x+\frac{(a+1)(b+1)}{(c+1)2!}x^2+$$

$$+\ldots+\frac{(a+1)\ldots(a+n-1)(b+1)\ldots(a+n-1)}{(c+1)\ldots(c+n-1)(n-1)!}x^{n-1}$$

Bu qatorni hadlab differensiallaymiz

$$\frac{d}{dx}F(a,b,c;x)=\frac{ab}{c}\left[1+\frac{(a+1)(b+1)}{(c+1)}x+\right.$$

$$\left.+\ldots+\frac{(a+1)\ldots(a+n-1)(b+1)\ldots(a+n-1)}{(c+1)\ldots(c+n-1)(n-2)!}x^{n-2}\right]\ldots$$

Bu esa

$$\frac{d}{dx}F(a,b,c;x)=\frac{ab}{c}F(a+1,b+1,c+1;x)$$

Formula isbotlandi.

(3) qatorni avval x^a ga ko‘paytirib, so‘ngra hadlab differensiallasak, quyidagi formulalar kelib chiqadi:

$$\frac{d}{dx}\left[x^aF(a,b,c;x)\right]=ax^{a-1}F(a+1,b,c;x),$$

Yuqoridagi gipergeometrik funksiyaning qator ko‘rinishiga x^a ko‘paytirib, hadlab differensiallaymiz

$$\frac{d}{dx}\left[x^aF(a,b,c;x)\right]=ax^{a-1}+\frac{ab}{c}(a+1)x^a+\frac{a(a+1)(a+2)b(b+1)}{c(c+1)}x^{a+1}+$$

$$+\ldots+\frac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{c(c+1)\ldots(c+n-1)(n-1)!}(a+n)x^{a+n}$$

Olingan hosiladan umumiy ko‘paytuvchi ax^{a-1} ni chiqarib, qolgan qismini gipergeometrik funksiya ko‘rinishda yozamiz:

$$\frac{d}{dx}\left[x^aF(a,b,c;x)\right]=ax^{a-1}\left[1+\frac{(a+1)b}{c}x+\frac{(a+1)(a+2)b(b+1)}{c(c+1)}x^2+\right.$$

$$+\ldots+\frac{(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{c(c+1)\ldots(c+n-1)(n-1)!}(a+n)x^n\ldots$$

Bundan

$$\frac{d}{dx}\left[x^a F(a,b,c;x)\right]=ax^{a-1}F(a+1,b,c;x)$$

tenglik kelib chiqadi.

Yuqoridagi isbotga ko‘ra quyidagi 2 ta differensiallash formulasini ham isbotlash mumkin.

$$\frac{d}{dx}\left[x^b F(a,b,c;x)\right]=bx^{b-1}F(a,b+1,c;x),$$

$$\frac{d}{dx}\left[x^{c-1}F(a,b,c;x)\right]=(c-1)x^{c-2}F(a,b,c-1;x)$$

Xuddi shu kabi quyidagi tengliklar ham o‘rinli:

1. $\dfrac{d^n}{dx^n}F(a,b,c;x)=\dfrac{(a)_n(b)_n}{(c)_n}F(a+n,b+n,c+n;x),$

2. $\dfrac{d^n}{dx^n}[x^{a+n-1}F(a,b,c;x)]=(a)_n x^{a-1}F(a+n,b,c;x),$

3. $\dfrac{d^n}{dx^n}[x^{c-1}F(a,b,c;x)]=(c-n)_n x^{c-n-1}F(a,b,c-n;x),$

4. $\dfrac{d^n}{dx^n}[(1-x)^{a+n-1}F(a,b,c;x)]=$

$$=(-1)^n\frac{(a)_n(c-b)_n}{(c)_n}(1-x)^{a-1}F(a+n,b,c+n;x),$$

5. $\frac{d^n}{dx^n}[(1-x)^{a+b-c}F(a,b,c;x)] =$

$= \frac{(c-a)_n(c-b)_n}{(c)_n}(1-x)^{a+b-c-n}F(a,b,c+n;x),$

6. $\frac{d^n}{dx^n}[x^{c-1}(1-x)^{b-c+n}F(a,b,c;x)] =$

$= (c-n)_n x^{c-n-1}(1-x)^{b-c}F(a-n,b;c-n;x),$

7. $\frac{d^n}{dx^n}[x^{c-1}(1-x)^{a+b-c}F(a,b,c;x)] =$

$= (c-n)_n x^{c-n-1}(1-x)^{a+b-c-n}F(a-n,b-n,c-n;x)$

8. $\frac{d^n}{dx^n}[x^{c-a+n-1}(1-x)^{a+b-c}F(a,b,c;x)] =$

$= (c-a)_n x^{c-a-1}(1-x)^{a+b-c-n}F(a-n,b,c;x).$

Bu tengliklar to‘g’riligiga, masalan, matematik induksiya usuli bilan ishonch hosil qilish mumkin.

Bu tengliklarni birinchisini o‘rinli ekanligini ko‘rsatib o‘tamiz. Yuqorida gipergeometrik funksiyani bir marta differensiallab

$$\frac{d}{dx}F(a,b,c;x) = \frac{ab}{c}F(a+1,b+1,c+1;x)$$

tenglikni hosil qilgan edik. Bundan foydalanib $\frac{d^2}{dx^2}F(a,b,c;x)$ ni topishimiz mumkin.

$$\frac{d^2}{dx^2}F(a,b,c;x)=\frac{a(a+1)b(b+1)}{c(c+1)}F(a+2,b+2,c+2;x).$$

Bu jarayonni n marta qo‘llasak,

$$\frac{d^n}{dx^n}F(a,b,c;x)=\frac{(a)_n(b)_n}{(c)_n}F(a+n,b+n,c+n;x)$$

kelib chiqadi.

Odatda ushbu oltita $F(a\pm1,b,c;x)$, $F(a,b\pm1,c;x)$, $F(a,b,c\pm1;x)$ funksiyalar $F(a,b,c;x)$ *funksiyaga qo‘shni funksiyalar deyiladi.* $F(a,b,c;x)$ va ixtiyoriy unga ana shunday qo‘shni ikki funksiyalar orasida koeffitsientlari x ga bog’liq chiziqli funksiya bo‘lgan chiziqli kombinatsiya mavjud. Bunday kombinatsiyalar 15 ta bo‘lib, ularni Gauss topgan. Quyida biz ularning to‘la ro‘yxatini isboti bilan keltiramiz. Bunda F, $F(a\pm1)$, $F(b\pm1)$, $F(c\pm1)$ orqali mos ravishda $F(a,b,c;x)$, $F(a\pm1,b,c;x)$, $F(a,b\pm1,c;x)$, $F(a,b,c\pm1;x)$ funksiyalar tushuniladi. *F, F(a+1),F(a-1)* larning qator ko’rinishidan foydalanib, x^n ishtirok etgan hadlarni yig’amiz:

1. $$\left[c-2a-(b-a)x\right]F+a(1-x)F(a+1)-(c-a)F(a-1)=0,$$

Isbot:

$$(c-2a)\frac{(a)_n(b)_n}{(c)_n n!}x^n-(b-a)\frac{(a)_n(b)_n}{(c)_n n!}x^{n+1}a\frac{(a+1)_n(b)_n}{(c)_n n!}x^n-$$

$$-a\frac{(a+1)_n(b)_n}{(c)_n n!}x^{n+1}-(c-a)\frac{(a-1)_n(b)_n}{(c)_n n!}x^n=0$$

Buning hadlarini soddalashtiramiz

$$(c-2a)\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)n!}x^n-(b-a)\frac{\Gamma(a+n-1)\Gamma(b+n-1)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n-1)(n-1)!}x^n+$$

$$+a\frac{\Gamma(a+n+1)\Gamma(b+n)\Gamma(c)}{\Gamma(a+1)\Gamma(b)\Gamma(c+n)n!}x^n-$$

$$-a\frac{\Gamma(a+n)\Gamma(b+n-1)\Gamma(c)}{\Gamma(a+1)\Gamma(b)\Gamma(c+n-1)(n-1)!}x^n=$$

$$=(c-a)\frac{\Gamma(a+n-1)\Gamma(b+n)\Gamma(c)}{\Gamma(a-1)\Gamma(b)\Gamma(c+n)n!}x^n$$

Hosil bo‘lgan tenglamada Gamma funksiyaning xossalaridan foydalanib, tenglikni har ikki tomonini $\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo‘lib,

$$c-a-(b-a)\frac{c+n-1}{(a+n-1)(b+n-1)}n+n-\frac{c+n-1}{b+n-1}n=(c-a)\frac{a-1}{a+n-1}$$

hosil qilamiz. Ifodani soddalashtirish natijasida

$$[n(b+n-1)(c+n-1)-n(c+n-1)(b+n-1)]x^n=0$$

ekanligi kelib chiqadi. Formula isbotlandi.

