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KOMPLEKS ANALIZDAN MUSTAQIL ISHLAR TO’PLAMI

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Ushbu uslubiy qo’llanma oliy ta’lim muassasalarining “Matematika”, ”Amaliy matematika va informatika” hamda ”Fizika” ta’lim yo’nalishlarida tahsil olayotgan talabalar uchun mo’ljallangan. Qo’llanmada asosan kompleks analiz fanidan mavzular bo’yicha bir qancha tipik masalalarning namunaviy yechimlari va talabalar mustaqil bajarishlari uchun ko’plab masalalar keltirilgan.

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KIRISH

”Kompleks analiz” fani matematik analiz fanining uzviy davomi bo’lib, oliy matematikaning asosiy bo’limlaridan biri hisoblanadi. Unda matematik analiz fanida o’rganilgan obyektlar kompleks analiz nuqtai nazaridan tahlil qilinadi va chuqurroq o’rgatiladi. Kompleks o’zgaruvchili funksiyalarning o’zigagina xos bo’lgan xossalari keltiriladi.

Ushbu uslubiy qo’llanma Kompleks analiz fani dasturidagi “Kompleks tekislik. Kompleks o’zgaruvchi funksiyalar. Kompleks o’zgaruvchili elementar funksiyalar. Funksiya integrali” modullarida ko’zda tutilgan mavzularga doir misol va masalalarini qamrab olgan bir qo’llanmadir. U oliy ta’lim muassasalarining “Matematika”, ”Amaliy matematika va informatika”, ”Fizika” ta’lim yo’nalishlarida tahsil olayotgan talabalar uchun mo’ljallab yozilgan.

Mazkur qo’llanmada ta’riflar, teoremalar, tasdiqlar berilmasada, ularga doir misol va masalalar dastlab sodda va muayyan tasavvur hosil qilinadigan, so’ngra murakkabroq masalalarni yechishga alohida e’tibor qaratilgan. Misol va masalalarni sharhlab, ularni yechib ko’rsatishdan ko’zlangan maqsad kompleks analiz kursidan olingan nazariy bilimlardan misol va masalalarni yechishda foydalana borilishini namoyish qilishdir. Talabalar namuna sifatida yechib ko’rsatilgan masalalarda qo’llanilgan usullardan foydalanib mustaqil bajarishlari uchun ko’plab misol va masalalar keltirilgan.

Ma’lumki, haqiqiy va kompleks analiz orasida o’xshash va farqli jihatlari mavjud. Qo’llanmada keltirilgan ma’lumotlarda kompleks analizga xos bo’lgan usullarga alohida e’tibor qaratilgan va ular yordamida algebra hamda haqiqiy o’zgaruvchili funksiyalar nazariyasining ayrim masalalarini (Koshi integral formulasi yordamida integrallarni hisoblash) sodda hal etilishi ko’rsatilgan.

Qo’llanmani o’qish jarayonida talabalar o’zlarining matematik analiz, chiziqli algebra va analitik geometriyadan olgan bilimlarini to’ldiradilar. Undan matematikaning ko’plab sohalari bo’yicha ilmiy-tadqiqot ishlari olib borayotgan magistrantlar, tayanch doktorantlar va mustaqil izlanuvchilar ham foydalanishlari mumkin.

I BOB. KOMPLEKS TEKISLIK. KOMPLEKS O'ZGARUVCHILI FUNKSIYALAR

1. KOMPLEKS SONLAR, ULAR USTIDA ARIFMETIK AMALLAR.

1.1-Misol. i ning darajalarini hisoblash fornulalarini keltiring.

Yechish. i ning kiritilishidan va natural ko'rsatkichli darajaning ta'rifidan foydalanib quyidagi tengliklarga ega bo'lamiz:

$$i^1 = i = \sqrt{-1}, \quad i^2 = -1; \quad i^3 = i^2 \cdot i = -i; \quad i^4 = (i^2)^2 = (-1)^2 = 1;$$

$$i^5 = i^4 \cdot i = i; \; i^6 = i^5 \cdot i = -1 \text{ va hokazo.}$$

Umumiy holda ixtiyoriy $k = 0, \quad 1, \quad 2, \ldots$ uchun quyidagi tengliklar o'rinlidir:

$$i^{4k} = (i^4)^k = 1^k = 1, \quad i^{4k+1} = i^{4k} \cdot i = 1 \cdot i = i,$$

$$i^{4k+2} = i^{4k} \cdot i^2 = 1 \cdot (-1) = -1,$$

$$i^{4k+3} = i^{4k} \cdot i^3 = 1 \cdot (-i) = -i\,.$$

1.2-Misol. Muavr formulasi yordamida $\cos 3\varphi$ va $\sin 3\varphi$ larni φ ning trigonometrik funksiyalari orqali ifodalang.

Yechish. Muavr formulasi va qisqa ko'paytirish formulasidan foydalanamiz:

$$z = \cos 3\varphi + i\sin 3\varphi = (\cos\varphi + i\sin\varphi)^3 = \cos^3\varphi + 3i\cos^2\varphi\sin\varphi +$$

$$+3i^2\cos\varphi\sin^2\varphi + i^3\sin^3\varphi = \cos^3\varphi - 3\cos\varphi\sin^2\varphi +$$

$$+i(3\cos^2\varphi\sin\varphi - \sin^3\varphi)$$

Ikkala qismdagi sonlarninig tengligidan ularning haqiqiy va mavhum qismlari tengligi kelib chiqadi:

$$\operatorname{Re} z = \cos 3\varphi = \cos^3\varphi - 3\cos\varphi\sin^2\varphi,$$

$$\operatorname{Im} z = \sin 3\varphi = 3\cos^2\varphi\sin\varphi - \sin^3\varphi\,.$$

J: $\cos 3\varphi = \cos^3\varphi - 3\cos\varphi\sin^2\varphi$, $\quad \sin 3\varphi = 3\cos^2\varphi\sin\varphi - \sin^3\varphi.$

1.3-Misol. $\sqrt[6]{\dfrac{\sqrt{3}-i}{-1+i}}$ ildizning qiymatlarini toping.

Yechish. Dastlab ildiz ostidagi kasrni trigonometrik shaklga keltirib olamiz:

$$\sqrt{3}-i=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right),\ -1+i=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right).$$

Kompleks sonlarni bo'lishga asosan quyidagi munosabatlarni olamiz:

$$\frac{\sqrt{3}-i}{-1+i}=\frac{2}{\sqrt{2}}\cdot\frac{\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}}{\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}}=\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right).$$

Chunki, bo'linma argumentiga asosan $\frac{5\pi}{6}-\frac{3\pi}{4}=\frac{\pi}{12}$.

Ildiz chiqarish formulasini qo'llab, berilgan ifodaning qiymartlarini olamiz:

$$\sqrt[6]{\frac{\sqrt{3}-i}{-1+i}}=z_k=\sqrt[6]{\sqrt{2}}\left(\cos\frac{\frac{\pi}{12}+2k\pi}{6}+i\sin\frac{\frac{\pi}{12}+2k\pi}{6}\right)=$$

$$=\sqrt[12]{2}\left(\cos\frac{(24k+1)\pi}{72}+i\sin\frac{(24k+1)\pi}{72}\right),\ k=0,1,2,3,4,5.$$

1.4-Misol. Quyidagi $z_1=\sqrt{3}+i\sqrt{2},\quad z_2=\sqrt{3}-i\sqrt{2}$ kompleks sonlarning yig'indisi, ayirmasi, ko'paytmasi, nisbati hamda $z_1+\frac{1}{z_2}$ ni toping.

Yechish.

$$z_1+z_2=(\sqrt{3}+i\sqrt{2})+(\sqrt{3}-i\sqrt{2})=(\sqrt{3}+\sqrt{3})+i(\sqrt{2}-\sqrt{2})=2\sqrt{3}.$$

$$z_1-z_2=(\sqrt{3}+i\sqrt{2})-(\sqrt{3}-i\sqrt{2})=(\sqrt{3}-\sqrt{3})+i(\sqrt{2}+\sqrt{2})=2\sqrt{2}i.$$

$$z_1\cdot z_2=(\sqrt{3}+i\sqrt{2})\cdot(\sqrt{3}-i\sqrt{2})=(\sqrt{3})^2-(i\sqrt{2})^2=3-2i^2=5\ .$$

$$\frac{z_1}{z}=\frac{\sqrt{3}+i\sqrt{2}}{\sqrt{3}-i\sqrt{2}}=\frac{(\sqrt{3}+i\sqrt{2})\cdot(\sqrt{3}+i\sqrt{2})}{(\sqrt{3})^2-(i\sqrt{2})^2}=\frac{3+i2\sqrt{3}\cdot\sqrt{2}+2i^2}{3+2}=$$

$$=\frac{(3-2)+i\cdot 2\sqrt{6}}{5}=\frac{1}{5}+i\frac{2\sqrt{6}}{5}.$$

$$z_1+\frac{1}{z_2}=\sqrt{3}+i\sqrt{2}+\frac{1}{\sqrt{3}-i\sqrt{2}}=\sqrt{3}+i\sqrt{2}+\frac{\sqrt{3}+i\sqrt{2}}{3+2}=$$

$$=\sqrt{3}+i\sqrt{2}+\frac{\sqrt{3}}{5}+i\frac{\sqrt{2}}{5}=\frac{6\sqrt{3}}{5}+i\frac{6\sqrt{2}}{5}.$$

Mustaqil ishlash uchun topshiriqlar.

1- Misol. Quyidagi z_1 va z_2 kompleks sonlarning yig‘indisi, ayirmasi, ko‘paytmasi, nisbatini toping:

1.1. $z_1 = -3 + 4i,$ $\quad z_2 = 4 - 3i.$

1.2. $z_1 = 2 + i,$ $\quad z_2 = 1 - 2i.$

1.3. $z_1 = \sqrt{3} + \sqrt{2}i,$ $\quad z_2 = \sqrt{3} - i\sqrt{2}.$

1.4. $z_1 = 3i,$ $\quad z_2 = 2 - i.$

1.5. $z_1 = 2 + i\sqrt{3},$ $\quad z_2 = 2 - i\sqrt{3}.$

1.6. $z_1 = 3 + 4i,$ $\quad z_2 = 3 - 4i.$

1.7. $z_1 = 5 + 2i,$ $\quad z_2 = 5 - 2i.$

1.8. $z_1 = 2 + i\sqrt{3},$ $\quad z_2 = 3 + i\sqrt{2}.$

1.9. $z_1 = 2 - i\sqrt{3},$ $\quad z_2 = 3 - i\sqrt{2}.$

1.10. $z_1 = \sqrt{2} + i\sqrt{3},$ $\quad z_2 = \sqrt{3} + i\sqrt{2}.$

1.11. $z_1 = 3 + 4i,$ $\quad z_2 = 4 + 3i.$

1.12. $z_1 = 3 - 4i,$ $\quad z_2 = 4 - 3i.$

1.13. $z_1 = 1 + \sqrt{5}i,$ $\quad z_2 = 1 - \sqrt{5}i.$

1.14. $z_1 = 2 + \sqrt{5}i,$ $\quad z_2 = 2 - \sqrt{5}i.$

1.15. $z_1 = 2 + \sqrt{5}i,$ $\quad z_2 = \sqrt{5} + 2i.$

1.16. $z_1 = 2 - \sqrt{5}i,$ $\quad z_2 = \sqrt{5} + 2i.$

1.17. $z_1 = 3 - \sqrt{5}i,$ $\quad z_2 = 3 + \sqrt{5i}.$

1.18. $z_1 = \sqrt{3} + \sqrt{5}i,$ $\quad z_2 = \sqrt{3} - \sqrt{5}i.$

1.19. $z_1 = \sqrt{5} + \sqrt{3}i,$ $\quad z_2 = \sqrt{5} - \sqrt{3}i.$

1.20. $z_1 = 3 + \sqrt{5}i,$ $\quad z_2 = 3 - \sqrt{5}i.$

1.21. $z_1 = \sqrt{5} + 3i,$ $\quad z_2 = \sqrt{5} + 3i.$

1.22. $z_1 = \sqrt{3} + i\sqrt{2},$ $\quad z_2 = \sqrt{3} - i\sqrt{2}.$

1.23. $z_1 = \sqrt{5} + i\sqrt{3},$ $\quad z_2 = \sqrt{5} - i\sqrt{3}.$

1.24. $z_1 = i\sqrt{5 + \sqrt{3}},$ $\quad z_2 = i\sqrt{5 - \sqrt{3}} + 3.$

1.25. $z_1 = 12 + 5i,$ $\quad z_2 = 12 + 4i.$

1.26. $z_1 = \sqrt{3} + 2i,$ $\quad z_2 = \sqrt{3} - 3i.$

1.27. $z_1 = 7 + 24i,$ $\quad z_2 = 5 - 24i.$

1.28. $z_1 = \frac{2}{3} - \frac{1}{3}i,$ $\quad z_2 = \frac{1}{3} + \frac{4}{3}i.$

1.29. $z_1 = \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{4}i$, $\qquad z_2 = \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3}i$.

1.30. $z_1 = \frac{\sqrt{3}}{5} - \frac{1}{3}i$, $\qquad z_2 = \frac{2}{5} - \frac{1}{5}i$.

1.31. $z_1 = i\sqrt{5+\sqrt{3}},$ $\qquad z_2 = i\sqrt{5-\sqrt{3}} + 3.$

1.32. $z_1 = i\sqrt{5+\sqrt{3}},$ $\qquad z_2 = i\sqrt{5-\sqrt{3}} + 3.$

2. Kompleks sonning ko'rinishlari.

2.1-misol. $z = \frac{3-i}{2+3i}$ kompleks sonni algebraik shaklga keltirib, uning haqiqiy va mavhum qismlarini toping.

Yechish. Bu bo'linma (kasr) shaklida berilgan kompleks sonni dastlab soddalashtiramiz. Buning uchun uning surat va maxrajini maxrajdagi kompleks sonning qo'shmasiga ko'paytiramiz:

$$z = \frac{3-i}{2+3i} = \frac{(3-i)(2-3i)}{(2+3i)(2-3i)} = \frac{6-9i-2i+3i^2}{2^2+3^2} = \frac{6-11i-3}{13} = \frac{9}{13} - \frac{11}{13}i.$$

Bu tenglikning o'ng tomonidagi kompleks sonning haqiqiy va mavhum qismlari biz izlagan javob bo'ladi:

J: $\operatorname{Re} z = \frac{9}{13}; \quad \operatorname{Im} z = -\frac{11}{13}.$

2.2-misol. $z = 1 - i$ kompleks sonning moduli va argumentini toping.

Yechish. a) $z = 1 - i$ kompleks sonning haqiqiy va mavhum qismlari $\operatorname{Re} z = x = 1$ va $\operatorname{Im} z = y = -1$ bo'lib, uning moduli $|z| = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. $z = 1 - i$ nuqta to'rtinchi chorakda joylashganligi uchun $tg\varphi = \frac{1}{-1} = -1$ tenglamadan

$$\arg(1-i) = \frac{7\pi}{4} \ .$$

kelib chiqadi.

