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# BOUND AND GROUND STATES OF A SPIN-BOSON MODEL WITH AT MOST ONE PHOTON: NON-INTEGER LATTICE CASE

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***Abstract:*** A non-integer lattice model of radiative decay (so-called spin-boson model) of a two level atom and at most one photon is considered. The number and location of the eigenvalues (bound and ground states) of this model are studied.

## INTRODUCTION

In the theory of solid-state physics [1], quantum field theory [2] and statistical physics [3,4] are arise the problems related with the standard or "truncated" spin-boson models. In a well-known model of radiative decay (the so-called spin-boson model) it is assumed that an atom, which can be in two states - ground state with energy - $\varepsilon$ and excited state with energy $\varepsilon$ - emits and absorbs photons, going over from one state to the other [4-7].

For a fixed $h>0$ we define the set $T_h^d$ as the $d$-dimensional torus, the cube $(-\frac{\pi}{h};\frac{\pi}{h}]^d$ with appropriately identified sides equipped with its Haar measure.

Let $L^2(T_h^d)$ be the Hilbert space of square integrable (complex) functions defined on $T_h^d$.

We recall that the energy operator of model of a spin-boson model with at most one photon is given by the (formal) expression

$$\mathcal{A} := \begin{pmatrix} A_{00} & A_{01} \\ A_{01}^{*} & A_{11} \end{pmatrix}$$

and acts in the Hilbert space

$$H := C^2 \otimes F_b^{(2)}(L_2(T_h^d)).$$

Here matrix elements $A_{ij}$ are defined by

$$A_{00} f_0^{(\sigma)} = \varepsilon\sigma f_0^{(\sigma)}, \quad A_{01} f_1^{(\sigma)} = \alpha \int_{T_h^d} v(t) f_1^{(-\sigma)}(t)dt,$$

$$(A_{11} f_1^{(\sigma)})(k) = (\varepsilon\sigma + w(k)) f_1^{(\sigma)}(k), \quad f = \{\{f_0^{(\sigma)}, f_1^{(\sigma)}\} : \sigma = \pm\} \in H,$$

$C^2$ is the state of the two-level atom and $F_b^{(2)}(L_2(T_h^d)) := C \oplus L^2(T_h^d)$ is the two-particle cut subspace of the symmetric Fock space for bosons:

$$F_b(L_2(T_h^d)) := C \oplus L_2(T_h^d) \oplus L_2^{sym}((T_h^d)^2) \oplus \ldots \oplus L_2^{sym}((T_h^d)^n) \oplus \ldots,$$

where $L_2^{sym}((T_h^d)^n)$ is the Hilbert space of symmetric functions of $n$ variables.

We make the following assumptions: $\varepsilon > 0$ the dispersion of the free field $w(\cdot)$ is an analytic on $T_h^d$ and has a unique zero minimum at the point $0 \in T_h^d$; $v(\cdot)$ is a real-valued analytic function on $T_h^d$; the coupling constant $\alpha > 0$ is an arbitrary. Here $\alpha v(k)$ means the coupling between the atoms and the field modes. In general, the dispersion relation $w \geq 0$ and the coupling function $v$ are fixed by the physical of the problem.

We notice that this model is bounded and self-adjoint, and can be considered as a non-integer lattice analog of the truncated spin-boson Hamiltonian. In the "algebraic" sense, a non-integer lattice spin-boson Hamiltonian is similar to a standard one with only the difference is that $\mathcal{A}$ does not act in the Euclidean space $R^d$ but on a torus $T_h^d$.