2. $$(b-a)F+aF(a+1)-bF(b+1)=0,$$

Isbot:

$$(b-a)\frac{(a)_n(b)_n}{(c)_n n!}+a\frac{(a+1)_n(b)_n}{(c)_n n!}=$$

$$=(b-a)\frac{(a)_n(b)_n}{(c)_n n!}+a(a+n)\frac{(a)_n(b)_n}{a(c)_n n!}=$$

$$=\frac{(a)_n(b)_n}{(c)_n n!}(b-a+a+n)=\frac{(a)_n(b)_n}{(c)_n n!}(b+n)=b\frac{(a)_n(b+1)_n}{(c)_n n!}$$

3. $$(c-a-b)F+a(1-x)F(a+1)-(c-b)F(b-1)=0,$$

Isbot: Bu tenglikning qator ko'rinishidan foydalanib, quyidagiga ega bo'lamiz:

$$(c-a-b)\frac{(a)_n(b)_n}{(c)_n n!}x^n+a(1-x)\frac{(a+1)_n(b)_n}{(c)_n n!}x^n=(c-b)\frac{(a)_n(b-1)_n}{(c)_n n!}x^n.$$

Hosil bo'lgan tenglamada poxgammer ifodadan Gamma funksiyaga o'tib, hamda Gamma funksiyaning xossalaridan foydalanib, tenglikni har ikki tomonini $\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo'lib,

$$c-b+n-n\frac{c+n-1}{b+n-1}=\frac{(c-b)(b-1)}{b+n-1}$$

tenglikni hosil qilamiz. Ifodani soddalashtirish natijasida

$$cb+cn-c-b^2-bn+b+bn+n^2-n-cn-n^2+n-cb+c+b^2-b=0$$

ekanligi kelib chiqadi. Formula isbotlandi.

4. $$c\left[a-(c-b)x\right]F-ac(1-x)F(a+1)+(c-a)(c-b)xF(c+1)=0,$$

Isbot: Bu yerda quyidagiga egamiz:

$$ac\frac{(a)_n(b)_n}{(c)_n n!}x^n - c(c-b)\frac{(a)_n(b)_n}{(c)_n n!}x^{n+1} + (c-a)(c-b)\frac{(a)_n(b)_n}{(c+1)_n n!}x^{n+1} =$$

$$= ac\frac{(a+1)_n(b)_n}{(c)_n n!}x^n - ac\frac{(a+1)_n(b)_n}{(c)_n n!}x^{n+1}.$$

Hosil bo‘lgan tenglamada poxgammer ifodadan Gamma funksiyaga o‘tib, hamda Gamma funksiyaning xossalaridan foydalanib, tenglikni xar ikki tomonini $\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo‘lib,

$$ac - c(c-b)\frac{n(c+n-1)}{(a+n-1)(b+n-1)} + (c-a)(c-b)\frac{cn}{(a+n-1)(b+n-1)} =$$

$$= c(a+n) - c\frac{n(c+n-1)}{b+n-1}$$

ni hosil qilamiz. Ifodani soddalashtirish natijasida

$$(a+n+1)(c-b) - (c-b)(a+n-1) = 0$$

ekanligi kelib chiqadi. Formula isbotlandi.

5. $$(c-a-1)F + aF(a+1) - (c-1)F(c-1) = 0, \qquad (4)$$

Isbot:

$$(c-a-1)\frac{(a)_n(b)_n}{(c)_n n!} + a\cdot\frac{(a+1)_n(b)_n}{(c)_n n!} =$$

$$= (c-a-1)\frac{(a)_n(b)_n}{(c)_n n!} + a\cdot(a+n)\cdot\frac{(a)_n(b)_n}{a\cdot(c)_n n!} =$$

$$= \frac{(a)_n(b)_n}{(c)_n n!}\left[c-a-1+(a+n)\right] = \frac{(a)_n(b)_n}{(c)_n n!}\left[(c-1)+n\right] = \frac{(a)_n(b)_n}{(c-1)_n n!}.$$

Formula isbotlandi.

6. $$(c-a-b)F-(c-a)F(a-1)+b(1-x)F(b+1)=0,$$

Isbot:

$$(c-a-b)\frac{(a)_n(b)_n}{(c)_n n!}x^n+b\frac{(a)_n(b+1)_n}{(c)_n n!}x^n-b\frac{(a)_n(b+1)_n}{(c)_n n!}x^{n+1}=$$

$$=(c-a)\frac{(a-1)_n(b)_n}{(c)_n n!}x^n.$$

Hosil bo‘lgan tenglamada poxgammer ifodadan Gamma funksiyaga o‘tib, hamda Gamma funksiyaning xossalaridan foydalanib, tenglikni xar ikki tomonini $\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo‘lib,

$$(c-a-b)+(b+n)-\frac{(c+n-1)n}{a+n-1}=(c-a)\frac{a-1}{a+n-1}$$

ni hosil qilamiz. Ifodani soddalashtirish natijasida

$$(c-a+n)(a+n-1)-(c-a)(a-1)-n(c+n-1)=0$$

ekanligi kelib chiqadi. Formula isbotlandi.

7. $$(b-a)(1-x)F-(c-a)F(a-1)+(c-b)F(b-1)=0,$$

Isbot:

$$(b-a)\frac{(a)_n(b)_n}{(c)_n n!}x^n-(b-a)\frac{(a)_n(b)_n}{(c)_n n!}x^{n+1}+(c-b)\frac{(a)_n(b-1)_n}{(c)_n n!}x^n=$$

$$=(c-a)\frac{(a-1)_n(b)_n}{(c)_n n!}x^n.$$

Hosil bo‘lgan tenglamada poxgammer ifodadan Gamma funksiyaga o‘tib, hamda Gamma funksiyaning xossalaridan foydalanib, tenglikni har ikki tomonini $\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo‘lib,

$$(b-a)-(b-a)\frac{(c+n-1)n}{(a+n-1)(b+n-1)}+(c-b)\frac{b-1}{b+n-1}=$$

$$=(c-a)\frac{a-1}{a+n-1}$$

hosil qilamiz. Ifodani soddalashtirish natijasida

$$(a+n-1)\{(b-a)(b+n-1)+(c-b)(b-1)\}-n(b-a)(c+n-1)-$$

$$-(ca-c-a^2+a)(b+n-1)=0$$

ekanligi kelib chiqadi. Formula isbotlandi.