2.3 - misol. Amallarni bajaring, hosil bo'lgan kompleks sonning moduli va argumentini toping, uni kompleks tekislikda tasvirlang: $(1-i)^3 \cdot (1+i\sqrt{3})^8.$

Yechish. Oldin $z_1=1-i$, $z_2=1+i\sqrt{3}$ sonlarning moduli va argumentini hisoblash formulalardan foydalanib topib, so‘ng ularni trigonometrik shaklda yozamiz va Muavr formulasidan foydalanamiz:

$$z_1=1-i \Rightarrow |z_1|=\sqrt{1^2+(-1)^2}=\sqrt{2}.$$

$$\arg z_1 = arctg\frac{x}{y}+2\pi=-\frac{\pi}{4}+2\pi=\frac{7\pi}{4}\Rightarrow$$

$$z_1=\sqrt{2}\cdot(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4})\Rightarrow z_1^3=(1-i)^3=2\sqrt{2}(\cos\frac{21\pi}{4}+i\sin\frac{21\pi}{4})=$$

$$2\sqrt{2}(-\cos\frac{\pi}{4}-i\sin\frac{\pi}{4})=2\sqrt{2}\cdot(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2})=-2(1+i).$$

$$z_2=1+i\sqrt{3} \Rightarrow |z_2|=\sqrt{1^2+(\sqrt{3})^2}=2,\quad \arg z_2=arctg\sqrt{3}=\frac{\pi}{3}\Rightarrow$$

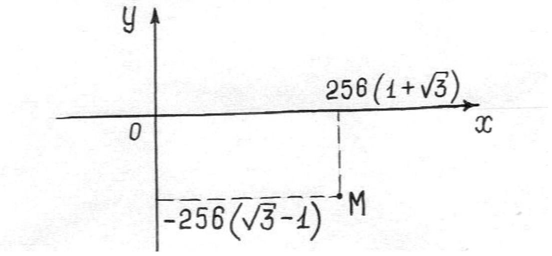
$$z_2=1+i\sqrt{3}=2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})\Rightarrow z_2^8=(1+i\sqrt{3})^8=2^8(\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3})=$$

$$=256(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})=256\cdot(-\frac{1}{2}+i\frac{\sqrt{3}}{2})=-128(1-i\sqrt{3}).$$

Demak, $z=(1-i)^3\cdot(1+i\sqrt{3})^8=z_1^3\cdot z_2^8=256(1+i)(1-i\sqrt{3})=$

$$=256\cdot[(1+\sqrt{3})+i(1-\sqrt{3})]=256\cdot(1+\sqrt{3})-i\cdot256\cdot(\sqrt{3}-1).$$

Bu kompleks son tekislikda $M(256(1+\sqrt{3}),-256(\sqrt{3}-1))$ nuqtani ifodalaydi (1-chizma)



1-chizma

2-misol. Kompleks sonni algebraik shaklga keltirib, kompleks sonlarning moduli va argumentini topib, ularni kompleks tekislikda tasvirlang.

2.1. $(\sqrt{3}+i\sqrt{3})^6\cdot(1+i)^3$.

2.2. $(\sqrt{3}+i\sqrt{3})^4\cdot(1-i)^4$.

2.3. $(-\sqrt{3}+3i)^6\cdot(3+i\sqrt{3})^4$.

2.4. $(-\sqrt{3}-3i)^3\cdot(3+i\sqrt{3})^6$.

2.5. $(\sqrt{3}+3i)^5\cdot(3+i\sqrt{3})^3$.

2.6. $(\sqrt{3}-3i)^4\cdot(3+i\sqrt{3})^6$.

2.7. $(\sqrt{3}+3i)^3\cdot(1+i)^5$.

2.8. $(\sqrt{3}+3i)^4\cdot(1-i)^5$.

2.9. $(3+i\sqrt{3})^4\cdot(1+i)^5$.

2.10. $(3+i\sqrt{3})^3\cdot(1-i)^5$.

2.11. $(-1+i\dfrac{\sqrt{3}}{3})^6\cdot(1+i)^3$.

2.12. $(-1+i\dfrac{\sqrt{3}}{3})^4\cdot(1-i)^4$.

2.13. $(1-i\dfrac{\sqrt{3}}{3})^6\cdot(1+i)^4$.

2.14. $(1-i\dfrac{\sqrt{3}}{3})^3\cdot(1+i)^6$.

2.15. $(1+i\dfrac{\sqrt{3}}{3})^5\cdot(1+i)^3$.

2.16. $(1+i\dfrac{\sqrt{3}}{3})^4\cdot(1-i)^6$.

2.17. $(-1+i)^3\cdot(1+i\sqrt{3})^5$.

2.18. $(-1+i)^4\cdot(1-i\sqrt{3})^5$.

2.19. $(1+i)^4\cdot(1+i\sqrt{3})^5$.

2.20. $(1-i)^3\cdot(1-i\sqrt{3})^5$.

2.21. $(1-i)^3\cdot(1+i\sqrt{3})^8$.

2.22. $z=\left(\dfrac{i^{17}+2}{i^{39}+1}\right)^2$.

2.23. $z=\dfrac{(1+i)^5}{(1-i)^3}$.

2.24. $(1+2i)^6$.

2.25. $\dfrac{(1+2i)^2-(1-i)^3}{(3+2i)^3-(2+i)^2}$.

2.26. $\left(-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right)^2$.

2.27. $\dfrac{(-2+2i)^5}{(-1+i)^3}+2i-5$.

2.28. $\dfrac{1+itg\alpha}{1-itg\alpha}$.

2.29. $(1-i\sqrt{3})^3(1+i)^2$.

2.30. $(1+i\sqrt{3})^{-6}(1-i)^4$.

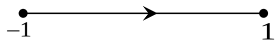
2.31. $(\dfrac{1}{2}-i\dfrac{\sqrt{3}}{2})^{10}$.

2.32. $\dfrac{\left(1+i\sqrt{3}\right)^{15}}{\left(1+i\right)^{10}}$.

3. KOMPLEKS TEKISLIKDA SOHALAR VA EGRI CHIZIQLAR.

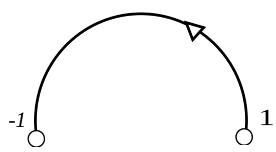
3.1-misol. $z=\cos t,\ \pi\le t\le 2\pi$ tenglama bilan ifodalangan chiziqning geometrik o'rnini aniqlang.

Yechish. Bu chiziq $z=-1$ nuqtadan $z=1$ nuqtaga yo'naltirilgan $[-1;1]$ kesmadan iborat (2- chizma).



2 chizma

3.2-misol. $z=e^{it},\ 0\le t\le\pi$ tenglama bilan ifodalangan chiziqning geometrik o'rnini aniqlang.



3 chizma

Yechish. Bu chiziq soat strelkasiga teskari yo'naltirilgan $|z|=1,\ \operatorname{Im}z\ge 0$ yarim aylanadan iborat (3-chizma).

3.3-misol. Ushbu to'plamlarni ochiq yoki yopiq ekanligini aniqlang.

a) $D=\{z\in C:|z-a|<r\}$; b) $D=\{z\in C:|z-a|\le r\}$.

Yechish. a) Bu to’plam markazi a nuqtada, radiusi r ga teng bo’lgan ochiq doiradan iboratdir. Bunda $a \in C$ berilgan nuqta, r esa musbat son.

b) Ushbu $D = \{z \in C : |z - a| \le r\}$ to’plam markazi a nuqtada, radiusi r ga teng bo’lgan yopiq doiradan iboratdir. Yopiq to’plam ta’rifga asosan bu yopiq to’plam bo’ladi.

3.4-misol. Bir bog’lamli sohaga misol keltiring.

Yechish. Kengaytirilgan kompleks tekislikning quyidagi sohalari bir bog’lamlidir: a) $|z| > 1$; b) butun kengaytirilgan kompleks tekislik; c) $z \ne a$ - kengaytirilgan kompleks tekislikdan a nuqtani chiqarib tashlangani.

3.5-misol. Bir bog’lamli bo’lmagan sohaga misol keltiring.

Yechish. Quyidagi sohalar bir bog’lamli bo’lmaydi:

a) $z \ne 1, i$ - kengaytirilgan kompleks tekislikdan 1 va i nuqtalarni chiqarib tashlangani;

b) butun kengaytirilgan kompleks tekislikdan $[0,1]$ va $[i, 2i]$ kesmalarni qirqib olingani;

c) $1 < |z| < \infty$.

3.6-misol. $|\operatorname{Re} z| < 1$ tengsizlikni qanoatlantiruvchi kompleks tekislikning barcha nuqtalari to’plamini geometrik tasvirlang.

Yechish. a) $|\operatorname{Re} z| < 1 \;\Rightarrow\; -1 < x < 1$. J: Mavhum o’qgacha bo’lgan masofasi birdan kichik nuqtalardan tashkil topgan yo’lak.

3.7-misol. Quyidagi $1 < |z - 2 + 3i| \le 3$ tengsizlikni qanoatlantiruvchi barcha nuqtalar to‘plamini C kompleks tekislikda tasvirlang.

Yechish.

$$|z - 2 + 3i| = |x + iy - 2 + 3i| = |(x-2) + i(y+3)| = \sqrt{(x-2)^2 + (y+3)^2}$$

bo‘lgani uchun berilgan $1 < |z - \;\; + \;\; 2| \le$ to‘plam $1 < (x-2)^2 + (y+3)^2 \le 9$ halqadan iborat bo‘ladi. Bu markazi (2;-3) nuqtada, radiuslari 1 va 3 ga teng bo‘lgan konsentrik aylanalar orasidagi nuqtalar va radiusi 3 ga teng aylana nuqtalarini o‘z ichiga olgan halqadir (4-chizma)

3.8 – misol. Ushbu $\mathrm{z} = \mathrm{z(t)} = \mathrm{z}_0 + \mathrm{re}^{\mathrm{it}}$ $(-\pi \leq t \leq \pi)$ funksiya aniqlagan egri chiziqni toping, bunda z_0 -kompleks son, $r > 0$ o'zgarmas son.

Yechish. Agar $z = x + iy$, $z_0 = x_0 + iy_0$ deb, $e^{it} = cost + isint$ bo'linishini e'tiborga olsak, u holda

$$x + iy = x_0 + iy_0 = r(cost + isint),$$

ya'ni,

$$x + iy = (x_0 + rcost) + i(y_0 = +rsint)$$

ko'rinishga keladi. Keyingi tenglikda haqiqiy va mavhum qismlarini bir-biriga tenglab,

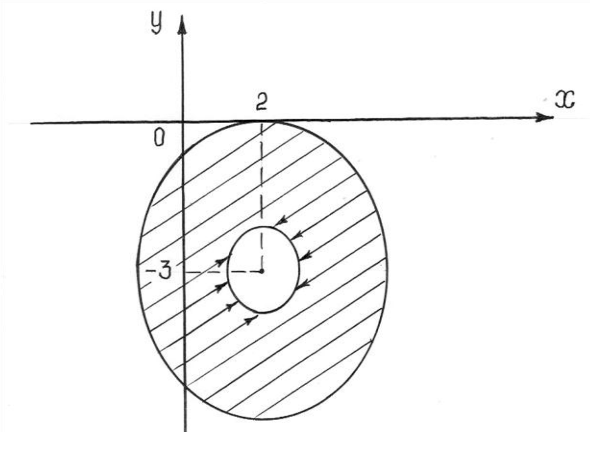
$$x = x_0 + rcost, \qquad y = y_0 + rsint \qquad (-\pi \leq \mathrm{t} \leq \pi)$$

tengliklarni hosil qilamiz. Bu markazi (x_0, y_0) nuqtada, radiusi esa r bo'lgan aylanadir. Demak,

$$\mathrm{z} = \mathrm{z(t)} = z_0 + \mathrm{r}e^{it}$$

funksiya markazi (x_0, y_0) nuqtada, radiusi r ga teng bo'lgan aylanani ifodalar ekan.

3-misol. Quyidagi tengsizliklarni qanoatlantiruvchi barcha nuqtalar to'plamini kompleks tekislik C da tasvirlang.



4 chizma

3.1. $1 < |z + 2 - 3i| \leq 3$.

3.2. $1 \leq |z + 1 + i| < 2$.

3.3. $1 < |z - 1 + i| \leq 2$.

3.4. $1 \leq |z + 1 - i| < 2$.

3.5. $1 < |z - 3 + 4i| \leq 3$.

3.6. $1 \leq |z + 3 - 4i| < 3$.

3.7. $1 < |z - 1 + 2i| \leq 2$.

3.8. $1 \le |z+1-2i| \le 3$.

3.9. $1 < |z-2+i| \le 2$.

3.10. $1 \le |z+2-i| \le 3$.

3.11. $1 < |z+2+i| \le 2$.

3.12. $1 < |z-2-3i| \le 3$.

3.13. $2 \le |z-1-3i| < 3$.

3.14. $2 < |z+1-3i| \le 3$.

3.15. $2 \le |z+1+3i| < 3$.

3.16. $2 < |z-1+3i| \le 3$.

3.17. $2 \le |z-3i| < 3$.

3.18. $2 < |z+3-i| \le 3$.

3.19. $2 \le |z+3| < 3$.

3.20. $2 < |z-3+i| \le 3$.

3.21. $1 < |z-2+3i| \le 3$.

3.22. $|\mathrm{Re}\, z| > 1$.

3.23. $1 < |z-1| < 3$.

3.24. $1 < |z+3i| \le 4$.

3.25. $|z+2-2i| \le 1$.

3.26. $0 < |z-2| \le 3$.

3.27. $1 < |\mathrm{i}\,\mathrm{Re}\, z + \mathrm{Im}\, z| \le 3$.

3.28. $2 > |\mathrm{Re}\, z| > 1$.

3.29. $-1 < |3z+3i| \le 3$.

3.30. $2 < \left|z - \dfrac{3}{i}\right| \le 3$.

3.31. $1 < \left|\dfrac{z+5-i}{3+2i}\right| \le 3$.

3.32. $1 < \left|\dfrac{z-4i}{4+5i}\right| \le 3$.

4.1-misol. $z = a(\cos^3 t + i\sin^3 t),\ 0 \le t \le 2\pi$ tenglama bilan ifodalangan chiziqning geometrik o'rnini aniqlang. Bunda $a-$ o'zgarmas musbat son.