We write elements $F$ of the space $C^2 \otimes F_b^{(2)}(L_2(T_h^d))$ in the form $F=\{f_0^{(\sigma)}, f_1^{(\sigma)}\}$, and a discrete variable $\sigma=\pm$. The norm in $C^2 \otimes F_b^{(2)}(L_2(T_h^d))$ is given by

$$\|F\|^2 := \sum_{\sigma=\pm}\left(|f_0^{(\sigma)}|^2 + \int_{T_h^d}\left|f_1^{(\sigma)}(k)\right|^2 dk\right).$$

It is easy to see that transformation $U: C^2 \otimes F_b^{(2)}(L_2(T_h^d)) \to F_b^{(2)}(L_2(T_h^d)) \oplus F_b^{(2)}(L_2(T_h^d))$ defined by

$$U:\left(\begin{pmatrix} f_0^{(+)} \\ f_0^{(-)} \end{pmatrix}, \begin{pmatrix} f_1^{(+)} \\ f_1^{(-)} \end{pmatrix}\right) \to \left(\begin{pmatrix} f_0^{(+)} \\ f_1^{(-)} \end{pmatrix}, \begin{pmatrix} f_0^{(-)} \\ f_1^{(+)} \end{pmatrix}\right),$$

is a unitary operator and block-diagonalizes the model $\mathcal{A}$, i.e.

$$U^* \mathcal{A} U = diag\{\mathcal{A}^{(+)}, \mathcal{A}^{(-)}\},$$

where the operator matrix

$$\mathcal{A}^{(\sigma)} := \begin{pmatrix} A_{00}^{(\sigma)} & A_{01} \\ A_{01}^* & A_{11}^{(\sigma)} \end{pmatrix}$$

acts in the truncated Fock space $F_b^{(2)}(L_2(T_h^d))$. The matrix elements of $\mathcal{A}^{(\sigma)}$ are given by

$$A_{00}^{(\sigma)} f_0 = \varepsilon\sigma f_0, \quad A_{01} f_1 = \alpha \int_{T_h^d} v(t) f_1(t) dt,$$

$$(A_{11}^{(\sigma)} f_1)(k) = (\varepsilon\sigma + w(k)) f_1(k), \quad f = \{f_0, f_1\} \in F_b^{(2)}(L_2(T_h^d))$$

For each $s=\pm$ we consider the model $\mathcal{A}_0^{(\sigma)}$ acting in the Hilbert space $F_b^{(2)}(L_2(T_h^d))$ as

$$\mathcal{A}_0^{(\sigma)} := \begin{pmatrix} 0 & 0 \\ 0 & A_{11}^{(\sigma)} \end{pmatrix}.$$

One can show that the essential spectrum of a model $\mathcal{A}^{(\sigma)}$ coincides with the essential spectrum of $\mathcal{A}_0^{(\sigma)}$, that is,

$$\sigma_{ess}(\mathcal{A}^{(\sigma)}) = \sigma_{ess}(\mathcal{A}_0^{(\sigma)}) = [-\sigma\varepsilon, -\sigma\varepsilon + M_h], \; M_h := \max_{k \in T_h^d} w(k).$$

One of the main result of the paper is the following theorem. It describes the spectrum, essential spectrum, point spectrum of the model $\mathcal{A}$ and gives the information about the the lower bound of the essential spectrum.

**Theorem 1.** For the the spectrum, essential spectrum and point spectrum of the model $\mathcal{A}$ the equalities hold:

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}^{(+)}) \cup \sigma(\mathcal{A}^{(-)});$$

$$\sigma_{ess}(\mathcal{A}) = \sigma_{ess}(\mathcal{A}^{(+)}) \cup \sigma_{ess}(\mathcal{A}^{(-)});$$

$$\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^{(+)}) \cup \sigma_p(\mathcal{A}^{(-)}).$$

Moreover,

$$\min \sigma_{ess}(\mathcal{A}) = -\varepsilon$$

**Remark 1.** Since the part of the set $\sigma_{disc}(\mathcal{A}^{(\sigma)})$ might be subset of $\sigma_{ess}(\mathcal{A}^{(-\sigma)})$, the assertions are hold:

$$\sigma_{disc}(\mathcal{A}) \subseteq \sigma_{disc}(\mathcal{A}^{(+)}) \cup \sigma_{disc}(\mathcal{A}^{(-)});$$

$$\sigma_{disc}(\mathcal{A}) = \{\sigma_{disc}(\mathcal{A}^{(+)}) \cup \sigma_{disc}(\mathcal{A}^{(-)})\} \setminus \sigma_{ess}(\mathcal{A}).$$

Moreover,

 

$$\sigma_{disc}(\mathcal{A}) = \bigcup_{\sigma=\pm} \{\sigma_{disc}(\mathcal{A}^{(\sigma)}) \setminus \sigma_{ess}(\mathcal{A}^{(-\sigma)})\}.$$

We note that the models $\mathcal{A}^{(\sigma)}$, $\sigma = \pm$, have a more simple structure than the operator $\mathcal{A}$, and hence, Theorem 1 plays an important role in the subsequent investigations of the essential spectrum of $\mathcal{A}$.