8. $$c(1-x)F-cF(a-1)+(c-b)xF(c+1)=0,$$

Isbot: x^n ishtirok etgan hadlarni yig’sak, quyidagiga ega bo’lamiz:

$$c\frac{(a)_n(b)_n}{(c)_n n!}x^n-c\frac{(a)_n(b)_n}{(c)_n n!}x^{n+1}+(c-b)\frac{(a)_n(b)_n}{(c+1)_n n!}x^{n+1}=c\frac{(a-1)_n(b)_n}{(c)_n n!}x^n.$$

Hosil bo‘lgan tenglamada poxgammer ifodadan Gamma funksiyaga o‘tib, hamda Gamma funksiyaning xossalarida foydalanib, tenglikni xar ikki tomonini $\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo‘lib,

$$c-c\frac{(c+n-1)n}{(a+n-1)(b+n-1)}+(c-b)\frac{cn}{(a+n-1)(b+n-1)}=c\frac{a-1}{a+n-1}$$

hosil qilamiz. Ifodani soddalashtirish natijasida

$$nc(b+n-1)-cn(b+n-1)=0$$

ekanligi kelib chiqadi. Formula isbotlandi.

9. $\left[a-1-(c-b-1)x\right]F+(c-a)F(a-1)-(c-1)(1-x)F(c-1)=0,$

Isbot:

$$(a-1)\frac{(a)_n(b)_n}{(c)_n n!}x^n-(c-b-1)\frac{(a)_n(b)_n}{(c)_n n!}x^{n+1}+(c-a)\frac{(a-1)_n(b)_n}{(c)_n n!}x^n=$$

$$=(c-1)\frac{(a)_n(b)_n}{(c-1)_n n!}x^n-(c-1)\frac{(a)_n(b)_n}{(c-1)_n n!}x^{n+1}.$$

Hosil bo‘lgan tenglamada poxgammer ifodadan Gamma funksiyaga o‘tib, hamda Gamma funksiyaning xossalaridan foydalanib, tenglikni har ikki tomonini $\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo‘lib,

$$(a-1)-(c-b-1)\frac{(c+n-1)n}{(a+n-1)(b+n-1)}+(c-a)\frac{a-1}{a+n-1}=$$

$$=(c-1)\frac{c+n-1}{c-1}-(c-1)\frac{(c+n-1)(c+n-2)}{(a+n-1)(b+n-1)(c-1)}n$$

hosil qilamiz. Ifodani soddalashtirish natijasida

$n(b+n-1)(c+n-1-n+1-c)=0$ ekanligi kelib chiqadi. Formula isbotlandi.

10. $\left[c-2b+(b-a)x\right]F+b(1-x)F(b+1)-(c-b)F(b-1)=0,$

Isbot:

$$(c-2b)\frac{(a)_n(b)_n}{(c)_n n!}x^n+(b-a)\frac{(a)_n(b)_n}{(c)_n n!}x^{n+1}+b\frac{(a)_n(b+1)_n}{(c)_n n!}x^n-$$

$$-b\frac{(a)_n(b+1)_n}{(c)_n n!}x^{n+1}=(c-b)\frac{(a)_n(b-1)_n}{(c)_n n!}x^n$$

Hosil bo‘lgan tenglamada poxgammer ifodadan Gamma funksiyaga o‘tib, hamda Gamma funksiyaning xossalaridan foydalanib, tenglikni har ikki tomonini $\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo‘lib,

$$(c-2b)+(b-a)\frac{(c+n-1)n}{(a+n-1)(b+n-1)}+b\frac{b+n}{b}-b\frac{(c+n-1)n}{b(a+n-1)}=$$

$$=(c-b)\frac{b-1}{b+n-1}$$

hosil qilamiz. Ifodani soddalashtirish natijasida

$$n(c+n-1)[a+n-1+1-a-n]=0$$

ekanligi kelib chiqadi. Formula isbotlandi.

11. $c\left[b-(c-a)x\right]F-bc(1-x)F(b+1)+(c-a)(c-b)xF(c+1)=0,$

Isbot:

$$cb\frac{(a)_n(b)_n}{(c)_n n!}x^n+bc\frac{(a)_n(b+1)_n}{(c)_n n!}x^{n+1}+(c-a)(c-b)\frac{(a)_n(b)_n}{(c+1)_n n!}x^{n+1}=$$

$$=c(c-a)\frac{(a)_n(b)_n}{(c)_n n!}x^{n+1}+bc\frac{(a)_n(b+1)_n}{(c)_n n!}x^n$$

Hosil bo‘lgan tenglamada poxgammer ifodadan Gamma funksiyaga o‘tib, hamda Gamma funksiyaning xossalaridan foydalanib, tenglikni har ikki tomonini

$\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo‘lib,

$$cb+bc\frac{(c+n-1)n}{b(a+n-1)}+(c-a)(c-b)\frac{cn}{(a+n-1)(b+n-1)}=$$

$$=c(c-a)\frac{(c+n-1)n}{(a+n-1)(b+n-1)}+bc\frac{b+n}{b}$$

hosil qilamiz. Ifodani soddalashtirish natijasida

$$(c-a)(b+n-1-b-n+1)=0$$

ekanligi kelib chiqadi. Formula isbotlandi.

12. $(c-b-1)F+bF(b+1)-(c-1)F(c-1)=0,$

Isbot:

$$(c-b-1)\frac{(a)_n(b)_n}{(c)_n n!}+b\frac{(a)_n(b+1)_n}{(c)_n n!}=$$

$$=(c-b-1)\frac{(a)_n(b)_n}{(c)_n n!}+b(b+n)\frac{(a)_n(b)_n}{b(c)_n n!}=$$

$$=\frac{(a)_n(b)_n}{(c)_n n!}(c-b-1+b+n)=\frac{(a)_n(b)_n}{(c)_n n!}(c-1)+n\frac{(a)_n(b)_n}{(c-1)_n n!}$$

13. $c(1-x)F-cF(b-1)+(c-a)xF(c+1)=0,$

Isbot:

$$c\frac{(a)_n(b)_n}{(c)_n n!}x^n - c\frac{(a)_n(b)_n}{(c)_n n!}x^{n+1} + (c-a)\frac{(a)_n(b)_n}{(c+1)_n n!}x^{n+1} = c\frac{(a)_n(b-1)_n}{(c)_n n!}x^n$$

Hosil bo‘lgan tenglamada poxgammer ifodadan Gamma funksiyaga o‘tib, hamda Gamma funksiyaning xossalaridan foydalanib, tenglikni har ikki tomonini $\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo‘lib,

$$c - c\frac{(c+n-1)n}{(a+n-1)(b+n-1)} + (c-a)\frac{cn}{(a+n-1)(b+n-1)} = c\frac{b-1}{b+n-1}$$

hosil qilamiz. Ifodani soddalashtirish natijasida

$$n(a+n-1)+n(1-a-n)=0$$

ekanligi kelib chiqadi. Formula isbotlandi.

14. $\left[b-1-(c-a-1)x\right]F+(c-b)F(b-1)-(c-1)(1-x)F(c-1)=0.$

Isbot: Bu yerda quyidagiga egamiz:

$$(b-1)\frac{(a)_n(b)_n}{(c)_n n!}x^n - (c-a-1)\frac{(a)_n(b)_n}{(c)_n n!}x^{n+1} +$$

$$+(c-b)\frac{(a)_n(b-1)_n}{(c)_n n!}x^n = (c-1)\frac{(a)_n(b)_n}{(c-1)_n n!}x^n - (c-1)\frac{(a)_n(b)_n}{(c-1)_n n!}x^{n+1}$$

Hosil bo‘lgan tenglamada poxgammer ifodadan Gamma funksiyaga o‘tib, hamda Gamma funksiyaning xossalaridan foydalanib, tenglikni har ikki tomonini $\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo‘lib,

$$b-1-(c-a-1)\frac{(c+n-1)n}{(a+n-1)(b+n-1)}+(c-b)\frac{b-1}{b+n-1}=$$

$$=(c-1)\frac{c+n-1}{c-1}-(c-1)\frac{n(c+n-2)(c+n-1)}{(a+n-1)(b+n-1)(c-1)}$$

hosil qilamiz. Ifodani soddalashtirish natijasida

$$(a+n-1)(b-c-n)(b+n-1)+(c-b)(b-1)+n(c+n-1)=0$$

ekanligi kelib chiqadi. Formula isbotlandi.