Yechish. Agar $z = x + iy,$ deb olsak, u holda

$$x + iy = a(\cos^3 t + i\sin^3 t) = a\cos^3 t + i\,a\sin^3 t$$

bo'lib,

$$\left.\begin{aligned} x &= a\cos^3 t \\ y &= a\sin^3 t \end{aligned}\right\}\ (0 \le t \le 2\pi)$$

bo'ladi. Oxirgi tengliklarni

$$\left.\begin{aligned} x^{\frac{2}{3}} &= a^{\frac{2}{3}}\cos^2 t \\ y^{\frac{2}{3}} &= a^{\frac{2}{3}}\sin^2 t \end{aligned}\right\}\ (0 \le t \le 2\pi)$$

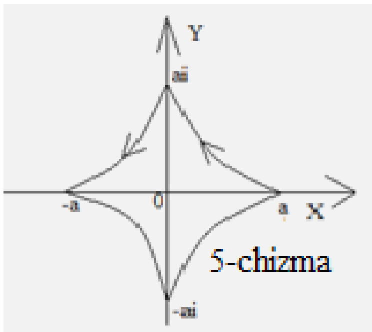
ko'rinishda yozsak, undan

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \quad (0 \le t \le 2\pi)$$

bo'lishi kelib chiqadi. Bu chiziq astpoidadir. Demak,

$$z = a(\cos^3 t + i\sin^3 t), \qquad 0 \le t \le 2\pi$$

astpoidaning parametrik tenglamasi ekan,(5-chizma)



5-chizma

4.2-misol. $z = e^{it},\ 0 \le t \le \pi$ tenglama bilan ifodalangan chiziqning geometrik o'rnini aniqlang.

Yechish. Bu chiziq soat strelkasiga teskari o'naltirilgan $|z| = 1,$ $\operatorname{Im} z \ge 0$ yarim aylanadan iborat.

4.3-misol. $z = \dfrac{a+b}{2}\cdot e^{it} + \dfrac{a-b}{2}\cdot e^{-it}\ (0 \le t \le 2\pi)$ tenglama bilan ifodalangan chiziqning geometrik o'rnini aniqlang. Bunda $a, b-$ o'zgarmas musbat son.

Yechish. Agar $z = x + iy$ deb olsak, u holda

$$e^{it} = \cos t + i\sin t,$$

$$e^{-it} = \cos t - i\sin t$$

munosabatlardan foydalanib,

$$z = \frac{a+b}{2}\cdot(\cos t + i\sin t) + \frac{a-b}{2}\cdot(\cos t + i\sin t),$$

ya'ni,

$$z = a\cos t + ib\sin t$$

bo'lishini topamiz. Oxirgi tenglikda haqiqiy va mavhum qismlarni bir biriga tenglashtirib,

$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \end{aligned}\right\} \qquad (0 \le t \le 2\pi)$$

tengliklarga kelamiz. Bu yarim o'qlari a va b bo'lgan ellipsdir. Demak,

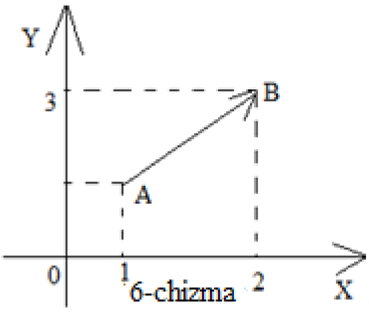
$$z = \frac{a+b}{2} \cdot e^{it} + \frac{a-b}{2} \cdot e^{-it} \qquad (0 \le t \le 2\pi)$$

chiziq ellipsni ifodalar ekan.

4.4-Masala. $z = (1+i) + [(2+3i) - (1+i)\}]t \qquad (0 \le t \le 1)$ tenglama bilan ifodalangan chiziqning geometrik o'rnini aniqlang.

Yechish. $z(t) = x(t) + y(t) = (1+i) + [(2+3i) - (1+i)]t =$ $1 + i + (1+2i)t = 1 + t + i(1+2t)$ tenglikdan, berilgan chiziqning parametric tenglamasi.

$$\begin{cases} x(t) = 1 + t, \\ y(t) = 1 + 2t \end{cases} \quad 0 \le t \le 1$$



6-chizma

ekanligini, bu yerdan esa $y = 1 + 2t = 2(1+t) - 1 = 2x - 1, \quad 1 \le x \le 2$ ekanligini topamiz. Demak, berilgan chiziq 6-chizmada tasvirlangan A(1;1) nuqtadan B(2;3) nuqtaga qarab yo'nalgan AB kesmadan iborat ekan.

4-Misol. Quyidagi funksiyalar aniqlagan egri chiziqlarni toping**.**

4.1. $z = t + it^2 \ (0 \le t < +\infty)$**.**

4.2. $z = 2t + it^2 \ (0 \le t < +\infty)$**.**

4.3. $z = t + i2t^2 \ (0 \le t < +\infty)$**.**

4.4. $z = t + \frac{i}{t} \ \ (-\infty < t < 0)$**.**

4.5. $z = t + \frac{i}{t} \ \ (0 < t < +\infty)$**.**

4.6. $z = 2t + \frac{i}{t} \ \ (-\infty < t < 0)$**.**

4.7. $z = 4t^2 + it^4$ $(-\infty < t < +\infty)$.

4.8. $z = t + \frac{i}{2t}$ $(0 < t < +\infty)$.

4.9. $z = t^2 + i\frac{t^4}{16}$ $(-\infty < t < +\infty)$.

4.10. $z = 9t^2 + it^4$ $(-\infty < t < +\infty)$.

4.11. $z = \mathrm{Re}\, e^{i2t}$ $(0 \le t \le \frac{\pi}{4})$.

4.12. $z = \mathrm{Re}\, e^{i3t}$ $(0 \le t \le \frac{\pi}{6})$.

4.13. $z = \mathrm{Im}\, e^{i2t}$ $(0 \le t \le \frac{\pi}{4})$.

4.14. $z = \mathrm{Im}\, e^{i3t}$ $(0 \le t \le \frac{\pi}{6})$.

4.15. $z = 2t$ $(0 \le t \le 3)$.

4.16. $z = 2 + it$ $(2 \le t \le 5)$.

4.17. $z = t + 3i$ $(1 \le t \le 2)$.

4.18. $z = 2 + i + [(3 + 2i) - (2 + i)]t$ $(0 \le t \le 1)$.

4.19. $z = 3 + 2i + [(5 + 4i) - (3 + 2i)]t$ $(0 \le t \le 1)$.

4.20. $z = 3 + 2i + [(4 + 4i) - (3 + 2i)]t$ $(0 \le t \le 1)$.

4.21. $z = 1 + i + [(2 + 3i) - (1 + i)]t$ $(0 \le t \le 1)$.

4.22. $z = \cos t,\ \pi \le t \le 2\pi.$

4.23. $z = e^{it},\ 0 \le t \le \pi.$

4.24. $z = i + 2e^{it}$ $(3\pi \le t \le 5\pi)$.

4.25. $z = it + 2$ $(-\infty < t < \infty)$.

4.26. $z = (1 - i)t + i$ $(-\infty < t < \infty)$.

4.27. $z = (1 + i)t^2 + 1$ $(-\infty < t < \infty)$.

4.28. $z = 1 + 3e^{it}$ $(\pi \le t \le 3\pi)$.

4.29. $z = i + 2t$ $(-\infty < t < \infty)$.

4.30. $z = (1 - i)t^2 + (1 + i)t$ $(-\infty < t < \infty)$.

4.31. $z = \sin t,\ 0 \le t \le \dfrac{3\pi}{2}.$

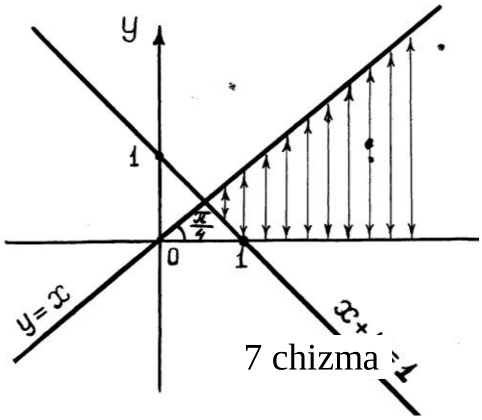
4.32. $z = \cos 2t,\ \pi \le t \le 3\pi.$

5.1-Misol. Quyidagi $\begin{cases} \operatorname{Re} z + \operatorname{Im} z > 1, \\ 0 < \arg z < \dfrac{\pi}{4} \end{cases}$ tengsizliklar sistemasini qanoatlantiruvchi nuqtalar to‘plamini C tekislikda tasvirlang.

Yechish. $\begin{cases} \operatorname{Re} z = x, \\ \operatorname{Im} z = y \end{cases},$

$0 < \arg z < \dfrac{\pi}{4} \Rightarrow \begin{cases} x + y > 1 \\ 0 < y < x \end{cases}$

Bu to‘plam 7-chizmada tasvirlangan.



7 chizma

5.2-misol. Ushbu

$$z = \frac{a+b}{2} e^{it} + \frac{a-b}{2} e^{it}\ (0 \le t \le 2\pi)$$

funksiya aniqlaydigan egri chiziqni toping, bunda a, b- o’zgarmas haqiqiy sonlar.

Yechish. z kompleks sonni $z = x + iy$ deb, so’ngra $e^{it} = \cos t + i\sin t$, $e^{-it} = \cos t - i\sin t$ munosabatlardan foydalanib,

$$x + iy = \frac{a+b}{2}(cost + isint) + \frac{a-b}{2}(cost - isint),$$

ya’ni

$$x + iy = acost + ibsint$$

bo’lishini topamiz. Keyingi tenglikda haqiqiy va mavhum qismlarini bir-biriga tenglashtirib,

$$x = acost \quad y = bsint \quad (0 \le t \le 2\pi)$$

tengliklarga kelamiz. Bu yarim o’qlari a va b bo’lgan ellipsdir. Demak,

$$z = \frac{\mathrm{a+b}}{2}\mathrm{e}^{\mathrm{it}} + \frac{\mathrm{a-b}}{2}\mathrm{e}^{\mathrm{it}} \quad (0 \le t \le 2\pi)$$

ellipsni ifodalar ekan.

5.3-Misol. Kompleks tekislik C da ushbu

$$0 < Re(iz) < 1$$

tengsizlikni qanoatlantiruvchi nuqtalarning geometrik o'rnini toping.

Yechish. $z = x + iy$ bo'lsin deylik. Unda

$$Re(iz) = Re(i(x + iy)) = Re(-y + ix) = -y$$

bo'lib, berilgan tengsizliklar $0 < -y < 1$, ya'ni $-1 < y < 0$ tengsizliklarga keladi, C tekislikning mavhum qismi $-1 < y < 0$ tengsizliklarni qanoatlantiruvchi z nuqtalari to'plami $y = -1$ va $y = 0$ gorizontal to'g'ri chiziqlar orasidagi tekislik qismidan iborat bo'ladi.

5.4-Misol. C da ushbu $|z - i| + |z + i| < 4$ tengsizlikni qanoatlantiruvchi nuqtalarning geometrik o'rnini toping.

Yechish. Ravshanki, quyidagi $\{z\epsilon C;\ |z - i| + |z + i| = 4\}$

to'plam sohaning chegarasi bo'ladi. Agar $z = x + iy$ deb olsak, u holda

$$|z - i| + |z + i| = |x + iy - i| + |x + iy + i| = |x + (y - 1)i| +$$

$$+|x + (y + 1)i| = \sqrt{x^2 + (y - 1)^2} + \sqrt{x^2 + (y + 1)^2}$$

bo'lib,

$$\sqrt{x^2 + (y - 1)^2} + \sqrt{x^2 + (y + 1)^2} = 4,$$

ya'ni

$$\frac{x^2}{3} + \frac{y^2}{4} = 1$$

bo'ladi. Bu esa yarim o'qlari $\sqrt{3}$ va 2 bo'lgan ellipsdir.

Demak, izlanayotgan nuqtalar to’plamining chegarasi ellips bo’lib, berilgan tengsizlikni qanoatlantiruvchi nuqtalarning geometrik o’rni shu ellips bilan o’ralgan.

5-misol. Quyidagi tengsizliklar sistemasini qanoatlantiruvchi nuqtalar to‘plamini C tekislikda tasvirlang.

5.1. $\begin{cases} (\operatorname{Im} z)^2 < 2\operatorname{Re} z, \\ (\operatorname{Re} z)^2 \le \operatorname{Im} z. \end{cases}$

5.2. $\begin{cases} (\operatorname{Im} z)^2 < \operatorname{Re} z, \\ \operatorname{Re} z + \operatorname{Im} z \le 3. \end{cases}$

5.3. $\begin{cases} (\operatorname{Im} z)^2 < \operatorname{Re} z, \\ |z| < 2. \end{cases}$

5.4. $\begin{cases} (\operatorname{Im} z)^2 < \operatorname{Re} z, \\ \frac{\pi}{4} \le \arg z < \frac{\pi}{2} \end{cases}$

5.5. $\begin{cases} (\operatorname{Im} z)^2 < \operatorname{Re} z, \\ |z| > 1. \end{cases}$

5.6. $\begin{cases} (\operatorname{Re} z)^2 \le \operatorname{Im} z, \\ \operatorname{Re} z + \operatorname{Im} z \le 4. \end{cases}$

5.7. $\begin{cases} (\operatorname{Re} z)^2 \le \operatorname{Im} z, \\ |z - i| < 1. \end{cases}$

5.8. $\begin{cases} (\operatorname{Re} z)^2 \le \operatorname{Im} z, \\ |z - 1| < 1. \end{cases}$

5.9. $\begin{cases} (\operatorname{Re} z)^2 \le \operatorname{Im} z, \\ \frac{\pi}{3} < \arg z < \pi \end{cases}$

5.10. $\begin{cases} |z - 1| < 1, \\ |z - i| < 1. \end{cases}$

5.11. $\begin{cases} |z - 1| < 1, \\ \frac{\pi}{4} \le \arg z < \frac{\pi}{2}. \end{cases}$

5.12. $\begin{cases} |z - i| > 2, \\ \frac{\pi}{4} \le \arg z < \frac{\pi}{3} \end{cases}$

5.13. $\begin{cases} \operatorname{Re} z + \operatorname{Im} z > 1, \\ |z| < 2. \end{cases}$

5.14 . $\begin{cases} \operatorname{Re} z + \operatorname{Im} z \le 3, \\ \frac{\pi}{3} < \arg z < \frac{\pi}{2}. \end{cases}$

5.15. $\begin{cases} \dfrac{\pi}{4} < \arg z \le \dfrac{\pi}{2}, \\ 1 < |z| \le 3. \end{cases}$

5.16. $\begin{cases} 1 < |z| \le 3, \\ (\operatorname{Im} z)^2 < \operatorname{Re} z. \end{cases}$

5.17. $\begin{cases} 1 < |z| \le 3, \\ (\operatorname{Re} z)^2 < \operatorname{Im} z. \end{cases}$

5.18. $\begin{cases} 1 < |z| \le 2, \\ \operatorname{Im} z > 0. \end{cases}$

5.19. $\begin{cases} 1 < |z| \le 2, \\ \operatorname{Re} z > 0. \end{cases}$

5.20. $\begin{cases} 1 < |z| \le 2, \\ \operatorname{Im} z > \operatorname{Re} z. \end{cases}$

5.21. $\begin{cases} \operatorname{Re} z + \operatorname{Im} z > 1, \\ 0 < \arg z < \dfrac{\pi}{4}. \end{cases}$

5.22. $\begin{cases} \operatorname{Re} z + \operatorname{Im} z > 1, \\ 0 < \arg z < \dfrac{3\pi}{4}. \end{cases}$

5.23. $\begin{cases} 0 < \operatorname{Re} z + \operatorname{Im} z < 1, \\ 0 < \arg z < \dfrac{\pi}{3}. \end{cases}$

5.24. $\begin{cases} 1 < \operatorname{Re} z - \operatorname{Im} z < 2, \\ 0 < \arg z < \pi. \end{cases}$

5.25 . $\begin{cases} 2 < |z-2i| < 3, \\ \dfrac{\pi}{3} < \arg z < \dfrac{3\pi}{2}. \end{cases}$

5.26. $\begin{cases} 1 < |z-1| < 3, \\ 0 < \arg z < 2\pi. \end{cases}$

5.27. $\begin{cases} |z-1+i| < 5, \\ 0 < \arg z < 2\pi. \end{cases}$

5.28. $\begin{cases} 2 < |z-1|, \\ \dfrac{3\pi}{2} < \arg z < 2\pi. \end{cases}$

5.29. $\begin{cases} 1 < |z-2i| < 3, \\ |z-2i| = 2. \end{cases}$

5.30. $\begin{cases} 3 < |z-2i|, \\ \operatorname{Re} z > 3,\ \operatorname{Im} z < 2. \end{cases}$

5.31. $\begin{cases} 2 < |z-2i|, \\ \operatorname{Re} z < 2,\ \operatorname{Im} z > 2. \end{cases}$

5.32. $\begin{cases} |z-2i| < 4, \\ \operatorname{Re} z < 4,\ \operatorname{Im} z < 3. \end{cases}$

6.1-misol. Ushbu $|z|^2+(4-3i)z+(4+3i)\bar{z}+3$ tenglama aylananing tenglamasi ekanligini isbotlang va bu aylana markazining koordinatlari hamda radiusini toping.

