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DESCRIPTION OF THE NUMERICAL RANGE OF A FRIEDRICHS MODEL WITH RANK TWO PERTURBATION

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Abstract: In the paper we consider a Friedrichs model A with rank two perturbations. It is associated with a system of two quantum particles on three-dimensional lattice. We establish the connection between the spectrum and numerical range of A.

INTRODUCTION

The numerical range is an important tool in the spectral analysis of bounded and unbounded linear operators in Hilbert spaces. First of all we give its definition. Let H be a complex Hilbert space with inner product $(.,.)$ and A be a bounded linear operator in H. Then the numerical range of an A is the subset of the complex numbers C, given by

$$W(A):=\{(Ax,x): x\in H, \|x\|=1\}.$$

It was first studied by O.Toeplitz in [1]; he proved that the numerical range of a matrix contains all its eigenvalues and that its boundary is a convex curve. In [2] F.Hausdorff showed that indeed the set $W(A)$ is convex. In fact, it turned out that this continues to hold for general bounded linear operators and that the spectrum is contained in the closure $\overline{W(A)}$ (see [3]). So, the numerical range of a bounded linear operator satisfies the so-called spectral inclusion property

$$W(A)\subset \sigma_p(A), \quad \overline{W(A)}\subset \sigma(A)$$

for the point spectrum $\sigma_p(A)$ (or set of eigenvalues) and the spectrum $\sigma(A)$ of A; note that $W(A)$ is closed if $\dim H<\infty$.

In this paper we consider a Friedrichs model A with rank two perturbation and we study its numerical range. We find the necessity conditions which guarantees that the numerical and spectrum of A are coincide.

FRIEDRICHS MODEL

Let T^3 be the three-dimensional torus, the cube $[-\pi,\pi)^3$ with appropriately identified sides equipped with its Haar measure and $L(T^3)$ be the Hilbert space of square integrable (complex) functions defined on T^3.

Let us consider an operator (Friedrichs model) A acting on the Hilbert space $L(T^3)$

$$A:=A_0-\mu_1 V_1+\mu_2 V_2, \tag{1}$$

where the operators A_0 and V_α, $\alpha=1,2$ are defined by

$$(A_0 f)(p)=u(p)f(p), \qquad (A_0 f)(p)=\upsilon_\alpha(p)\int_{T^3}\upsilon_\alpha(t)f(t)dt, \ \ \alpha=1,2.$$

Here $\mu_\alpha>0$, $\alpha=1,2$ are positive reals, $u(\cdot)$ and $\upsilon_\alpha(\cdot)$, $\alpha=1,2$ are real-valued continuous functions on T^3. Under these assumptions, operator A defined by (1) is bounded and self-adjoint.

Throughout this paper, we assume that the function $u(\cdot)$ has a unique non-degenerate minimum at the point $p_1\in T^3$ has a unique non-degenerate maximum at the point $p_2\in T^3$, the function $\upsilon_\alpha(\cdot)$ has the continuous partial derivatives up to the third-order inclusive at some neighborhood of $p_\alpha\in T^3$. Suppose that

$$mes(\sup p\{\upsilon_1(\cdot)\}\cap \sup p\{\upsilon_2(\cdot)\})=0. \tag{2}$$

In accordance with Weyl's theorem it is clear that the essential spectrum of the operator A coincides with the spectrum of A_0:

$$\sigma_{ess}(A)=[E_1;E_2],$$

where the numbers E_1 and E_2 are defined by

$$E_1:=\min_{p\in T^3} u(p), \quad E_2:=\max_{p\in T^3} u(p).$$

To study the spectral properties of the operator A, we introduce the following two bounded self-adjoint operators (Friedrichs model with rank one perturbation) A_0, acting on $L_2(T^3)$ by the rule

$$A_0 := A_0+(-1)^{\alpha}\mu_\alpha V_\alpha, \quad \alpha=1,2. \tag{3}$$

Let C be the field of complex numbers. We define an analytic function $\Delta_{\mu_\alpha}(\cdot)$ (the Fredholm determinant associated with the operator A_α) on $C\setminus[E_1;E_2]$ by

$$\Delta_{\mu_\alpha}(z):=1+(-1)^{\alpha}\mu_\alpha\int_{T^3}\frac{\upsilon_\alpha^2(t)dt}{u(t)-z}.$$

The following lemma is a simple consequence of the Birman-Schwinger principle and the Fredholm theorem.