15. $c[c-1-(2c-a-b-1)x]F+(c-a)(c-b)xF(c+1)-c(c-1)(1-x)F(c-1)=0$

Isbot:

$$c(c-1)\frac{(a)_n(b)_n}{(c)_n n!}x^n-c(2c-a-b-1)\frac{(a)_n(b)_n}{(c)_n n!}x^{n+1}+$$

$$+(c-b)(c-a)\frac{(a)_n(b)_n}{(c+1)_n n!}x^{n+1}=c(c-1)\frac{(a)_n(b)_n}{(c-1)_n n!}x^n-c(c-1)\frac{(a)_n(b)_n}{(c-1)_n n!}x^{n+1}$$

tenglamada poxgammer ifodadan Gamma funksiyaga o‘tib, hamda Gamma funksiyaning xossalarida foydalanib, tenglikni har ikki tomonini $\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(c+n)n!}$ ifodaga bo‘lib,

$$c(c-1)-\frac{cn(2c-a-b-1)(c+n-1)}{(a+n-1)(b+n-1)}+\frac{cn(c-a)(c-b)}{(a+n-1)(b+n-1)}=$$

$$=c(c-1)\frac{c+n-1}{c-1}-c(c-1)\frac{(c+n-2)(c+n-1)n}{(a+n-1)(b+n-1)(c-1)}$$

hosil qilamiz. Ifodani soddalashtirish natijasida

$$(c-a)(c-b)-(a+n-1)(b+n-1)+$$

$$+(c+n-1)[c+n-2-2c+a+b+1]=0$$

ekanligi kelib chiqadi. Formula isbotlandi.

Ikkita parametri o‘zgarmas bo‘lgan qo‘shni gipergeometrik funksiyalar orasida esa quyidagi bog’lanishlar mavjud:

$$(c-a)F(a-1)+(2a-c-ax+bx)F+a(1-x)F(a+1)=0,$$

$$(c-b)F(b-1)+(2b-c-bx+ax)F+b(x-1)F(b+1)=0,$$

$$c(c-1)(x-1)F(c-1)+c[c-1-(2c-a-b-1)x]F+$$

$$+(c-a)(c-b)xF(c+1)=0.$$

Bu tengliklarning to‘g’riligiga ishtirok etayotgan gipergeometrik funksiyalarning qatorga yoyilmasidan foydalanib, o‘zgaruvchi x ning mos darajalari koeffitsientlarini taqqoslash usuli bilan ishonch hosil qilish mumkin.

3-§. Gipergeometrik funksiyaning integral ko‘rinishi.

(3) qatorni

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$$

tenglikni e’tiborga olib, ushbu

$$F(a,b,c;x)=1+\sum_{n=1}^{\infty}\frac{\Gamma(c)\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)\Gamma(c+n)n!}x^n=$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}+\sum_{n=1}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}x^n\right]=$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} x^n$$

ko‘rinishda yozib olamiz.

Bundan formulaga asosan

$$B\left(b+n, c-b\right) = \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)}$$

bo‘lganligi sababli, avvalgi tenglik

$$F\left(a,b,c;x\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{n!} x^n B\left(b+n, c-b\right)$$

ko‘rinishda yoziladi yoki

$$F\left(a,b,c;x\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{n!} x^n \int_0^1 t^{n+b-1}\left(1-t\right)^{c-b-1} dt.$$

Bu yerdagi integral n ning barcha qiymatlarida yaqinlashuvchi bo‘lgani uchun

$$b > 0, \quad c-b > 0 \text{ yoki } c > b > 0 \tag{5}$$

shartlarning bajarilishi zarurdir.

Avvalgi tenglikni ushbu

$$F\left(a,b,c;x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{n!\Gamma(a)} \int_0^1 t^{n+b-1}\left(1-t\right)^{c-b-1} x^n dt =$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}\left(1-t\right)^{c-b-1} \left[\sum_{n=0}^{\infty} \frac{(a)_n}{n!} \left(xt\right)^n \right] dt$$

ko‘rinishda yozib olamiz. Integral ostidagi yig’indi $\left(1-xt\right)^{-a}$ funksiyaning cheksiz qatorga yoylmasidan iborat bo‘lgani uchun

$$F\left(a,b,c;x\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-xt\right)^{-a}dt \qquad (6)$$

formulaga ega bo‘lamiz. *Bu esa gipergeometrik funksiyaning integral ko‘rinishidir*.

(5) shartlarni bitta $c-a-b>0$ shart bilan almashtirish mumkin. haqiqatan ham, agar $a<0$ bo‘lsa, $(-a)>0$ bo‘ladi va bu tengsizlikning (5) tengsizlikni ikkinchisi bilan qo‘shib, $c-a-b>0$ tengsizlikni hosil qilamiz; agarda $a>0$ bo‘lsa, bu tengsizlikdan, (5) tengsizliklarning ikkinchisidan kuchliroq bo‘lgan $c-b>a$ tengsizlikka ega bo‘lamiz.

4-§. Gipergeometrik funksiyani analitik davom ettirishga oid va boshqa ba’zi formulalar.

Endi gipergeometrik funksiyaning $x=1$ dagi qiymatini hisoblaymiz. Shu maqsadda, (6) formuladagi integral $b>0,\ c>0$ va $|x|<1$ bo‘lganda tekis yaqinlashuvchi bo‘lgani sababli $x\to 1$ da limitga o‘tamiz:

$$\lim_{x\to 1}F\left(a,b,c;x\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\lim_{x\to 1}\int_0^1\left[t^{b-1}\left(1-t\right)^{c-b-1}\left(1-xt\right)^{-a}\right]dt=$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\lim_{x\to 1}\left[t^{b-1}\left(1-t\right)^{c-b-1}\left(1-xt\right)^{-a}\right]dt=$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}\left(1-t\right)^{c-a-b-1}dt=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}B\left(b,c-a-b\right)=$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Shunday qilib,

$$\lim_{x\to 1} F(a,b,c;x) = F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Agar (6) formuladagi integralda

$$t = \frac{1-s}{1-xs} \text{ yoki } s = \frac{1-t}{1-xt}$$

almashtirish bajarsak, integral quyidagi ko‘rinishda yoziladi:

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt =$$

$$= (1-x)^{c-a-b}\int_0^1 s^{c-b-1}(1-s)^{b-1}(1-xs)^{-(c-a)}\,ds =$$

$$= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}(1-x)^{c-a-b}F(c-a,c-b,c;x).$$

Demak,

$$F(a,b,c;x) = (1-x)^{c-a-b}F(c-a,c-b,c;x). \qquad (7)$$

Bu tenglik avtotransformatsiya formulasi deyiladi.

$F(a,b,c;x) = F(b,a,c;x)$ tenglikni e’tiborga olib, (6) tenglikni

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}\,dt$$

kabi yozib olamiz va integralda $t = 1 - s$ almashtirish bajaramiz:

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}(1-x)^{-b}\int_0^1 s^{c-a-1}(1-s)^{a-1}\left(1-\frac{x}{x-1}s\right)^{-b} ds.$$

Bu yerdagi integralni (6) bilan taqqoslab,

$$F(a,b,c;x) = (1-x)^{-b} F\left(c-a,b,c;\frac{x}{x-1}\right) \qquad (8)$$

formulaga ega bo‘lamiz. Bundan $c = a$ da

$$F(a,b,a;x) = (1-x)^{-b}$$

tenglik kelib chiqadi. Agar $x < 1/2$ bo‘lsa, $\left|x/(x-1)\right| < 1$ bo‘ladi. Bunda (8) tenglikning o‘ng tomonidagi gipergeometrik funksiya, tegishli gipergeometrik qatorning yig’indisi sifatida qaralishi mumkin [1]. Demak, (8) formula $F(a,b,c;x)$ funksiyani $-\infty < x < -1$ oraliqqa analitik davom ettiradi.