Yechish. $\left.\begin{array}{l} z = x + iy \\ \overline{z} = x - iy \\ |z| = \sqrt{x^2 + y^2} \end{array}\right\} \Rightarrow 0 = |z|^2 + (4-3i)z + (4+3i)\overline{z} + 21 =$

$= x^2 + y^2 + (4-3i)(x+iy) + (4+3i)\cdot(x-iy) + 21 = x^2 + y^2 + 8x + 6y + 21 =$
$= (x+4)^2 + (y+3)^2 - 4 \quad \Rightarrow \quad (x+4)^2 + (y+3)^2 = 2^2.$

Bu markazi (-4,-3) nuqtada va radiusi 2 ga teng bo‘lgan aylananing tenglamasi.

6-misol. Quyidagi tenglama aylananing tenglamasi ekanligini isbotlang va bu aylana markazining koordinatalari hamda radiusini toping**.**

6.1. $|z|^2 + (1-i)z + (1+i)\overline{z} + 1 = 0$**.**

6.2. $|z|^2 + (1-4i)z + (1+4i)\overline{z} + 6 = 0$**.**

6.3. $|z|^2 + (2-3i)z + (2+3i)\overline{z} + 11 = 0$**.**

6.4. $|z|^2 + (3-2i)z + (3+2i)\overline{z} + 12 = 0$**.**

6.5. $|z|^2 + (4-3i)z + (4+3i)\overline{z} + 20 = 0$**.**

6.6. $|z|^2 + (2-4i)z + (2+4i)\overline{z} + 9 = 0$**.**

6.7. $|z|^2 + (4-5i)z + (4+5i)\overline{z} + 21 = 0$**.**

6.8. $|z|^2 + (4-i)z + (4+i)\overline{z} + 16 = 0$**.**

6.9. $|z|^2 + (3-i)z + (3+i)\overline{z} + 9 = 0$**.**

6.10. $|z|^2 + (4-4i)z + (4+4i)\overline{z} + 24 = 0$**.**

6.11. $|z|^2 + (1-5i)z + (1+5i)\overline{z} + 25 = 0$**.**

6.12. $|z|^2 + (5-i)z + (5+i)\overline{z} + 25 = 0$**.**

6.13. $|z|^2 + (2-5i)z + (2+5i)\overline{z} + 28 = 0$**.**

6.14. $|z|^2 + (5-2i)z + (5+2i)\overline{z} + 28 = 0$**.**

6.15. $|z|^2 + (3-5i)z + (3+5i)\overline{z} + 25 = 0$**.**

6.16. $|z|^2 + (5-3i)z + (5+3i)\bar{z} + 25 = 0$.

6.17. $|z|^2 + (5-4i)z + (5+4i)\bar{z} + 25 = 0$.

6.18. $|z|^2 + (2-2i)z + (2+2i)\bar{z} + 7 = 0$.

6.19. $|z|^2 + (3-3i)z + (3+3i)\bar{z} + 16 = 0$.

6.20. $|z|^2 + (4-4i)z + (4+4i)\bar{z} + 28 = 0$.

6.21. $|z|^2 + (4-3i)z + (4+3i)\bar{z} + 21 = 0$.

6.22. $|z|^2 + (5-i)z + (5+i)\bar{z} + 10 = 0.$

6.23. $|z|^2 + 4z + 3\bar{z} + 6 = 0.$

6.24. $x + 2iy - 3y + 6ix = -1 + 8i.$

6.25. $(5-8i)x + (7+3i)y = 2 - i.$

6.26. $x^2 + 4 = 0.$

6.27. $y^2 + 3 = 0.$

6.28. $(7+2i)x - (5-4i)y = -1 - i.$

6.29. $x^2 - (4+3i)x + (1+5i) = 0.$

6.30. $x - iy + 4y + ix = 3 + 5i.$

6.31. $|z|^2 - (2-4i)z + (3-2i)\bar{z} + 15 = 0.$

6.32. $|z|^2 - 6iz + (3+2i)\bar{z} + 24 = 0.$

7.1-misol. C kompleks tekisligidagi $z = \frac{1+i}{\sqrt{2}}$ nuqtaning C Riman sferasidagi obrazini toping.

Yechish. Bu masalani yechishda almashtirish bajarib olamiz.

$z = \frac{1+i}{\sqrt{2}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \Rightarrow x = \frac{\sqrt{2}}{2}$, $y = \frac{\sqrt{2}}{2}$, $|z| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$. Bu yerdan formulaga ko’ra $\xi = \frac{x}{1+|z|^2} = \frac{\sqrt{2}}{4}$; $\eta = \frac{y}{1+|z|^2} = \frac{\sqrt{2}}{4}$; $\zeta = \frac{|z|^2}{1+|z|^2} = \frac{1}{2}$

ekanligini topamiz. Demak, berilgan nuqtaning Riman sferasidagi obrazi $(\frac{\sqrt{2}}{4};\frac{\sqrt{2}}{4};\frac{1}{2})$ ekan.

7-misol. **C** kompleks tekisligidagi z nuqtaning **C** Riman sferasidagi obrazini toping**.**

7.1. $1+i$.
7.2. $1-i$.
7.3. $2+i$.
7.4. $2i+1$.
7.5. $2-i$.
7.6. $-2+i$.
7.7. $2+2i$.
7.8. $2-2i$.
7.9. $3+i$.
7.10. $3-i$.
7.11. $-1+i$.
7.12. $1+3i$.
7.13. $1-3i$.
7.14. $3+2i$.
7.15. $3-2i$.
7.16. $2+3i$.
7.17. $2-3i$.
7.18. $-2+3i$.
7.19. $-2-3i$.
7.20. $3-3i$.
7.21. $\frac{1+i}{\sqrt{2}}$.
7.22. $\frac{1}{1+i}$.
7.23. $(\frac{1}{2}-i\frac{\sqrt{3}}{2})^3$.
7.24. $\frac{1-i}{1+i}$.
7.25. $(1+i)^2$
7.26. $(1-i)^4$.
7.27. $(1-i\sqrt{3})^3$.
7.28 $z=\left(\frac{i^{17}+2}{i^{39}+1}\right)^2$.
7.29 $z=\frac{(1+i)^5}{(1-i)^3}$.
7.30 $z=\frac{1}{1+i}$.
7.31 $(1-i\sqrt{3})^3(1+i)^2$.
7.32. $(1+i\sqrt{3})^{-6}(1-i)^4$.

8. KOMPLEKS O'ZGARUVCHILI FUNKSIYA LIMITI VA UZLUKSIZLIGI

8.1-misol. Hisoblang. $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{2^k}\cos\frac{k\pi}{4}$.

Yechish. Berilgan limitni hisoblash uchun oldin

$$z_n=\sum_{k=0}^{n}\frac{1}{2^k}(\cos\frac{k\pi}{4}+i\sin\frac{k\pi}{4})=\sum_{k=0}^{n}\frac{e^{i\frac{k\pi}{4}}}{2^k}$$

ketma-ketlikning limitini topamiz:

$$\lim_{n\to\infty}z_n=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{e^{i\frac{k\pi}{4}}}{2^k}=\lim_{n\to\infty}\sum_{k=0}^{n}(\frac{e^{i\frac{\pi}{4}}}{2})^k=\lim_{n\to\infty}\frac{1-\frac{e^{\frac{in\pi}{4}}}{2^n}}{1-\frac{e^{i\frac{\pi}{4}}}{2}}=((\left|e^{\frac{in\pi}{4}}\right|=$$

$$=\left|\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right|=1))=\frac{1}{1-\frac{e^{i\frac{\pi}{4}}}{2}}.$$

Bundan

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{2^k}\cos\frac{k\pi}{4}=\lim_{n\to\infty}\operatorname{Re}z_n=\operatorname{Re}\frac{1}{1-\frac{e^{\frac{i\pi}{4}}}{2}}=\operatorname{Re}\frac{2}{2-e^{\frac{i\pi}{4}}}=$$

$$=\operatorname{Re}\frac{2}{2-(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})}=\operatorname{Re}\frac{4}{4-\sqrt{2}-i\sqrt{2}}=$$

$$=\frac{4(4-\sqrt{2})}{(4-\sqrt{2})^2+(\sqrt{2})^2}=\frac{4-\sqrt{2}}{5-2\sqrt{2}}$$

ekanligi kelib chiqadi.

8.2-Misol. Ushbu

$$\mathrm{F(z)}=\frac{z}{|z|},\qquad (\mathrm{z}\neq 0)$$

funksiyaning $z\to 0$ dagi limiti mavjud bo’ladimi?

Yechish. Avvalo berilgan funksiyaning haqiqiy va mavhum qismlarini topaylik:

$$\mathrm{F(z)} = \frac{z}{|z|} = \frac{x+iy}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}},$$

$$\mathrm{u(x,y)} = \frac{x}{\sqrt{x^2+y^2}}, \qquad \mathrm{v(x,y)} = \frac{y}{\sqrt{x^2+y^2}}.$$

$x \to 0,\ y \to 0$da $\mathrm{u(x,y)} = \frac{y}{\sqrt{x^2+y^2}}$ funksiyaning limiti mavjud emas, chunki $x \to 0,\ y = kx\ \ x \to 0$ $(\mathrm{k-const})$ da $\mathrm{u(x,y)} = \frac{y}{\sqrt{x^2+y^2}} \to \frac{1}{\sqrt{1+k^2}}$ bo'lib, k ning turli qiymatida funksiya limiti turlicha bo'ladi.

Yuqorida keltirilganga ko'ra $z \to 0$ da berilgan funksiyaning limiti mavjud bo'lmaydi

8.3 - misol. Ushbu $\mathrm{F(z)} = \frac{zRez}{|z|}$ (z≠0) funksiyaning $z \to 0$ dagi limitini toping.

Yechish. Berilgan $f(z)$ funksiyaning haqiqiy va mavhum qismlarini topamiz:

$$\mathrm{F(z)} = \frac{zRez}{|z|} = \frac{(x+iy)x}{\sqrt{x^2+y^2}} = \frac{x^2}{\sqrt{x^2+y^2}} + i\frac{yx}{\sqrt{x^2+y^2}},$$

$$\mathrm{u(x,y)} = \frac{x^2}{\sqrt{x^2+y^2}}, \qquad \mathrm{v(x,y)} = \frac{xy}{\sqrt{x^2+y^2}}.$$

Ravshanki, $\lim\limits_{\substack{x\to 0\\ y\to 0}} \mathrm{u(x,y)} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{\sqrt{x^2+y^2}} = 0,\quad \lim\limits_{\substack{x\to 0\\ y\to 0}} \mathrm{v(x,y)} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{\sqrt{x^2+y^2}} = 0$

ya'ni $\lim\limits_{z\to 0} \mathrm{f(z)} = \lim\limits_{z\to 0} \frac{zRez}{|z|} = 0$ bo'ladi.

Mustaqil ishlash uchun topshiriqlar

8-misol. Hisoblang**.**

8.1. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{1}{2^k} \sin(\frac{k\pi}{4}).$

8.2. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{1}{2^k} \sin(\frac{k\pi}{3}).$

8.3. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{1}{2^k} \cos(\frac{k\pi}{3}).$

8.4. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{2^k}{3^k} \cos(\frac{k\pi}{4}).$

8.5. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{2^k}{3^k}\sin(\frac{k\pi}{4})$.

8.6. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{2^k}{3^k}\cos(\frac{k\pi}{3})$.

8.7. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{2^k}{3^k}\sin(\frac{k\pi}{3})$.

8.8. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{2^k}{3^k}\cos(\frac{k\pi}{6})$.

8.9. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{2^k}{3^k}\sin(\frac{k\pi}{6})$.

8.10. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{3^k}\cos(\frac{k\pi}{3})$.

8.11. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{3^k}\sin(\frac{k\pi}{3})$.

8.12. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{3^k}\cos(\frac{k\pi}{4})$.

8.13. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{3^k}\sin(\frac{k\pi}{4})$.

8.14. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{3^k}\cos(\frac{k\pi}{6})$.

8.15. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{3^k}\sin(\frac{k\pi}{6})$.

8.16. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{4^k}\cos(\frac{k\pi}{3})$.

8.17. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{4^k}\sin(\frac{k\pi}{3})$.

8.18. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{4^k}\cos(\frac{k\pi}{4})$.

8.19. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{4^k}\sin(\frac{k\pi}{4})$.

8.20. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{3^k}{4^k}\cos(\frac{k\pi}{4})$.

8.21. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{2^k}\cos(\frac{k\pi}{4})$.

8.22. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{2^k}\cos 2k\alpha$.

8.23. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{2^{k+1}}\sin 2(k+1)\alpha$.

8.24. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{3^{k+1}}\cos(2k+1)\alpha$.

8.25. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{3^{k+1}}\sin(2k+1)\alpha$.