**Theorem 2.** For any $\alpha > 0$ the operator $\mathcal{A}$ at least one and at most four eigenvalues (bound states). Moreover, two of them are smaller than $-\varepsilon$ and other two of them are bigger than $M_h + \varepsilon$.

**Remark 2.** In Theorem 2, the eigenvalue $E_0$ of $\mathcal{A}$ which exists for any $\alpha > 0$ and $h > 0$ is usually called ground state. The corresponding eigenvector-function has a form $F = (f_0^{(+)}, f_0^{(-)}, f_1^{(+)}, f_1^{(-)})$, where

$$f_0^{(+)} = 0, \quad f_0^{(-)} = const \neq 0, \quad f_1^{(+)}(k) = -\frac{\alpha v(k) f_0^{(-)}}{\varepsilon + w(k) - E_0}, \quad f_1^{(-)}(k) = 0.$$

We remark that the spectral properties of similar models to $\mathcal{A}^{(\sigma)}$**,** $\sigma = \pm$, are studied in many papers, see, for example, [8-13].

## REFERENCES


[1] A. I. Mogilner. Hamiltonians in solid state physics as multiparticle discrete Schrödinger operators: problems and results. Advances in Sov. Math., **5** (1991), 139-194.

[2] K. O. Friedrichs. Perturbation of spectra in Hilbert space. Amer. Math. Soc. Providence, Rhole Island, 1965.

[3] V. A. Malishev, R. A. Minlos. Linear infinite-particle operators. Translations of Mathematical Monographs. 143, AMS, Providence, RI, 1995.

[4] R. A. Minlos, H. Spohn. The three-body problem in radioactive decay: the case of one atom and at most two photons. Topics in Statistical and Theoretical Physics. Amer. Math. Soc. Transl., Ser. 2, **177**, AMS, Providence, RI, 1996, 159-193.

[5] M. Hübner, H. Spohn. Spectral properties of the spin-boson Hamiltonian. Ann. Inst. Henri Poincaré, **62**:3 (1995), 289-323.

[6] H. Spohn. Ground states of the spin-boson Hamiltonian. Comm. Math. Phys., **123** (1989), 277-304.

[7] Yu. Zhukov, R. Minlos. Spectrum and scattering in a "spin-boson" model with not more than three photons. Theor. Math. Phys., **103**:1 (1995), 398-411.

[8] M.Muminov, H.Neidhardt, T.Rasulov. On the spectrum of the lattice spin-boson Hamiltonian for any coupling: 1D case. Journal of Mathematical Physics, 56 (2015), 053507.

[9] T.H.Rasulov. On the finiteness of the discrete spectrum of a 3x3 operator matrix. Methods of Functional Analysis and Topology, 22:1 (2016), 48-61.

[10] T.Kh.Rasulov. Branches of the essential spectrum of the lattice spin-boson model with at most two photons. Theoretical and Mathematical Physics, 186:2 (2016), 251-267.

[11] M.Muminov, H.Neidhardt, T.Rasulov. On the spectrum of the lattice spin-boson Hamiltonian for any coupling: 1D case. Journal of Mathematical Physics, 56 (2015), 053507.

[12] M.I.Muminov, T.H.Rasulov. On the eigenvalues of a 2x2 block operator matrix. Opuscula Mathematica. 35:3 (2015), 369-393.

[13] M.I.Muminov, T.Kh.Rasulov. An eigenvalue multiplicity formula for the Schur complement of a 3x3 block operator matrix. Siberian Math. J., 56:4 (2015), P. 878-895.


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