(8) tenglikda x ni $(1-x)$ ga almashtirib, uning boshqa ko‘rinishiga ega bo‘lamiz:

$$F(a,b,c;1-x) = x^{-b}F\left(c-a,b,c;\frac{x-1}{x}\right). \qquad (9)$$

Gipergeometrik funksiyani $-1 \le x \le 1$ kesmadan tashqarida aniqlashga xizmat qiluvchi boshqa formulalarni ham keltirib chiqaraylik.

Avval x va $1-x$ argumentli gipergeometrik funksiyalar orasidagi munosabatni topamiz. $|x| < 1$ va $|1-x|$ intervallar kesishmasida (1) tenglamaning $y_1(x) = F(a,b,c;-x)$ yechimi uning y_3 va y_4 yechimlari chiziqli kombinatsiyasi sifatida ifodalanadi, ya’ni

$$F(a,b,c;x) = AF(a,b,a+b-c+1;1-x) +$$

$$+B(1-x)^{c-a-b}F(c-a,c-b,c-a-b+1;1-x), \qquad c-a-b\notin \mathrm{Z}. \quad (10)$$

Bu yerda a, b, c parametrlarning (10) tenglikdagi barcha gipergeometrik funksiyalar mahnoga ega bo‘ladigan qiymatlari qaraladi.

$x=1$ bo‘lganda (10) tenglamaning o‘ng tomoni a, b, c parametrlarning ixtiyoriy qiymatlarida mahnoga ega, chap tomoni chekli bo‘lishi esa $c-a-b$ ning ishorasiga bog’liq. Agar $c-a-b>0$ bo‘lsa, (10) dan $x=1$ da

$$A=F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (11)$$

kelib chiqadi. Agar $c-a-b<0$ bo‘lsa, (10) ning chap tomoniga (7) formulani qo‘llab, so‘ngra hosil bo‘lgan tenglikni $(1-x)^{a+b-c}$ ga ko‘paytirib va $x=1$ deb

$$B=F(c-a,c-b,c;1)=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \quad (12)$$

ekanligini topamiz.

(11) va (12) tengliklarga asosan (10) tenglikni

$$F(a,b,c;x)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1,1-x)+$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}F(c-a,c-b,c-a-b+1;1-x),$$

$$c-a-b\notin \mathrm{Z} \quad (13)$$

ko‘rinishda yozish mumkin. (8) ning o‘ng tomoniga (13) formulani qo‘llab, quyidagi formulaga ega bo‘lamiz:

$$F(a,b,c;x)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(1-x)^{-a}F\left(a,c-b,a-b+1;\frac{1}{1-x}\right)+$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(1-x)^{-b}F\left(c-a,b,b-a+1;\frac{1}{1-x}\right),a-b\notin \mathrm{Z}. \qquad (14)$$

Agar (8) va (13) tengliklarning o‘ng tomoniga mos ravishda (14) va (9) formulalarni qo‘llasak,

$$F(a,b,c;x)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-x)^{-a}F\left(a,a-c+1,a-b+1;\frac{1}{x}\right)+$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(-x)^{-b}F\left(b,b-c+1,b-a+1;\frac{1}{x}\right),a-b\notin \mathrm{Z}; \qquad (15)$$

$$F(a,b,c;x)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}x^{-a}F\left(a,a-c+1,a+b-c+1;\frac{x-1}{x}\right)+$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}x^{a-c}(1-x)^{c-a-b}F\left(c-a,1-a,c-a-b+1;\frac{x-1}{x}\right),$$

$$c-a-b\notin \mathrm{Z} \qquad (16)$$

tengliklar kelib chiqadi.

(13), (14), (15), (16) tengliklar gipergeometrik funksiyani mos ravishda $|1-x|<1$, $|1-x|>1$, $|x|>1$, $|(1-x)/x|<1$ tengsizliklar bilan *aniqlanuvchi oraliqlarga analitik davomini beradi.* (13) - odatda *Bolts formulasi* deb ataladi.

Quyidagi tenglik $c=a+b$ bo‘lganda gipergeometrik funksiyaning $x=1$ nuqta atrofidagi xulqini ifodalaydi [1]:

$$F\left(a,b,a+b;1-x\right)=-\frac{\Gamma\left(a+b\right)}{\Gamma\left(a\right)\Gamma\left(b\right)}F\left(a,b,1;x\right)\ln x+$$

$$+\frac{\Gamma\left(a+b\right)}{\Gamma^2\left(a\right)\Gamma^2\left(b\right)}\sum_{k=0}^{\infty}\frac{\Gamma\left(a+k\right)\Gamma\left(b+k\right)}{\left(k!\right)^2}\times$$

$$\times\left[2\cdot\frac{\Gamma'\left(1+k\right)}{\Gamma\left(1+k\right)}-\frac{\Gamma'\left(a+k\right)}{\Gamma\left(a+k\right)}-\frac{\Gamma'\left(b+k\right)}{\Gamma\left(b+k\right)}\right]x^k.$$

Gipergeometrik funksiyalar uchun quyidagi tengliklar ham o‘rinli [1]:

$$F\left(a,1-a,c;x\right)=\left(1-x\right)^{c-1}F\left(\frac{c-a}{2},\frac{c+a-1}{2},c;4x\left(1-x\right)\right);$$

$$F\left(a,1-a,c;-x\right)=\left(1+x\right)^{c-1}\left(\sqrt{1+x}+\sqrt{x}\right)^{2-2a-2c}\times$$

$$\times F\left[c+a-1,c-1/2,2c-1;4\sqrt{x\left(1+x\right)}\left(\sqrt{1+x}+\sqrt{x}\right)^{-2}\right];$$

$$F\left(2a,2b,a+b+\frac{1}{2};x\right)=F\left[a,b,a+b+\frac{1}{2};4x\left(1-x\right)\right];$$

$$F\left(a,b,a+b+\frac{1}{2};x\right)=F\left(2a,2b,a+b+\frac{1}{2};\frac{1}{2}-\frac{1}{2}\sqrt{1-x}\right)=$$

$$=\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{-2a}F\left(2a,a-b+\frac{1}{2},a+b+\frac{1}{2};\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right).$$

$$F\left(a-\frac{1}{2},a,2a;x\right)=\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2a}$$

$$F\left(2a,2b,a+b+\frac{1}{2};\frac{1}{2}\right)=\frac{\Gamma\left(a+b+1/2\right)\Gamma\left(1/2\right)}{\Gamma\left(a+1/2\right)\Gamma\left(b+1/2\right)},\ a+b+\frac{1}{2}\neq 0,-1,-2,...$$

Izoh. 2-, 3- va 4- bandda keltirilgan formulalarni $\{z\}=\{x+iy\}$ kompleks o‘zgaruvchi tekisligida ham qarash mumkin. Bunda ba’zi formulalarga qo‘shimcha cheklanishlar kelib chiqadi [1,6].

5-§. Umumlashgan gipergeometrik funksiyalar va qatorlar.

Gaussning gipergeometrik funksiyasi va qatorini parametrlar soni bo‘yicha umumlashtiruvchi ushbu funksiya (qator) lar

$${}_mF_n\left(a_1,a_2,...,a_m;c_1,c_2,...,c_n,x\right)=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\cdot...\cdot\left(a_m\right)_k}{\left(c_1\right)_k\cdot...\cdot\left(c_n\right)_k}\cdot\frac{x^k}{k!}$$

umumlashgan gipergeometrik funksiya (qator) lar deb ataladi, bu yerda a_j, $b_s\neq 0,\ -1,-2,....,\quad j=\overline{1,m},\quad s=\overline{1,n}.$ Bu belgilashlarga asosan $F\left(a,b,c;x\right)={}_2F_1\left(a,b,c;x\right)$ bo‘ladi.