8.26. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{3^k}\cos\frac{k\pi}{2}$.

8.27. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{2^k}\sin\frac{k\pi}{2}$.

8.28. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{2^k}\sin\frac{k\pi}{3}$.

8.29. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{3^k}\sin\frac{(2k+1)\pi}{4}$.

8.30. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{4^k}\cos\frac{k\pi}{3}$.

8.31. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{3^k}\cos\frac{(2k+1)\pi}{3}$.

8.32. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{2^k}\cos\frac{(2k-1)\pi}{4}$.

9.1-Misol. $f(z)=\frac{1}{2}(z+\frac{1}{z})$ funksiyani $E=\{|z|<2\}$ sohada bir yaproqlikka tekshiring.

Yechish. Faraz qilaylik, $z_1, z_2 \in E$ lar uchun $f(z_1)=f(z_2)$, ya'ni

$$\frac{1}{2}(z_1+\frac{1}{z_1})=\frac{1}{2}(z_2+\frac{1}{z_2})$$

bo‘lsin. U holda

$$\frac{(z_2-z_1)(1-z_1z_2)}{z_1z_2}=0$$

bo'lishi kelib chiqadi.

Berilgan funksiya E to‘plamda bir yaproqli bo‘lishi uchun shu to‘plamning $z_1z_2=1$ tenglikni qanoatlantiruvchi z_1, z_2 nuqtalarni o‘zida saqlamasligi zarur va yetarli. Lekin, $z_1=i\in E$, $z_2=-i\in E$ va $z_1\cdot z_2=1$. Demak $f(z)$ funksiya E sohada bir yaproqli bo‘lmaydi.

9.2-misol. $f(z)=z^2$ funksiyalrni bir yaproqlikka tekshiring.

Yechish. a) $f(z)=z^2$ funksiyani qaraylik. Bir yaproqlilik ta'rifidan agar $z_1^2=z_2^2$ bo'lsa, bundan $z_1=z_2$ yoki $z_1=-z_2$. Oxirgi tenglik bilan bog'langan ikki nuqta koordinata boshiga nisbatan simmetrikdir. $f(z)=z^2$ funksiya 0 nuqtaga nisbatan simmetrik bo'lgan nuqtalar juftini saqlamaydigan D sohada bir yaproqlidir. Xususiy holda, $f(z)=z^2$ funksiya $\mathrm{Im}\,z>0$ yuqori yarim tekislikda bir yaproqli bo'ladi.

9.3-misol. $f(z)=e^z$ funksiyani bir yaproqlikka tekshiring.

Yechish. $f(z)=e^z$ akslantirish bir yaproqli bo'lishi uchun D soha qanoatlantirishi kerak bo'lgan shartni topamiz. Agar $e^{z_1}=e^{z_2}$, ya'ni

$$e^{z_1-z_2}=1,\quad z_1-z_2=2k\pi i \qquad (k=0,\pm1,\pm2,\ldots) \qquad (1)$$

Bu akslantirish bir yaproqli bo'lishi uchun D soha (1) shartni qanoatlantiruvchi turli xil nuqtalar juftini saqlamasligi zarur va yetarlidir. Xususiy holda, $f(z)=e^z$ funksiya gorizontal $a<\mathrm{Im}\,z<b,\ 0<b-a\le 2\pi$ yo'lakda bir yaproqlidir.

9.4-misol. C_z tekislikdagi $1;\ i;\ -1$ nuqtalarni mos ravishda C_w tekislikdagi 1; 0; -1 nuqtalarga akslantiruvchi kasr-chiziqli funksiyani toping.

Yechish. Kasr-chiziqli akslantirishning xossasida keltirilgan

$$\frac{w-w_1}{w-w_2}\cdot\frac{w_3-w_1}{w_3-w_2}=\frac{z-z_1}{z-z_2}\cdot\frac{z_3-z_1}{z_3-z_2}.$$

tenglikda

$$z_1=1,\ z_2=i,\ \ z_3=-1,$$
$$w_1=-1,\ w_2=0,\ \ w_3=1,$$

deb topamiz:

$$\frac{w-(-1)}{w-0}\cdot\frac{1-0}{1-(-1)}=\frac{z-1}{z-i}\cdot\frac{-1-i}{-1-1}\Rightarrow w=\frac{z-i}{zi-1}.$$

Demak, izlanayotgan kasr - chiziqli funksiya

$$w=\frac{z-i}{zi-1}$$

bo'ladi.

9.5-misol. Kompleks tekislik C_z da $z_1=1+i$ nuqta uchun ushbu $\{z\in C_z:|z|=1\}$ aylanaga nisbatan simmetrik bo'lgan nuqtani toping.

Yechish. Izlanayotgan nuqtani z_1^* deylik. Bu nuqtani topishda

$$z_1^*-z_0=\frac{r^2}{\overline{z_1-z_0}}$$

formuladan foydalanamiz. $z_0=0$, $r=1$ ekanligini e'tiborga olib

$$z_1^*=\frac{1}{\overline{z_1}}$$

bo'lishini topamiz. Demak,

$$z_1^*=\frac{1}{\overline{z}_1}=\frac{1}{\overline{1+i}}=\frac{1}{1-i}=\frac{1}{2}+\frac{1}{2}i$$

ekan.

9-misol.Quyidagi $f(z)$ funksiyalarni berilgan sohalarda bir yaproqlikka tekshiring.

9.1. $f(z)=z^2;\ E=\{\operatorname{Re}z>0\}$.

9.2. $f(z)=z^2;\ E=\{\operatorname{Im}z>0\}$.

9.3. $f(z)=z^3;\ E=\{0<\arg z<\frac{\pi}{2}\}$.

9.4. $f(z)=z^2;\ E=\{|z|<1\}$.

9.5. $f(z)=z^2;\ E=\{|z|<1, 0<\arg z<\frac{3\pi}{2}\}$.

9.6. $f(z)=z^2;\ E=\{|z|>2\}$.

9.7. $f(z)=\frac{1}{2}(z+\frac{1}{z});\ E=\{|z|<1\}$.

9.8. $f(z)=\frac{1}{2}(z+\frac{2}{z});\ E=\{|z|<2\}$.

9.9. $f(z)=\frac{1}{2}(z+\frac{2}{z});\ E=\{\operatorname{Im} z>0\}$.

9.10. $f(z)=\frac{1}{2}(z+\frac{2}{z});\ E=\{\operatorname{Re} z>0\}$.

9.11. $f(z)=\frac{1}{2}(z+\frac{2}{z});\ E=\{\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\}$.

9.12. $f(z)=\frac{1}{z+3};\ E=\{|z|<3\}$.

9.13. $f(z)=\frac{1}{z+3};\ E=\{|z|>3\}$.

9.14. $f(z)=\frac{1}{z+4};\ E=\{|z|<4\}$.

9.15. $f(z)=\frac{1}{z+3};\ E=\{\operatorname{Re} z>3\}$.

9.16. $f(z)=\frac{1}{z+i};\ E=\{\operatorname{Re} z>1\}$.

9.17. $f(z)=e^{2x}(\cos 2y+i\sin 2y);\ E=\{\operatorname{Im} z>0\}$.

9.18. $f(z)=e^{2x}(\cos 2y+i\sin 2y);\ E=\{0<\operatorname{Im} z<\pi\}$.

9.19. $f(z)=e^{2x}(\cos 2y+i\sin 2y);\ E=\{|z|<1\}$.

9.20. $f(z)=e^{2x}(\cos 2y+i\sin 2y);\ E=\{0<\operatorname{Re} z<\frac{1}{2}\}$.

9.21. $f(z)=\frac{1}{2}(z+\frac{1}{z});$ $E=\{|z|<2\}$.

9.22. $f(z)=z^2$**;** $E=\{|z-i|<2\}$.

9.23. $f(z)=\frac{1}{z^2}$**;** $E=\{|z|>2\}$.

9.24. $f(z)=z$**;** $E=\{|z+2i|<1\}$.

9.25. $f(z)=z^3$.**;** $E=\{|z-(1+\mathrm{i})|<3\}$.

9.26. $f(z)=z^2+\frac{1}{z}$**;** $E=\{|z-2|>1\}$.

9.27. $f(z)=z\,\mathrm{Re}\,z$**;** $E=\{|z|<3\}$.

9.28. $f(z)=z|z|^2$**;** $E=\{|z-i|>4\}$.

9.29. $f(z)=\frac{1}{\bar{z}}$**;** $E=\{|z-2+3i|<1\}$.

930. $f(z)=\bar{z}z$**;** $E=\{|z-i|>0\}$.

931. $f(z)=z^2+i|z|^2$**;** $E=\{|z|<1\}$.

9.32. $f(z)=\frac{e^z}{z}$**;** $E=\{|z|<3\}$.

10.1-misol. $f(z)=\frac{1}{z^2+1}$ funksiyani uzluksizlikka tekshiring.

Yechish. $z^2+1=0 \Rightarrow z^2=-1 \Rightarrow z=\pm i$ nuqtalar funksiyaning uzilish nuqtalari. Qolgan barcha nuqtalarda funksiyaning uzluksiz ekanligini ko‘rsatamiz. $\forall z\in C\setminus\{-i, i\}$ uchun

$$\Delta f(z)=f(z+\Delta z)-f(z)=\frac{1}{(z+\Delta z)^2+1}-\frac{1}{z^2+1}=\frac{-\Delta z\cdot(2z+1)}{[(z+\Delta z)^2+1]\cdot(z^2+1)}$$

bo‘lib, bu tenglikdan $\lim\limits_{\Delta z\to 0}\Delta f(z)=0$ ekanligi kelib chiqadi. Bu esa $f(z)=\frac{1}{z^2+1}$ funksiyaning $\forall z\in C\setminus\{-i, i\}$ nuqtada uzluksiz ekanligini anglatadi.

10.2-Misol. ***1*** Ushbu

$$f(z)=z^3$$

funksiyaning ixtiyoriy z nuqtada uzluksizligini isbotlang.

Yechish. $f(z)-f(z_0)$ ayirmani qaraylik:

$$f(z) - f(z_0) = z^3 - z_0^3 = (z - z_0)(z^2 + zz_0 + z_0^2)$$

$z \to 0$bo'lgani uchun shunday $M > 0$ son topiladiki

$$|z| < M \quad |z_0| < M$$

tengsizliklar o'rinli bo'ladi.

Endi ∀ ε > 0 songa qarab δ ni $\delta = \frac{\varepsilon}{3M2}$ deb olsak u holda $|z - z_0| < \delta$ tengsizlikni qanoatlantiruvchi barcha z lar uchun

$$|z^3 - z_0^3| = |(z - z_0)||(z^2 + zz_0 + z_0^2)| < 3M^2|z - z_0| < 3M^2\delta = \varepsilon$$

munosabat bajariladi. Bu $f(z)=z^3$ funksiyaning z_0 nuqtada uzluksiz ekanligini bildiradi.

10.3-Misol. Ushbu $F(z) = \frac{1}{z}$ $(z \neq 0)$ funksiyani uzluksizlikka tekshiring.

Yechish. $\forall\, z_o \epsilon C$ $(z_0 \neq 0)$ nuqtani olaylik. Bunda Δz orttirma, funksiya orttirmasini topamiz:

$$\Delta f = f(z_0 + \Delta z) - f(z_0) = \frac{1}{z_0+\Delta z} - -\frac{1}{z_0} = \frac{-\Delta z}{z_0(z_0+\Delta z)}.$$

Endi $\Delta z \to 0$ da $\Delta f = \lim_{\Delta z \to 0}\left[\frac{-\Delta z}{z0(z0+\Delta z)}\right] = 0.$

Demak, berilgan funksiya $\forall\, z_o \epsilon C$ $(z_0 \neq 0)$ nuqtada uzluksiz bo'ladi.

10-Masala. Berilgan funksiyalarni uzluksizlikka tekshiring**.**

10.1. $f(z) = \dfrac{1}{z^2 - 1}$**.**

10.2. $f(z) = \dfrac{1}{(z-1)(z+i)}$**.**

10.3. $f(z) = \dfrac{1}{(z+1)(z+i)}$**.**

10.4. $f(z) = \dfrac{1}{(z-2)(z+i)}$**.**

10.5. $f(z)=\dfrac{1}{z^2+4}$.

10.6. $f(z)=\dfrac{1}{(z+2)(z+i)}$.

10.7. $f(z)=\dfrac{1}{(z-2)(z-1)}$.

10.8. $f(z)=\dfrac{1}{(z+2)(z-1)}$.

10.9. $f(z)=\dfrac{1}{(z-2)(z-i)}$.

10.10. $f(z)=\dfrac{1}{(z+2)(z-i)}$.

10.11. $f(z)=\dfrac{z}{(z+2)(z+i)}$.

10.12. $f(z)=\dfrac{z}{(z-2)(z+i)}$.

10.13. $f(z)=\dfrac{1}{(2z+1)(z+i)}$.

10.14. $f(z)=\dfrac{1z}{(2z+1)(z-i)}$.

10.15. $f(z)=\dfrac{1}{(2z-1)(z+i)}$.

10.16. $f(z)=\dfrac{1}{(2z-1)(z-i)}$.

10.17. $f(z)=\dfrac{z}{(2z-i)(z-1)}$.

10.18. $f(z)=\dfrac{z}{(2z-i)(z+1)}$.

10.19. $f(z)=\dfrac{z}{(2z+i)(z-1)}$.

10.20. $f(z)=\dfrac{z}{(3z+i)(z-1)}$.

10.21. $f(z) = \dfrac{1}{z^2+1}$.

10.22. $f(z) = \begin{cases} \dfrac{1}{z}, & |z| > 1; \\ z, & |z| = 1; \\ z^2, & |z| < 1. \end{cases}$

10.23. $f(z) = \begin{cases} \dfrac{1}{z^2}, & |z| > 1; \\ z^2, & |z| = 1; \\ z^3, & |z| < 1. \end{cases}$

10.24. $f(z) = \begin{cases} \dfrac{z^2(z^2+1+i)}{\sin^2 z}, & z \neq 0; \\ 1+i, & z = 0. \end{cases}$

10.25. $f(z) = \dfrac{z^3-1}{(z-4)(z^2-1)}$.

10.26. $f(z) = \dfrac{z^2+1}{|z| - \mathrm{Im} z + 1}$.

10.27. $f(z) = \dfrac{z(z^5-1)}{(z-1)(e^z-1)}$.

10.28. $f(z) = \begin{cases} \dfrac{z^3+8}{z+2}, & z \neq -2 \\ 1+i, & z = -2. \end{cases}$

10.29. $f(z) = \dfrac{1}{1-z^2}$, $|z| < 1$.

10.30. $f(z) = \dfrac{1+z}{z^3+i+1}$, $|z| < 1$.

10.31. $f(z) = \dfrac{1}{z+1}$, $0 < |z| < 1$.