Lemma 1. The operator A_α has an eigenvalue $z\in C\setminus[E_1;E_2]$ if and only if $\Delta_{\mu_\alpha}(z)=0$.

From Lemma 1 it follows that for the discrete spectrum of A_α the equality

$$\sigma_{disc}(A_\alpha)=\{z\in C\setminus[E_1;E_2]:\Delta_{\mu_\alpha}(z)=0\}$$

holds.

Lemma 2. The number $z\in C\setminus[E_1;E_2]$ is an eigenvalue of A if and only if z is an eigenvalue one of the operators $A_\alpha, \alpha=1,2.$

From Lemma 2 we conclude that

$$\sigma_{disc}(A)=\sigma_{disc}(A_1)\cup\sigma_{disc}(A_2).$$

Then using the monotonic property of the function $\Delta_{\mu_\alpha}(\cdot)$ on $(-\infty,E_1)$ and (E_1,∞) and Lemma 1 we obtain that the operator A has no more than one simple eigenvalue lying on the left-hand side of E_1 and right-hand side of E_2.

MAIN RESULTS

We set

$$\mu_\alpha^0:=(-1)^{\alpha+1}\left(\int_{T^3}\frac{\upsilon_\alpha^2(t)dt}{u(t)-z}\right)^{-1}, \ \alpha=1,2.$$

One of the main results of the present paper is the following assertion.

Lemma 3. If $\mu_\alpha=\mu_\alpha^0, \alpha=1,2,$ then the operator A has no eigenvalues lying outside of its essential spectrum.

The following theorems describe the numerical range $W(A)$ of A for the special case $\mu_\alpha=\mu_\alpha^0, \alpha=1,2.$

Theorem 4. If $\mu_\alpha=\mu_\alpha^0$ and $\upsilon_\alpha(p_\alpha)=0$ for $\alpha=1,2,$ then

$$W(A)=[E_1;E_2](=\sigma(A)).$$

Theorem 5. If $\mu_\alpha=\mu_\alpha^0$ and $\upsilon_\alpha(p_\alpha)\neq 0$ for $\alpha=1,2,$ then $W(A)=(E_1;E_2).$

From theorems 4 and 5 we obtain

Corollary 6. Let $\mu_\alpha=\mu_\alpha^0, \alpha=1,2$.

(i) If $\upsilon_1(p_1)=0$ and $\upsilon_2(p_2)\neq 0$, then $W(A)=[E_1;E_2);$

(ii) If $\upsilon_1(p_1)\neq 0$ and $\upsilon_2(p_2)=0$, then $W(A)=(E_1;E_2].$

Note that [4,5], if $\mu_\alpha=\mu_\alpha^0$ and $\upsilon_\alpha(p_\alpha)=0$ for some $\alpha\in\{1,2\},$ then the number $z=E_\alpha\in\sigma_{ess}(A)$ is a threshold eigenvalue of A; if $\mu_\alpha=\mu_\alpha^0$ and $\upsilon_\alpha(p_\alpha)\neq 0$ for for some $\alpha\in\{1,2\},$ then the number $z=E_\alpha\in\sigma_{ess}(A)$ is a virtual level of A Spectral properties of the generalized Friedrichs model, in particular, related with the usual eigenvalues, threshold eigenvalues and threshold energy resonances of the generalized Friedrichs model are studied in many works, see for example, [6-10]. So, using the methods of threshold analysis one can prove the same results for the generalized Friedrichs model.

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