Amaliyotda biz $F\left(a,b,c;x\right)$ funksiya bilan bir qatorda ${}_1F_1\left(a,c;x\right)$ va ${}_2F_2\left(a,b;c,d;x\right)$ funksiyalardan ham foydalanamiz. Jumladan, bu funksiyalar uchun quyidagi tengliklar o‘rinli [1,10]:

$$\int_0^x z^{a+c-1}\left(x-z\right)^{\delta-1}e^{-cz}\,\bar{I}_{\gamma}\left(cz\right)dz=x^{\alpha+\delta+\gamma+1}\frac{\Gamma\left(\alpha+\gamma\right)\Gamma\left(\delta\right)}{\Gamma\left(\alpha+\delta\right)\Gamma\left(\gamma\right)}\times$$

$$\times {}_2F_2\left(\gamma+\frac{1}{2},\alpha+\gamma;2\gamma+1,\alpha+\delta+\gamma;-2cx\right),\ x,\mathrm{Re}\,\delta,\mathrm{Re}(\alpha+\delta)>0, \qquad (17)$$

$$\int_x^{+\infty} z^{2\beta-2}e^{-cz}\bar{I}_{\beta-1}(cz)dz=\frac{x^{2\beta-1}}{1-2\beta}\,{}_1F_1(\beta-1/2;2\beta;-2cx),\quad c\geq 0,x>0,$$

$${}_1F_1(a,c;x)=e^x\,{}_1F_1(c-a,c;-x),$$

$${}_1F_1\left(\frac{1}{2}+\nu,1+2\nu;2x\right)=e^x\bar{I}_\nu(x),$$

$${}_1F_1(a,a;x)=e^x.$$

II BOB. IKKI ARGUMENTLI GIPERGEOMETRIK FUNKSIYALAR.

Bu bobda ikki argumentli gipergeometrik funksiyalarga oid ta'riflarni berib,ularga oid ba'zi formulalarni isbotsiz keltiramiz. Bu yerdagi ba'zi formulalar dunyo olimlarining oxirgi yillarda olgan ilmiy natijalarini e'lon qilgan ilmiy maqolalaridan olingan.

1-§. Ikki argumentli gipergeometrik funksiyalarning ta'riflari.

$$F_1(\alpha,\beta,\beta',\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}\,m!n!}x^m y^n,$$

$$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n\,m!n!}x^m y^n,$$

$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}\,m!n!}x^m y^n,$$

$$H_2(\alpha,\beta,\gamma,\delta,\varepsilon;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\varepsilon)_m\,m!n!}x^m y^n,$$

$$\Xi_2(\alpha,\beta,\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^m y^n,\ |x|<1,$$

$$\mathrm{H}_3(\alpha,\beta,\delta,e;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_m}{(\delta)_m m!n!}x^m y^n,\quad |x|<1.$$

Bulardan dastlabki to‘rttasi Gorn funksiyalari, oxirgi ikkitasi esa *buzilgan gipergeometrik funksiya (qator) lar* deyiladi. Ξ_2 - *Gumbert funksiyasi* deb ham ataladi.

Bu qatorlarning yaqinlashish sohasi quyidagicha:

$$F_1,F_3-\{(x,y):0\le x<1,0\le y<1\};$$

$$F_2-\{(x,y):0\le x<1,0\le y<1-x\};$$

$$H_2-\{(x,y):0\le x<1,0\le y\le 1/2\}\cup\{(x,y):0\le x<1,0\le y<1-x/2\};$$

$$\Xi_2,\mathrm{H}_3-\{(x,y):-1<x<1,-\infty<y<+\infty\}.$$

2-§. Ikki argumentli gipergeometrik funksiyalarning differensial tenglamalari.

$$\left.\begin{aligned}&x(1-x)r+y(1-x)s+\left[\gamma-(\alpha+\beta+1)x\right]p-\beta yq-\alpha\beta z=0,\\&y(1-y)t+x(1-y)s+\left[\gamma-(\alpha+\beta'+1)y\right]q-\beta' xp-\alpha\beta' z=0;\end{aligned}\right\}\ F_1$$

$$\left.\begin{aligned}&x(1-x)r-xys+\left[\gamma-(a+b+1)x\right]p-byq-abz=0,\\&y(1-y)t-xys+\left[\gamma-(a+b+1)y\right]q-bxp-abz=0;\end{aligned}\right\}\ F_2$$

$$\left.\begin{aligned}&x(1-x)r+ys+\left[\gamma-(a+b+1)x\right]p-abz=0,\\&y(1-y)t+xs+\left[\gamma-(a+b+1)y\right]q-abz=0;\end{aligned}\right\}\ F_3$$

$$\begin{cases} x(x-1)r - xys + \left[(a+b+1)x - e\right]p - byq + abz = 0, \\ y(y+1)t - xs + \left[a + (g+d+1)y\right]q + gdz = 0; \end{cases} \quad H_2$$

$$\begin{cases} x(1-x)r + ys + \left[g - (a+b+1)x\right]p - abz = 0, \\ yt + xs + gq - z = 0; \end{cases} \quad X_2$$

$$\begin{cases} x(1-x)r + xys + [d - (a+b+1)x]p + byq - abz = 0, \\ yt - xs + (1-a)q + z = 0. \end{cases} \quad H_3$$

Bu yerda $p = z_x$, $q = z_y$, $r = z_{xx,}$, $s = z_{xy}$, $t = z_{yy}$.

3-§. Ikki argumentli gipergeometrik funksiyalarni $F(a,b,c,;x)$ va $J_\nu(x)$ funksiyalar bo‘yicha yoyilmalari.

$$F_1(\alpha,\beta,\beta',\gamma;x,y) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m m!} x^m F(\alpha+m,\beta',\gamma+m;y) =$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta')_n}{(\gamma)_n n!} y^n F(\alpha+n,\beta,\gamma+n;x);$$

$$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m m!} x^m F(\alpha+m,\beta',\gamma';y) =$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta')_n}{(\gamma')_n n!} y^n F(\alpha+n,\beta,\gamma;x);$$

$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m m!} x^m F(\alpha',\beta',\gamma+m;y) =$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha')_n (\beta')_n}{(\gamma)_n n!} y^n F(\alpha, \beta, \gamma + n, x);$$

$$H_2(\alpha, \beta, \gamma, \delta, \varepsilon; x, y) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\varepsilon)_m m!} x^m F(\gamma, \delta, 1 - \alpha - m; -y) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_n (\delta)_n}{(1-\alpha)_n n!} y^n F(\alpha - n, \beta, \varepsilon; x);$$

$$\Xi_2(\alpha, \beta, \gamma; x, y) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m m!} x^m \overline{J}_{\gamma+m-1}\left(2i\sqrt{y}\right) =$$

$$= \sum_{n=0}^{\infty} \frac{y^n}{(\gamma)_n n!} F(\alpha, \beta, \gamma + n; x);$$

$$\mathrm{H}_3(\alpha, \beta, \delta; x, y) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\delta)_m m!} x^m \overline{J}_{-\alpha-m}\left(2\sqrt{y}\right) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(1-\alpha)_n} \cdot \frac{y^n}{n!} F(\alpha - n, \beta, \delta; x).$$

4-§. Ikki argumentli gipergeometrik funksiyalarning integral ko‘rinishlari.