10.32. $f(z)=\sin\frac{\pi}{1+z},\quad 0<|z|<1.$

5. KOMPLEKS O'ZGARUVCHILI FUNKSIYA HOSILASI. KOSHI-RIMAN SHARTLARI

11.1-misol. $f(z)=\frac{1}{z+2}$ $(z\neq -2)$ funksiyaning hosilasi ta'rif yordamida hisoblansin.

Yechish. $\forall\ z\in C\backslash\{-2\}$ uchun funksiya hosilasi ta'rifi formulasidan foydalanib topamiz:

$$f'(z)\ :=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{\frac{1}{z+\Delta z+2}-\frac{1}{z+2}}{\Delta z}=$$

$$\lim_{\Delta z\to 0}\frac{-1}{(z+\Delta z+2)(z+2)}=-\frac{1}{(z+2)^2}\ .$$

11.2-misol. Ushbu $f(z)=z^2$ funksiyaning hosilasi mavjudligini tekshiring.

Yechish. Ravshanki, $f(z)=(x+iy)=(x^2-y^2)+2ixy$ bo'lib, $u(x,y)=x^2-y^2$, $v(x,y)=2xy$ funksiyalar (x,y) bo'yicha differensiallanuvchi.

$$\frac{\partial u}{\partial x}=2x,\quad \frac{\partial u}{\partial y}=-2y,$$

$$\frac{\partial v}{\partial x}=2y,\ \frac{\partial v}{\partial y}=2x,$$

tengliklardan Koshi-Riman shartlarni bajarilishini ko'ramiz. Bu esa funksiya tekislikning har bir nuqtasida hosilaga ega ekanligini ko'rsatadi.

11.3-Misol. 2Ushbu $f(z)=z^2$ funksiyaning hosilasi mavjudligini tekshiring.

Yechish. Qaralayotgan $f(z)=z^2=x^2-y^2-2ixy$ funksiya uchun $u(x,y)=x^2-y^2$, $v(x,y)=-2xy$ bo'lib, Koshi-Riman shartlari $(0,0)$ nuqtadan boshqa hech bir nuqtada bajarilmaydi. Demak ,

$f(z) = z^2$ funksiya $z \neq 0$ nuqtalarda hosilaga mavjud emas, $z = 0$ nuqtada esa uning hosilasi mavjud va $f'(0) = 0$.

11- Misol. Funksiya hosilasini ta'rif yordamida hisoblang.

11.1. $f(z) = \dfrac{1}{z+i}\ (z \neq -i)$.

11.2. $f(z) = \dfrac{1}{z+1}\ (z \neq -1)$.

11.3. $f(z) = \dfrac{1}{z-1}\ (z \neq 1)$.

11.4. $f(z) = \dfrac{1}{z-i}\ (z \neq i)$.

11.5. $f(z) = \dfrac{1}{2z-1}\ (z \neq \dfrac{1}{2})$.

11.6. $f(z) = \dfrac{1}{2z+1}\ (z \neq -\dfrac{1}{2})$.

11.7. $f(z) = \dfrac{1}{2z-i}\ (z \neq \dfrac{i}{2})$.

11.8. $f(z) = \dfrac{1}{2z+i}\ (z \neq -\dfrac{i}{2})$.

11.9. $f(z) = z^2$.

11.10. $f(z) = z^3$.

11.11. $f(z) = z^2 + 2z$.

11.12. $f(z) = z^3 - z + 1$.

11.13. $f(z) = 1 - 3z^2$.

11.14. $f(z) = z + 2z^2$.

11.15. $f(z) = 3z \quad -1$.

11.16. $f(z) = 2z \quad +3$.

11.17. $f(z) = \dfrac{1}{z}\ (z \neq 0)$.

11.18. $f(z) = \dfrac{z}{2} + 5$.

11.19. $f(z)=\frac{2z}{3}$.

11.20. $f(z)=e^{x}(\cos y+i\sin y)$.

11.21. $f(z)=\frac{1}{z+2}\ (z\neq -2)$.

11.22. $f(z)=|z|^{2}$.

11.23. $f(z)=\operatorname{Im} z$.

11.24. $f(z)=z|z|^{2}$.

11.25. $f(z)=z^{2}+i|z|^{2}$.

11.26. $f(z)=x^{2}+iy^{2}$.

11.27. $f(z)=y\cdot x+i(x^{2}-y^{2})$.

11.28. $f(z)=\frac{1}{z}$.

11.29. $f(z)=\bar{z}z$.

11.30. $f(z)=x^{3}-3xy^{2}+i(3x^{y}-y^{3})$.

11.31. $f(z)=x^{2}+iy^{2}$.

11.32. $f(z)=x^{2}y^{2}$.

12.1-Misol. $f(z)=z\cdot\operatorname{Im} z$ funksiyani C – differensiallanuvchanlikka tekshiring.

Yechish. Bu funksiyani C – differensiallanuvchanlikka tekshirishda Koshi - Riman shartidan foydalanamiz.

$$f(z)=z\cdot\operatorname{Im} z=(x+iy)\cdot y=xy+iy^{2}\Rightarrow u(x,y)=xy,\ v(x,y)=y^{2}.$$

Bu funksiyalar $\forall(x,y)\in R^{2}$ nuqtada haqiqiy analiz ma'nosida differensiallanuvchi. Endi bu funksiyalar uchun Koshi-Riman shartlarini tekshiramiz. Ushbu $\frac{\partial u}{\partial x} = y$; $\frac{\partial u}{\partial y} = x$; $\frac{\partial v}{\partial x} = 0$; $\frac{\partial v}{\partial y} = 2y$

tengliklardan $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

Koshi-Riman shartlari faqat (0,0) nuqtada bajarilishi kelib chiqadi. Demak, $f(z) = z \cdot \mathrm{Im}\, z$ funksiya faqat $z = 0$ nuqtada C–differensiallanuvchi.

12.2-Misol. 3 Ushbu

$$f(z) = |z|^2 + i[Rez * Imz]^2$$

funksiyani C-differensiallanuvchanlikka tekshiring.

Yechish. Bu funksiya uchun $u(x,y) = |z|^2 = x^2 + y^2$ $v(x,y) = [Rez * Imz]^2 = x^2y^2$, bo`lib, $u$ va $v$ funksiyalar $R^2$ da differensiallanuvchi. Endi Koshi-Riman shartlarni tekshiraylik: $\begin{cases} 2x = 2x^2y, \\ 2y = -2xy^2 \end{cases}$ tengliklardan ko`rinadiki, Koshi-Riman shartlari faqat $x = 0,\ y = 0$ nuqtada bajariladi. Demak, berilgan funksiya faqat $z_0 = 0$ nuqtada C – differensiallanuvchi.

12.3-Misol. 4 Ushbu $f(z) = |z|^2[Rez]^2$ funksiyani C – differensiallanuvchanlikka tekshiring.

Yechish. Ravshanki, $\begin{cases} u(x,y) = (x^2 + y^2)x^2 \\ v(x,y) = 0\,, \end{cases}$ bo`lib, bu funksiyalar haqiqiy analiz ma`nosida differensiallanuvchi bo`lib,

$$\begin{cases} \frac{\partial u}{\partial x} = 4\text{x}3 + 2\text{y}2\text{x}, \\ \frac{\partial u}{\partial y} = 2x^2y \end{cases}$$

bo`lganligidan Koshi-Riman shartlari $x = 0$ tog`ri chiziq nuqtalari uchungina bajariladi. Demak, berilgan funksiya faqat $\{x = 0\}$ to`plamda $C -$ differensiallanuvchi bo`ladi.

12.4-Misol. Ushbu $f(z) = z$ funksiyani C-differensial-lanuvchanlikka tekshiring.

Yechish. Ravshanki, $\frac{\partial f}{\partial z} = 1$ bo'lib, qaralayotgan funksiyani tekislikning birorta nuqtasida ham C-differensiallanuvchi emasligini ko'rsatadi.

12.5-Misol. Ushbu $\mathrm{f(z)} = \sqrt[3]{Re\, z\, Im\, z}$ funksiyani C-differensiallanuvchanlikka tekshiring.

Yechish. Berigan funksiya uchun $\mathrm{u(x,y)} = \sqrt[3]{xy}$, $v(x,y) = 0$ bo'lib, z=0 nuqtada Koshi-Riman shartlari bajariladi:

$$\frac{\partial u(0,0)}{\partial x} = \frac{\partial v(0,0)}{\partial y} = 0$$

$$\frac{\partial u(0,0)}{\partial y} = -\frac{\partial v(0,0)}{\partial x}$$

Biroq,

$$\lim_{\Delta y=0} \frac{\Delta f(0)}{\Delta z} = \lim_{\Delta y=0} \frac{f(\Delta z) - F(0)}{\Delta z}) = \lim_{\Delta y=0} \frac{\sqrt[3]{\Delta x \Delta y}}{\Delta z} = 0.$$

$$\lim_{\Delta x=\Delta y} \frac{\Delta f(0)}{\Delta z} = \lim_{\Delta x\to 0} \frac{\sqrt[3]{(\Delta x)}}{(1+i)\Delta x} = \infty$$

bo'lgani sababli $\Delta z \to 0$ da $\frac{\Delta f(0)}{\Delta z}$ ning limiti mavjud emas. Binobarin qaralayotgan funksiya $z = 0$ nuqtada $C -$ differensiallanuvchi emas. $u = \sqrt[3]{xy}$ funksiya (0,0) nuqtada haqiqiy analiz ma'nosida differensiallanuvchi emas.

12.6-Misol. Ushbu $f(z) = x^2 - y^2 + 2i|xy|$ funksiyani C-differensiallanuvchanlikka tekshiring.

Yechish. Berilgan funksiyaning haqiqiy qismi $u(x,y)$ hamda mavhum qismi $v(x,y)$ larni topamiz.

$$f(z) = u(x,y) + iv(x,y) = x^2 - y^2 + 2i|xy| =$$

$$= \begin{cases} agar \ \ xy > 0 \ \ bolsa, x^2 - y^2 + 2ixy \\ agar \ \ xy < 0 \ \ bolsa, x^2 - y^2 - 2ixy \end{cases}.$$

Demak

$$u(x,y) = x^2 - y^2, \quad v(x,y) = \begin{cases} \text{agar } \text{xy} > 0 \ bo'lsa, & 2xy, \\ \text{agar xy} < 0 \ bo'lsa, & -2xy. \end{cases}$$

Endi $u(x,y)$ va $v(x,y)$ funksiyalar uchun Koshi-Riman shartlarini tekshiramiz

$$\frac{\partial u}{\partial x} = 2\text{x}, \qquad \frac{\partial y}{\partial y} = \begin{cases} \text{agar } \text{xy} > 0 \ bo'lsa, \ 2x, \\ \text{agar } \text{xy} < 0 \ bo'lsa, \ -2x \ , \end{cases}$$

$$\frac{\partial u}{\partial x} = 2\text{x}, \qquad \frac{\partial y}{\partial x} = \begin{cases} \text{agar } \text{xy} > 0 \ bo'lsa, \ 2y, \\ \text{agar } \text{xy} < 0 \ bo'lsa, \ -2y, \end{cases}$$

Ravshanki, $xy > 0$ bo’lganda, ya’ni I va III choraklarda

$$\frac{\partial u}{\partial x} = \frac{\partial y}{\partial y} \ , \ \frac{\partial u}{\partial y} = -\frac{\partial y}{\partial x}$$

bo’ladi. Demak, berilgan funksiya

$$\text{E} = \left\{ z \in C: 0 < argz < \frac{\pi}{2} \right\} \cup \left\{ z \in C: \pi < argz < \frac{3\pi}{2} \right\}$$

da golomorf bo’ladi. $xy < 0$ bo’lgangda, ya’ni II va IV choraklarda funksiya Koshi-Riman shartlarini bajarmaydi. Demak bu choraklarda funksiya C −differensiallanuvchi bo’la olmaydi.

12-Misol. Quyidagi funksiyalarni C-differensiallanuvchanlikka tekshiring

12.1. $f(z) = \operatorname{Re} z$.

12.2. $f(z) = z^2 \operatorname{Re} z$.

12.3 $f(z) = (\operatorname{Re} z)^2$.

12.4. $f(z) = z^2 \operatorname{Im} z$.

12.5. $f(z) = \operatorname{Re} z^2$.

12.6. $f(z) = z \cdot (\operatorname{Re} z)^2$.

12.7. $f(z)=[\operatorname{Re} z]^2 \cdot \operatorname{Im} z$.

12.8. $f(z)=[\operatorname{Im} z]^2 \cdot \operatorname{Re} z$.

12.9. $f(z)=z(\operatorname{Re} z+\operatorname{Im} z)$.

12.10. $f(z)=\operatorname{Im} z^2$.

12.11. $f(z)=|z|^2$.

12.12. $f(z)=|\bar{z}|^2$.

12.13. $f(z)=z \operatorname{Re} z$.

12.14. $f(z)=\bar{z} \cdot \operatorname{Im} z$.

12.15. $f(z)=\operatorname{Im} z$.

12.16. $f(z)=z$.

12.17. $f(z)=\bar{z}$.

12.18. $f(z)=2xy-i(x^2+y^2)$.

12.19. $f(z)=2xy+i(x^2+y^2)$.

12.20. $f(z)=2xy+i(x^2-y^2)$.

12.21. $f(z)=z \operatorname{Im} z$.

12.22. $f(z)=\dfrac{1}{tgz+ctgz}$.

12.23. $f(z)=z\operatorname{Im} z$.

12.24. $f(z)=\bar{z} \operatorname{Im} z$.

12.25. $f(z)=\dfrac{tg2z}{z^3-8}$.

12.26. $f(z)=\dfrac{e^z+1}{e^z-1}$.

12.27. $f(z)=\dfrac{\operatorname{Re} z}{|z|^2}$.

12.28. $f(z)=\dfrac{1}{z}$.

12.29. $f(z)=\dfrac{e^z+1}{e^z-1}$.

12.30. $f(z)=\dfrac{z\cos z}{1+z^2}$.

12.31. $f(z)=\dfrac{e^z}{z}$.

12.32. $f(z)=ze^{-z}$.

5. GOLOMORF FUNKSIYA. HOSILA ARGUMENTI VA MODULINING GEOMETRIK MA'NOSI

13.1-misol. $f(z)=x+ay+i(bx+cy)$ funksiyani golomorflikka tekhiring.

Yechish $f(z)=x+ay+i(bx+cy)$ funksiya hamda $u(x,y)=x+ay$, va $v(x,y)=bx+cy$ funksiyalar R^2 da haqiqiy analiz ma'nosida differensiallanuvchi. Bu funksiyalar uchun Koshi-Riman shartlarini tekshiramiz.

$\frac{\partial u}{\partial x}=1$; $\frac{\partial u}{\partial y}=a$; $\frac{\partial v}{\partial x}=b$; $\frac{\partial v}{\partial y}=c$ va

$$\begin{cases} \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} c=1 \\ b=-a \end{cases} \quad \Rightarrow b=-a \quad \text{ва} \quad c=1$$

shartlar bajarilsa, $f(z)$ funksiya C da golomorf bo'ladi va $f(z)=x+ay+i(-ax+y)=(1-ai)z$ tenglik bajariladi.