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 \frac{u^{\alpha-1}(1-u)^{\gamma-\alpha-1}}{(1-ux)^{\beta}(1-uy)^{\beta'}} du,$$

$$\mathrm{Re}\alpha > 0, \mathrm{Re}(\gamma - \alpha) > 0;$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma - \beta)\Gamma(\gamma' - \beta')} \times$$

$$\times\int_0^1\int_0^1 u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-ux-vy)^{-\alpha}\,dudv,$$

$$Re\beta>0,\ Re\beta'>0,\ \mathrm{Re}(\gamma-\beta)>0,\ \mathrm{Re}(\gamma'-\beta')>0;$$

$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}\times$$

$$\times\int_0^1\int_0^{1-v} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux)^{-\alpha}(1-vy)^{-\alpha'}\,dudv,$$

$$Re\beta>0,\ Re\beta'>0,\ \mathrm{Re}(\gamma-\beta-\beta')>0;$$

$$H_2(\alpha,\beta,\gamma,\delta,\varepsilon;x,y)=\frac{\Gamma(\varepsilon)}{\Gamma(\beta)\Gamma(\varepsilon-\beta)}\times$$

$$\times\int_0^1 u^{\beta-1}(1-u)^{\varepsilon-\beta-1}(1-xu)^{-\alpha}F\left[\gamma,\delta,1-\alpha;y(xu-1)\right]du,$$

$$\mathrm{Re}\varepsilon>\mathrm{Re}\beta>0;$$

$$\Xi_2(\alpha,\beta,\gamma;x,y)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\times$$

$$\times\int_0^1 v^{\alpha-1}(1-v)^{\gamma-\alpha-1}(1-vx)^{-\beta}\,\bar{I}_{\gamma-\alpha-1}\left[2\sqrt{y(1-v)}\right]dv,$$

$$\mathrm{Re}\gamma>\mathrm{Re}\alpha>0;$$

$$\mathrm{H}_3(\alpha,\beta,\delta;x,y)=\frac{\Gamma(\delta)}{\Gamma(\beta)\Gamma(\delta-\beta)}\times$$

$$\times\int_0^1 u^{\beta-1}(1-u)^{\delta-\beta-1}(1-ux)^{-\alpha}\,\overline{J}_{-\alpha}\left[2\sqrt{y(1-ux)}\right]du,$$

$$Re\delta > Re\beta > 0.$$

5-§. Ikki argumentli gipergeometrik funksiyalar parametrlarning xususiy qiymatlaridagi ko‘rinishi.

$$F_1(\alpha,\beta,\beta',\beta+\beta';x,y) = (1-y)^{-\alpha}F\left(\alpha,\beta,\beta+\beta';\frac{x-y}{1-y}\right);$$

$$F_2(\alpha,\beta,\beta',\beta,\gamma';x,y) = (1-x)^{-\alpha}F\left(\alpha,\beta',\gamma';\frac{y}{1-x}\right);$$

$$F_2(\alpha,\beta,\beta',\alpha,\alpha;x,y) = (1-x)^{-\beta}(1-y)^{-\beta'}F\left(\beta,\beta',\alpha;\frac{xy}{(1-x)(1-y)}\right);$$

$$F_3(\alpha,\gamma-\alpha,\beta,\gamma-\beta,\gamma;x,y) = (1-y)^{\alpha+\beta-\gamma}F(\alpha,\beta,\gamma;x+y-xy);$$

$$H_2(\alpha,\beta,\gamma,\delta,\beta;x,y) = (1-x)^{-\alpha}F[\gamma,\delta,1-\alpha;y(x-1)];$$

$$\mathrm{H}_3(\alpha,\beta,\beta;x,y) = (1-x)^{-\alpha}\overline{J}_{-\alpha}\left[2\sqrt{y(1-x)}\right];$$

6-§. Ikki argumentli gipergeometrik funksiyalarning o‘zgaruvchilarning xususiy qiymatlaridagi ko‘rinishi.

$$F_1(\alpha,\beta,\beta',\gamma;x,1) =$$

$$= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta')}F(\alpha,\beta,\gamma-\beta';x),\ \mathrm{Re}(\gamma-\alpha-\beta')>0,$$

$$F_1(\alpha,\beta,\beta',\gamma;1,y)=$$

$$=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta',\gamma-\beta;y),\ \mathrm{Re}(\gamma-\alpha-\beta)>0,$$

$$F_1(\alpha,\beta,\beta',\gamma;x,x)=F(\alpha,\beta+\beta',\gamma;x),$$

$$\mathrm{H}_3(\alpha,\beta,\delta;1,y)=\frac{\Gamma(\delta)\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)}\,{}_1F_2(\delta-\alpha-\beta,1-\alpha,\delta-\alpha;-y),$$

$$\mathrm{Re}(\delta-\alpha-\beta)>0.$$

7-§. Ikki argumentli gipergeometrik funksiyalarning differensiallash formulalari.

$$\frac{\partial^{m+n}}{\partial x^m\partial\gamma^n}F_1(\alpha,\beta,\beta',\gamma;x,y)=$$

$$=\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}}F_1(\alpha+m+n,\beta+m,\beta'+n;\gamma+m+n;x,y);$$

$$\frac{\partial^{m+n}}{\partial x^m\partial y^n}F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y)=$$

$$=\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n}F_2(\alpha+m+n;\beta+m,\beta'+n;\gamma+m,\gamma'+n;x,y);$$

$$\frac{\partial^{m+n}}{\partial x^m\partial y^n}F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)=$$

$$=\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}}F_3(\alpha+m,\alpha'+n,\beta+m,\beta'+n,\gamma+m+n;x,y).$$

$$\frac{x}{\beta}\frac{\partial}{\partial x}F_1(\alpha,\beta,\beta',\gamma;x,y)=F_1(\alpha,\beta+1,\beta',\gamma;x,y)-F_1(\alpha,\beta,\beta',\gamma;x,y);$$

$$\frac{y}{\beta'}\frac{\partial}{\partial y}F_1(\alpha,\beta,\beta',\gamma;x,y)=F_1(\alpha,\beta,\beta'+1,\gamma;x,y)-F_1(\alpha,\beta,\beta',\gamma;x,y);$$

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \mathrm{H}_3(\alpha,\beta,\delta;x,y) =$$

$$= (-1)^n \frac{(\alpha)_m (\beta)_m}{(1-\alpha)_n (\delta)_m} \mathrm{H}_3(\alpha+m-n,\beta+m,\delta+m;x,y);$$

$$\frac{\partial^n}{\partial x^n}\left[x^{\delta-1}\mathrm{H}_3(\alpha,\beta,\delta;x,y)\right] = (-1)^n (1-\delta)_n x^{\delta-n-1}\mathrm{H}_3(\alpha,\beta,\delta-n;x,y);$$

$$\frac{\partial^n}{\partial x^n}\left[x^{\beta+n-1}\mathrm{H}_3(\alpha,\beta,\beta+n;x,y)\right] = (\beta)_n x^{\beta-1}(1-x)^{-\alpha}\overline{J}_{-\alpha}\left[2\sqrt{y(1-x)}\right].$$

8-§. Ikki argumentli gipergeometrik funksiyalarning qo‘shni funksiyalar orasidagi munosabatlar.

$$(\gamma-\beta-\beta'-1)F_1(\alpha,\beta,\beta',\gamma;x,y) + \beta F_1(\alpha,\beta+1,\beta',\gamma;x,y) +$$

$$+\beta' F_1(\alpha,\beta,\beta'+1,\gamma;x,y) = (\gamma-1)F_1(\alpha,\beta,\beta',\gamma-1;x,y);$$

$$(\gamma-\alpha-1)F_1(\alpha,\beta,\beta',\gamma;x,y) + \alpha F_1(\alpha+1,\beta,\beta',\gamma;x,y) =$$

$$= (\gamma-1)F_1(\alpha,\beta,\beta',\gamma,\gamma-1;x,y);$$

$$\mathrm{H}_3(\alpha,\beta,\delta;x,y) - \mathrm{H}_3(\alpha,\beta,\delta-1;x,y) =$$

$$= \frac{\alpha\beta}{\delta(1-\delta)} x\mathrm{H}_3(\alpha+1,\beta+1,\delta+1;x,y);$$

$$\mathrm{H}_3(\alpha,\beta+1,\delta;x,y) - \mathrm{H}_3(\alpha,\beta,\delta;x,y) = \frac{\alpha}{\delta} x\mathrm{H}_3(\alpha+1,\beta+1,\delta+1;x,y).$$