13.2-misol. $f(z)=e^z -$ funksiyani golomorflikka tekshiring.

Yechish. $u(x,y)=e^x cosy,\quad v(x,y)=e^x siny$ bo'lib, $C-$ tekislikning barcha nuqtalarida Koshi-Riman shartlarini bajaradi, demak funksiya golomorf ekanligini ko'ramiz.

13.2-misol. $f(z)=\bar{z}z$ funksiyani golomorflikka tekshiring.

Yechish. $f=\bar{z}z$ Funksiya faqat $z=0$ nuqtada $C-$ differensiallanuvchi bo'lib,

$$\frac{\partial f}{\partial z}=\mathrm{z} \qquad \frac{\partial f}{\partial z}|\mathrm{z}=0$$

u bu nuqtada golomorf emas.

13.2-misol. $f(z)=\bar{z}$ funksiyani golomorflikka tekshiring.

Yechish. $f(z)=\bar{z}$ funksiyasi uchun $u(x,y)=x$ va $v(x,y)=-y$ funksiyalar garmoniklik shartnini qanoatlantiradi, demak garmonik funksiya. Ammo qo'shma garmonik funksiyalar emas.

13-misol. Berilgan funksiyalarni golomorflikka tekshiring.

13.1. $f(z)=x+y+i(ax+by)$.

13.2. $f(z)=x^2-y^2+ibxy$.

13.3. $f(z)=\frac{x}{x^2+y^2}-i\frac{ay}{x^2+y^2}$.

13.4. $f(z)=x+2y+i(ax-by)$.

13.5. $f(z)=x-y+i(ax-by)$.

13.6. $f(z)=x+y+i(ax-y)$.

13.7. $f(z)=a(x^2-y^2)+2ixy$.

13.8. $f(z)=x^2+ay^2+ibxy$.

13.9. $f(z)=x+y+i(x+ay)$.

13.10. $f(z)=\dfrac{ax}{x^2+y^2}+i\dfrac{y}{x^2+y^2}$.

13.11. $f(z)=x^2+ay^2-ibxy$.

13.12. $f(z)=x-y+i(ay+bx)$.

1313. $f(z)=x^2-y^2+iaxy$.

13.14. $f(z)=ax+by+icy$.

13.15. $f(z)=ax+y+i(bx+cy)$.

13.16. $f(z)=x^2-ay^2+i2xy$.

13.17. $f(z)=\dfrac{ax}{x^2+y^2}+i\dfrac{by}{x^2+y^2}$.

13.18. $f(z)=x-2y+i(bx+cy)$.

13.19. $f(z)=ax+i(bx+cy)$.

13.20. $f(z)=ax+y+i(bx+cy)$.

13.21. $f(z)=x+ay+i(bx+cy)$.

13.22. $f(z)=x^2-iy^2$.

13.23. $f(z)=ix\cdot y$.

13.24. $f(z)=ix^2+y^2$.

13.25. $f(z)=\dfrac{x}{x^2+y^2}+i\dfrac{y}{x^2+y^2}$.

13.26. $f(z)=e^{ix}\cos y$.

13.27. $f(z)=x\cos y shx-iy\sin y chx$.

13.28. $f(z)=x^2-y^2+ixy$.

13.29. $f(z)=x^2+iy$.

13.30. $f(z)=x^3-3xy^2+i(3x^y-y^3)$.

13.31. $f(z)=ze^{-z}$.

13.32. $f(z)=y\cdot x+i(x^2-y^2)$.

14.1-misol. Faraz qilaylik γ chiziq i nuqtadan chiquvchi $\arg(z-i)=\varphi$ nur bo‘lsin. $w=\frac{z-i}{z+i}$ akslantirish uchun i nuqtadagi cho‘zilish koeffitsienti $R(\varphi)$ va burilish burchagi $\alpha(\varphi)$ ni toping.

Yechish. $w(z)=\frac{z-i}{z+i}\Rightarrow \forall z\in C\setminus\{i\}$ uchun $w'(z)=\left(\frac{z-i}{z+i}\right)'=$

$=\frac{2i}{(z+i)^2}\Rightarrow w'(z)=-\frac{1}{2}$. Demak,

$$R(\varphi)=|w'(i)|=\left|-\frac{i}{2}\right|=\frac{1}{2}$$

va

$$\alpha(\varphi)=\arg w'(i)=\arg\left(-\frac{i}{2}\right)=\tfrac{3\pi}{2}.$$

14-misol. γ chiziq z_0 nuqtadan chiquvchi $\arg(z-z_0)=\varphi$ nur bo‘lsin. Quyidagi misollardagi akslantirishlar uchun z_0 nuqtadagi cho‘zilish koeffitsenti $R(\varphi)$ va burilish burchagi $\alpha(\varphi)$ ni toping.

14.1. $w=z^2$, $z_0=i$.

14.2. $w=\overline{z^2}$, $z_0=1$.

14.3. $w=\bar{z}+2z$ $z_0=0$.

14.4. $w=z^2$, $z_0=\frac{i}{4}$.

14.5. $w=z^2$, $z_0=1-i$.

14.6. $w=z^2$, $z_0=-1+i$.

14.7. $w=z^3$, $z_0=i$.

14.8. $w=z^3$, $z_0=-\frac{i}{4}$.

14.9. $w=z^2+2\bar{z}$, $z_0=1$.

14.10. $w=z^2-2\bar{z}$, $z_0=i$.

14.11. $w = e^{2x}(\cos 2y + \sin 2y);\ z_0 = 0$.

14.12. $w = e^{2x}(\cos 2y - i\sin 2y);\ z_0 = 0$.

14.13. $w = \dfrac{z-1}{z+1}, \quad z_0 = 1$.

14.14. $w = \dfrac{z-(1+i)}{z+1+i}, \quad z_0 = 1+i$.

14.15. $w = \dfrac{z-2+i}{z+2-i}, \quad z_0 = 2-i$.

14.16. $w = \dfrac{z-2i}{z+2i}, \quad z_0 = 2i$.

14.17. $w = \dfrac{z+2}{z-2}, \quad z_0 = -2$.

14.18. $w = \dfrac{z-2}{z+2}, \quad z_0 = 2$.

14.19. $w = \dfrac{z+2i}{z-2i}, \quad z_0 = -2i$.

14.20. $w = \dfrac{z+1-i}{z-1+i}, \quad z_0 = -1+i$.

14.21. $w = \dfrac{z-i}{z+i}, \quad z_0 = i$.

14.22. $w = 1-2iz,\ z_0 = 1$.

14.23. $w = \dfrac{z+1}{z-2},\ z_0 = i$.

14.24. $w = \dfrac{4z}{z+1},\ z_0 = 0$.

14.25. $w = \dfrac{1-z}{1+z},\ z_0 = \pi$.

14.26. $w = \dfrac{i}{z+i},\ z_0 = i$.

14.27. $w = \dfrac{2z}{z-1},\ z_0 = 0$.

14.28. $w = (1+i)\dfrac{z+i}{z-1},\ z_0 = 1$.

14.29. $w = 1 + 2i - z,\ z_0 = i.$

14.30. $w = \dfrac{z - i}{zi - 1},\ z_0 = -i.$

14.31. $w = \dfrac{z - i}{2i},\quad z_0 = 0.$

14.32. $w = \dfrac{z^2 + i}{z},\quad z_0 = i.$

15.1-Misol. Berilgan $u(x,y) = 2(x^2 - y^2) - 1$ garmonik funksiyaga berilgan sohada qo‘shma garmonik bo‘lgan $v(x,y)$ funksiyani toping va ular yordamida golomorf $f(z) = u(x,y) + iv(x,y)$ funksiyani quring.

Yechish. $v(x,y)$ funksiya $u(x,y)$ funksiyaga qo‘shma garmonik funksiya bo‘lgani uchun ular Koshi-Riman shartlarini bajarishi kerak:

$$\begin{cases} \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = 4y \\ \\ \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 4x \end{cases} \Rightarrow \begin{cases} v(x,y) = \int 4y dx + \varphi(y) = 4xy + \varphi(y) \\ \\ \dfrac{\partial v}{\partial y} = 4x + \varphi'(y) = \dfrac{\partial u}{\partial x} = 4x \Rightarrow 4x + \varphi'(y) = 4x \Rightarrow \end{cases}$$

$$\Rightarrow \varphi'(y) = 0 \Rightarrow \varphi(y) = const = c. \Rightarrow v(x,y) = 4xy + c$$

qo‘shma garmonik funksiya. Demak

$$f(z) = u + iv = 2(x^2 - y^2) - 1 + i(4xy + c) = 2(x + iy)^2 - 1 + ic = 2z^2 - 1 + ic.$$

15-MISOL. QUYIDA BERILGAN $u(x,y)$ GARMONIK FUNKSIYALARGA KO‘RSATILGAN SOHALARDA QO‘SHMA GARMONIK BO‘LGAN $v(x,y)$ FUNKSIYALARNI TOPING VA ULAR YORDAMIDA GOLOMORF $f(z) = u(x,y) + iv(x,y)$ FUNKSIYANI QURING.

15.1. $u(x,y) = 4xy \qquad E = C.$

15.2. $u(x,y) = 2x - 3y + 5,\ E = C.$

15.3. $u(x,y) = \dfrac{2x + y}{3(x^2 + y^2)},\quad E = \{0 < |z| < \infty\}.$

15.4. $u(x,y) = 2(x^2 - y^2) + 4xy \qquad E = C.$

15.5. $u(x,y) = x^2 - y^2 - 2xy, \qquad E = C.$

15.6. $u(x,y) = \dfrac{x - 2y}{2(x^2 + y^2)},\ E = \{0 < |z| < \infty\}.$

15.7. $u(x,y)=x^2-y^2+x, \quad E=C$.

15.8. $u(x,y)=3(x^2-y^2)-6xy, \quad E=C$.

15.9. $u(x,y)=x+2y-1, \quad E=C$.

15.10. $u(x,y)=\dfrac{x}{x^2+y^2}$, $E=\{0<|z|<\infty\}$.

15.11. $u(x,y)=\dfrac{x+y}{4(x^2+y^2)}$, $E=\{0<|z|<\infty\}$.

15.12. $u(x,y)=y^2-x^2+2xy, \quad E=C$.

15.13. $u(x,y)=x^2-3xy^2, \quad E=C$.

15.14. $u(x,y)=-x+4y-5, \quad E=C$.

15.15. $u(x,y)=xy+1, \quad E=C$.

15.16. $u(x,y)=\dfrac{x+2y}{x^2+y^2}$, $E=\{0<|z|<\infty\}$.

15.17. $u(x,y)=2x^3-6xy^2, \quad E=C$.

15.18. $u(x,y)=x^2-y^2+xy, \; E=C$.

15.19. $u(x,y)=2x+4y-1, \quad E=C$.

15.20. $u(x,y)=y^2-x^2-4xy, \quad E=C$.

15.21. $u(x,y)=2(x^2-y^2)-1, \quad E=C$.

1522. $u(x,\ y)=x^2-y^2, \quad E=C$.

15.23 .$u(x,\ y)=x\cdot y, \quad E=C$.

15.24. $u(x,\ y)=x^2+y^2, \quad E=C$.

15.25. $u(x,\ y)=\dfrac{x}{x^2+y^2}, \quad E=C$.

15.26. $u(x,\ y)=\dfrac{y}{x^2+y^2}, \quad E=C$.

15.27. $u(x,y)=x^2-y^2+xy, \; E=C$.

15.28. $u(x,y)=e^x\cos y, \quad E=C$.

15.29. $u(x,y)=x\cos y shx-y\sin y chx, \quad E=C$.

15.30. $u(x,y)=2e^x\cos y, \quad E=C$.

15.31. $u(x,y)=\ln(x^2+y^2)+x-2y, \quad E=C$.

15.32. $v(x,y)=2(\mathrm{ch}x\sin y-xy), \quad E=\mathrm{J}$.

6. KONFORM AKSLANTIRISHLAR

16.1-misol. Quyidagi $f(z)=4z^2-8z$ funksiyaning konformlik sohasi topilsin.

Yechish. Bu masalani konform akslantirish haqidagi teoremadan foydalanib yechamiz.

$$f'(z)=(4z^2-8z)'=8(z-1)\neq 0 \Rightarrow z\neq 1.$$

Funksiyani bir yaproqlikka tekshiramiz. Faraz qilaylik, $f(z_1)=f(z_2)$ bo‘lsin. U holda $4z_1^2-8z_1=4z_2^2-8z_2 \Rightarrow 4(z_1-z_2)(z_1+z_2-2)=0$.
Berilgan funksiyaning E to‘plamda bir yaproqli bo‘lishi uchun shu to‘plamning $z_1+z_2=2$ tenglikni qanoatlantiruvchi z_1, z_2 nuqtalarni o‘zida saqlamasligi zarur va yetarli.
Shunday qilib, $f(z)=4z^2-8z$ funksiya $z=1$ nuqtani va $z_1+z_2=2$ tenglikni qanoatlantiruvchi nuqtalarni o‘zida saqlamaydigan ixtiyoriy $E\subset C$ sohada konform bo‘lar ekan.

16.2-Misol. Ushbu $f(z)=z^3$ funksiyasi yordamida berilgan akslantirishni konformlilikka tekshiring.

Yechish. Bu funksiya tekislikning barcha nuqtalarida golomorf bo’lib, uning hosilasi $w'=3z^2$ koordinatalar boshidan tashqari barcha $z_0=0$ nuqtada bu akslantirish konform emas: $|z|=r$ aylana $|w|=r^3$ aylanaga o’tadi, lekin $\lambda_1:\{y=0\}$ to’g’ri chiziq bilan $\lambda^2:\{y=\frac{x}{\sqrt{3}}\}$ to’g’ri chiziqlar orasidagi burchak $\frac{\pi}{6}$ bo’lgani holda ularning aksari $\Gamma_1:\{y=0\}$ va $\Gamma_2:\{x=0\}$ lar orasidagi burchak $\frac{\pi}{2}$ ga tengdir. Demak, akslantirishi $z=0$ nuqtada burchak saqlanishi xossasiga ega emas.
$f(z)=z^3$ akslantirish $E_1:\{0<arg\, z<\frac{\pi 3}{3}\}E_2:\{\frac{2\pi}{3}<arg\, z<\frac{4\pi}{3}\}$ va $E_3:\{\frac{4\pi}{3}<arg\, z<2\pi\}$ sohalarda bir yaproqli. Demak bu akslantirish shu sohada komformdir.

16-misol. Quyidagi funksiyalarning konformlik sohalari topilsin.

16.1. $f(z)=z+\dfrac{1}{z}$.

16.2. $f(z)=\dfrac{2z+1}{z-1}$.