9-§. Ikki argumentli gipergeometrik funksiyalarning analitik davom ettirish formulalari.

$$F_1(\alpha,\beta,\beta',\gamma;x,y)=(1-x)^{-\beta}(1-y)^{-\beta'}F_1\left(\gamma-\alpha,\beta,\beta',\gamma;\frac{x}{x-1},\frac{y}{y-1}\right)=$$

$$=(1-x)^{-\alpha}F_1\left(\alpha,\gamma-\beta-\beta',\beta',\gamma;\frac{x}{x-1},\frac{y-x}{1-x}\right)=$$

$$=(1-y)^{-\alpha}F_1\left(\alpha,\beta,\gamma-\beta-\beta',\gamma;\frac{y-x}{y-1},\frac{y}{y-1}\right)=$$

$$=(1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'}F_1\left(\gamma-\alpha,\gamma-\beta-\beta',\beta',\gamma;x,\frac{x-y}{1-y}\right)=$$

$$=(1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'}F_1\left(\gamma-\alpha,\beta,\gamma-\beta-\beta',\gamma;\frac{x-y}{x-1},y\right).$$

$$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y)=(1-x)^{-\alpha}F_2\left(\alpha,\gamma-\beta,\beta',\gamma,\gamma';\frac{x}{x-1},\frac{y}{1-x}\right)=$$

$$=(1-y)^{-\alpha}F_2\left(\alpha,\beta,\gamma'-\beta',\gamma,\gamma',\frac{x}{1-y},\frac{y}{y-1}\right)=$$

$$=(1-x-y)^{-\alpha}F_2\left(\alpha,\gamma-\beta,\gamma'-\beta',\gamma,\gamma';\frac{x}{x+y-1},\frac{y}{x+y-1}\right),$$

$$H_2(\alpha,\beta,\gamma,\varepsilon;x,y)=(1-x)^{-\alpha}H_2\left[\alpha,\varepsilon-\beta,\gamma,\delta,\varepsilon;\frac{x}{x-1},y(1-x)\right],$$

$$\mathrm{H}_3(\alpha,\beta,\delta;x,y)=(1-x)^{-\alpha}\mathrm{H}_3\left[\alpha,\delta-\beta,\delta;\frac{x}{x-1},y(1-x)\right].$$

Izoh. Bu yerda qaralgan funksiyalarning ta’riflari va keltirilgan formulalar yordamida yana ko‘plab formulalar keltirib chiqarish mumkin.

Xulosa

Ma’lumki, aksariyat maxsus funksiyalar ma’lum bir differensial tenglamaning maxsus nuqtasi atrofidagi yyechimidan iborat bo‘ladi. Shulardan biri bo‘lgan gipergeometrik funksiya Gauss tenglamasining yechimi bo‘lib, u o‘z argumenti $-1 < x < 1$ bo‘lganda aniq bir qatorning yig‘indisidan iboratdir. Lekin matematikada bu funksiyaning bu oraliqdan tashqaridagi qiymatlaridan foydalanish qulay bo‘ladi. Buni amalga oshirish uchun $-1 < x < 1$ oraliqda aniqlangan funksiyani bu oraliqdan chetga analitik davom ettirishga to‘g‘ri keladi. Bunda funksiyaning $-1 < x < 1$ oraliqda o‘rinli bo‘lgan xossalaridan foydalangan holda amalga oshiriladi.

Mazkur bitiruv malakaviy ishda ana shu fikrlarga amal qilgan holda $F(a,b,c;x)$ gipergeometrik funksiyaning turli xossalari bayon qilingan. Jumladan, differensiallash qoidalari, qo‘shish qoidalari, integral ko‘rinishlari va h.k. Albatta matematikaning rivojida bu funksiyani turli yo‘nalishlardagi umumlashmalarini bilish ham qoniqarlidir.

Shuning uchun bu yyerda $F(a,b,c;x)$ funksiyaning turli umumlashmalari keltirilganligi, ularning har biriga o‘ziga hos differensial tenglamalar sistemalarining yyechimlaridan iboratdir. Xulosa qilib aytganda differensial tenglamalar nazariyasi juda keng tarqalgan bo‘lib, uni bilish, tabiatni o‘rganuvchi ko‘plab fanlar muammolarini hal qilishga yordam beradi. Qolaversa, differensial tenglamalar nazariyasida olib borilayotgan ilmiy – tatqiqot ishlarida gipergeometrik va boshqa maxsus funksiyalar ko‘p foydalanilgani uchun mavzuni o‘rganish Farg‘ona davlat univesitetidagi ilmiy – tatqiqot ishlarini rivojiga xizmat qiladi deb o‘ylayman.

FOYDALANILGAN ADABIYOTLAR

1. Karimov I.A. Yuksak ma’naviyat-yengilmas kuch. T., O’zbekiston, 2008 y.
2. Karimov I.A.”Barkamol avlod O’zbekiston taraqqiyotining poydevori” Toshkent 1997 y.
3. Karimov I.A. “O’zbekiston XXI asr bo’sag’asida” O’zbekiston 1992 y.
4. Karimov I.A. “2009-yilning yakunlari 2010-yilning boshlari O’zbekistonni ijtimoiy, iqtisodiy rivojlanishining eng muhim ustivor yo’nalishlariga bag’ishlangan Vazirlar Mahkamasining majlisidagi ma’ruzasi” Farg’ona. Farg’ona haqiqati gazetasi 2010-yil 3-fevral.
5. Бейтмен Г., Эрдейи А. Высшие трансцендентные функции. Т.1. -М.: Наука, 1965. -296 с.
6. Ватсон Дж.Н. Теория бесселевых функций. Т.1. -М.: Издательство ИЛ, 1949. -798 с.
7. Капилевич М.Б. О сингулярных задачах Коши и Трикоми // Докл. АН СССР, 1967, Т. 177, ќ6. -С. 1265-1268.
8. Капилевич М.Б. Об одном классе гипергеометрических функций Горна // Дифференциалные уравнения. -Минск, 1968, Т. 4, ќ8. -С.1465-1483.
9. Кузнецов Д.С. Специальные функции. - М.: Высшая школа, 1962. - 424 с.
10. Нахушев А.М. Дробное исчисление и его применение. - М.: Физматлит, 2003. -272 с.
11. Прудников А.П., Брычков Ю.А., Маричев О.И. Интегралы и ряды. Специальные функции. -М.: Наука, 1983. -752 с.
12. Прудников А.П., Брычков Ю.А., Маричев О.И. Интегралы и ряды. Допольнительные главы - М.: Наука, 1986. -800 с.
13. Salaxiddinov M.S. Matematik fizika tenglamalari. -Toshkent: O’qituvchi, 2002. -448 bet.
14. Салахитдинов М.С., Уринов А.К. К спектральной теории уравнений

смешанного типа. - Ташкент: Mumtoz So'z, 2010. -355 с.

15. Смирнов М.М. Уравнения смешанного типа. -М.: Высшая школа, 1985. - 304 с.

16. Фихтенгольц Г.М. Основы математического анализа. Т. II. -М.: Физматгиз, 1960. - 440 с.

17. Http://ZiyoNet.uz

18. Http://Ref.uz

19. Http://MatNet.ru

20. Http://Mircing.com

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