16.3. $f(z)=z^2+1$.

16.4. $f(z)=z^2-1$.

16.5. $f(z)=2z^2+z-1$.

16.6. $f(z)=z^2-2z$

16.7. $f(z)=z^3-1$.

16.8. $f(z)=z^3+1$.

16.9. $f(z)=z^3+3z$

16.10. $f(z)=z^3-3z$.

16.11. $f(z)=\dfrac{z-3}{2z+1}$

16.12. $f(z)=\dfrac{z+4}{2z-5}$.

16.13. $f(z)=e^x(\cos y+\sin y)$.

16.14. $f(z)=e^{2x}(\cos 2y+i\sin 2y)$.

16.15. $f(z)=e^{-x}(\cos y-\sin y)$.

16.16. $f(z)=e^{-2x}(\cos 2y-i\sin 2y)$.

16.17. $f(z)=\dfrac{1}{2}(z+\dfrac{1}{z})$.

16.18. $f(z)=\dfrac{1}{2}(z-\dfrac{1}{z})$.

16.19. $f(z)=3z^2-6z$.

16.20. $f(z)=z^3-8$.

16.21. $f(z)=4z^2-8z$.

16.22. $f(z)=\dfrac{ax}{x^2+y^2}+i\,\dfrac{y}{x^2+y^2}$.

16.23. $f(z)=x^2+ay^2-ibxy$.

16.24. $f(z)=x-y+i(ay+bx)$.

16.25. $f(z)=x^2-y^2+iaxy$.

16.26. $f(z) = ax + by + icy$.

16.27. $f(z) = ax + y + i(bx + cy)$.

16.28. $f(z) = x^2 - ay^2 + i2xy$.

16.29. $f(z) = \frac{ax}{x^2 + y^2} + i\frac{by}{x^2 + y^2}$.

16.30. $f(z) = x - 2y + i(bx + cy)$.

16.31. $f(z) = z^2 - 2z$.

16.32. $f(z) = ax + y + i(bx + cy)$.

II BOB. KOMPLEKS O'ZGARUVCHILI ELEMENTAR FUNKSIYALAR. FUNKSIYA INTEGRALI.

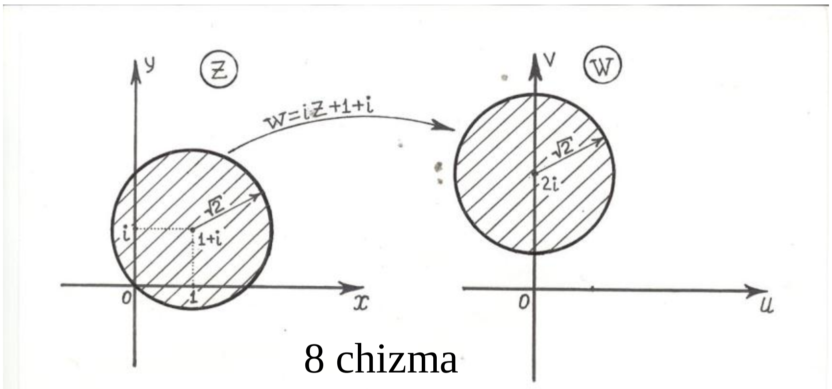
7. DARAJALI FUNKSIYA VA UNING XOSALARI KO'RSATKICHLI VA LOGARIFMIK FUNKSIYALAR. JUKOVSKIY FUNKSIYASI

1.1-misol. Berilgan $D=\{|z-1-i|<2\}$ sohaning $w=iz+1+i$ funksiya yordamidagi aksini toping.

Yechish. $w=iz+1+i$ tenglamani z ga nisbatan yechamiz:

$z=-iw+i-1 \Rightarrow |z-1-i|=|-iw-2|=|-i(w-2i)|=|-i|\cdot|w-2i|=|w-2i| \Rightarrow$

D doiraning aksi $G=w(D)=\{|w-2i|<\sqrt{2}\}$ doira ekan (8-chizma).



8 chizma

1.2-misol. Ushbu $w=z^3$ darajali funksiya yordamida C_z tekislikdagi

$$E=\left\{z\in C_z : 0<\arg z<\frac{\pi}{4}\right\}$$

to'plamning C_w tekislikdagi aksini toping.

Yechish. Berilgan E to'plamni

$$E=\left\{z\in C_z : \arg z=\frac{\pi}{4}\right\}=\left\{\varphi=\frac{\pi}{4}, 0<r<+\infty\right\}$$

kabi yozib olib,

$$w(\mathrm{E})=\left\{w\in C_w : \psi=3\cdot\frac{\pi}{4},\ 0<\rho<+\infty\right\}=\left\{w\in C_w : \arg w=\frac{3\pi}{4}\right\}$$

bo'lishini topamiz.

1.3-misol. Jukovskiy funksiyasi yordamida C_z tekislikdagi

$$l=\left\{z\in C_z:|z|=1,\frac{5\pi}{4}<\arg z<\frac{7\pi}{4}\right\}$$

yoyning aksini toping.

Yechish. Ravshanki,

$$l=\left\{z\in C_z:|z|=1,\frac{5\pi}{4}<\arg z<\frac{7\pi}{4}\right\}=l=\left\{r=1\frac{5\pi}{4}<\varphi<\frac{7\pi}{4}\right\}.$$

Jukovskiy funksiyasi xossalaridan uning haqiqiy va mavhum qismlari uchun quyidagi tenglikni olamiz:

$$u=\frac{1}{2}(r+\frac{1}{r})\cos\varphi=\cos\varphi,\ v=\frac{1}{2}(r-\frac{1}{r})\sin\varphi=0.$$

Agar $\frac{5\pi}{4}<\varphi<\frac{7\pi}{4}$ bo'lganda $-\frac{\sqrt{2}}{2}<\cos\varphi<\frac{\sqrt{2}}{2}$ bo'lishini e'tiborga olsak, u holda

$$w(l)=\left\{-\frac{\sqrt{2}}{2}<u<\frac{\sqrt{2}}{2},v=0\right\}=\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$$

ekanligini topamiz.

1-misol. Berilgan D sohaning $w=f(z)$ chiziqli funksiya yordamidagi aksini toping.

1.1. $D=\{|z-1|<2\}, w=1-2iz$.

1.2. $D=\{|z-i|<2\}, w=1-2iz$.

1.3. $D=\{|z+1|<2\}, w=1-2iz$.

1.4. $D=\{|z+i|<2\}, w=1-2iz$.

1.5. $D=\{|z-1|<2\}, w=1+2iz$.

1.6. $D=\{|z-i|<2\}, w=1+2iz$.

1.7. $D=\{|z+1|<2\}, w=1+2iz$.

1.8. $D=\{|z+i|<2\}, w=1+iz$.

1.9. $D=\{|z-1|<2\}, w=iz+1+i$.

1.10. $D=\{|z-i|<2\}, w=iz+1+i$.

1.11. $D=\{|z+1|<2\}, w=iz+1+i$.

1.12. $D=\{|z+i|<2\}, w=iz+1+i$.

1.13. $D=\{|z-1|<2\}, w=iz-1+i$.

1.14. $D=\{|z-i|<2\}, w=iz-1+i$.

1.15. $D=\{|z+1|<2\}, w=iz-1+i$.

1.16. $D=\{|z+i|<2\}, w=iz-1+i$.

1.17. $D=\{|z-1|<2\}, w=iz+1-i$.

1.18. $D=\{|z-i|<2\}, w=iz+1-i$.

1.19. $D=\{|z+1|<2\}, w=iz+1-i$.

1.20. $D=\{|z+i|<2\}, w=iz+1-i$.

1.21. $D=\{|z-1-i|<2\}, w=iz+1+i$.

1.22. $D=\{|z+3|<2\}, w=iz+1-i$.

1.23. $D=\{|z-4i|<2\}, w=iz-1-i$.

1.24. $D=\{|z+3|<3\}, w=iz+2+i$.

1.25. $D=\{|z+i-2|<3\}, w=iz-1-i$.

1.26. $D=\{|z|<1\}, w=iz+i$.

1.27. $D=\{|z+3i-1|<4\}, w=iz+1$.

1.28. $D=\{|z-3i|<1\}, w=iz-1-i$.

1.29. $D=\{|z+2i|<4\}, w=iz-1+i$.

1.30. $D=\{|z-i+2|<3\}, w=iz+1+i$.

1.31. $D=\{|z-1-i|<3\}, w=z+1-i$.

1.32. $D=\{|z-2i|<2\}, w=iz+2-i$.

2.1-misol. Berilgan $z_0=1+2i$ nuqtani qo‘zg‘almas qoldirib, $z_1=i$ nuqtani $w_1=-i$ nuqtaga o‘tkizadigan chiziqli akslantirishni toping.

Yechish. Ma’lumki, chiziqli akslantirishning umumiy ko‘rinishi $w=az+b$. Bu yerdagi $a,b\in C$ noma’lumlarni masala shartidan foydalanib topamiz:

$$\begin{cases} a(1+2i)+b=1+2i, \\ ai+b=-i. \end{cases} \Rightarrow a=2+i\ ,\ b=1-3i.$$

Demak, $w=(2+i)z+1-3i.$

2-misol. Berilgan z_0 nuqtani qo‘zg‘almas qoldirib, z_1 nuqtani w_1 nuqtaga o‘tkazadigan chiziqli akslantirishni toping.

2.1. $z_0 = 1+i$, $z_1 = i$, $w_1 = -i$.

2.2. $z_0 = 1-i$, $z_1 = i$, $w_1 = -i$.

2.3. $z_0 = 1+i$, $z_1 = 2+i$, $w_1 = i$.

2.4. $z_0 = 1-i$, $z_1 = 1+i$, $w_1 = i$.

2.5. $z_0 = 1+i$, $z_1 = 1-i$, $w_1 = i$.

2.6. $z_0 = 1-i$, $z_1 = 2-i$, $w_1 = i$.

2.7. $z_0 = 1+i$, $z_1 = 2+i$, $w_1 = 1-i$.

2.8. $z_0 = 1+i$, $z_1 = 2+i$, $w_1 = 1+i$.

2.9. $z_0 = 1+i$, $z_1 = 2-i$, $w_1 = 1-i$.

2.10. $z_0 = 1+i$, $z_1 = 2-i$, $w_1 = 1+i$.

2.11. $z_0 = 1+i$, $z_1 = 2+i$, $w_1 = 2-i$.

2.12. $z_0 = 1+i$, $z_1 = 2-i$, $w_1 = 2+i$.

2.13. $z_0 = 1+i$, $z_1 = 1+2i$, $w_1 = 2-i$.

2.14. $z_0 = 1+i$, $z_1 = 1-2i$, $w_1 = 2-i$.

2.15. $z_0 = 1+i$, $z_1 = 1+2i$, $w_1 = 2+i$.

2.16. $z_0 = 1+i$, $z_1 = 1-2i$, $w_1 = 2+i$.

2.17. $z_0 = 1+i$, $z_1 = 1+2i$, $w_1 = i$.

2.18. $z_0 = 1+i$, $z_1 = 1-2i$, $w_1 = i$.

2.19. $z_0 = 1+i$, $z_1 = 1+2i$, $w_1 = -i$.

2.20. $z_0 = 1+i$, $z_1 = 1-2i$, $w_1 = -i$.

2.21. $z_0 = 1+2i$, $z_1 = i$, $w_1 = -i$.

2.22. $z_0 = 3-i$, $z_1 = -i$, $w_1 = 2-i$.

2.23. $z_0 = 4+3i$, $z_1 = 3+2i$, $w_1 = 2-2i$.

2.24. $z_0 = 2+i$, $z_1 = 2+3i$, $w_1 = 2+i$.

2.25. $z_0 = 2i$, $z_1 = 4+i$, $w_1 = 3+i$.

2.26. $z_0 = 2-3i$, $z_1 = 3-2i$, $w_1 = 3+i$.

2.27. $z_0 = 3-i$, $z_1 = -2i$, $w_1 = 2i+3$.

2.28. $z_0 = 4+2i$, $z_1 = 3-2i$, $w_1 = 2i-1$.

2.29. $z_0 = 3+2i$, $z_1 = 4+i$, $w_1 = 3-i$.

2.30. $z_0 = 3+2i$, $z_1 = 2-3i$, $w_1 = 4-i$.

2.31. $z_0 = 3+2i$, $z_1 = 4+2i$, $w_1 = 3-3i$.

2.32. $z_0 = 1-i$, $z_1 = 3+i$, $w_1 = -i$.

3.1-misol. Quyidagi $w = 2z+1-3i$ uchun chekli qo‘zgalmas nuqta z_0 (agar u mavjud bo‘lsa), burilish burchagi φ va cho‘zilish koeffisentini toping. Akslantirishni $w - z_0 = \lambda(z - z_0)$ kanonik ko‘rinishga keltiring.

Yechish. Qo’zgalmas nuqtani $w(z_0) = z_0$ tenglikdan foydalanib topamiz:

$$2z_0 + 1 - 3i = z_0 \Rightarrow z_0 = -1+3i \Rightarrow w - z_0 = 2z+1-3i - z_0 =$$

$$= 2z+1-3i+1-3i = 2(z+1-3i).$$

Demak, $w+1-3i = 2(z+1-3i)$. Bu yerdan

$$z_0 = -1+3i,\ \phi = 0,\ k = 2;\ w+1-3i = 2(z+1-3i)$$

natijaga kelamiz.

3-misol. Quyidagi akslantirishlar uchun chekli qo‘zg‘almas nuqta z_0 (agar u mavjud bo‘lsa), burilish burchagi φ va chuzilish koeffisentini toping. Akslantirishni $w - z_0 = \lambda(z - z_0)$ kanonik ko‘rinishiga keltiring.

3.1. $w = z+1-2i$.

3.2. $w = z+1+2i$.

3.3. $w = z-1-2i$.

3.4. $w = z-1+2i$.

3.5. $w = z+2-i$.

3.6. $w = z+2+i$.

3.7. $w = z-2+i$.

3.8. $w = z-2-i$.

3.9. $w = z-2+2i$.

3.10. $w = z+2-2i$.

3.11. $w = 2z+1-2i$.

3.12. $w = 2z+1+2i$.

3.13. $w = 2z-1-2i$.

3.14. $w = 2z-1+2i$.

3.15. $w = 2z+2-i$.

3.16. $w = 2z+2+i$.

3.17. $w = 2z-2+i$.

3.18. $w = 2z-2-i$.

3.19. $w = 2z-1+i$.

3.20. $w = 2z+1-i$.

3.21. $w = 2z+1-3i$.

3.22. $w = 3z-1-i$.

3.23. $w = z-3+2i$.

3.24. $w = 3z+3-4i$.

3.25. $w = 4z+1-2i$.

3.26. $w = 2z+3-4i$.

3.27. $w = 3z+i$.

3.28. $w = 4z-3i+2$.

3.29. $w = 3z+4-2i$

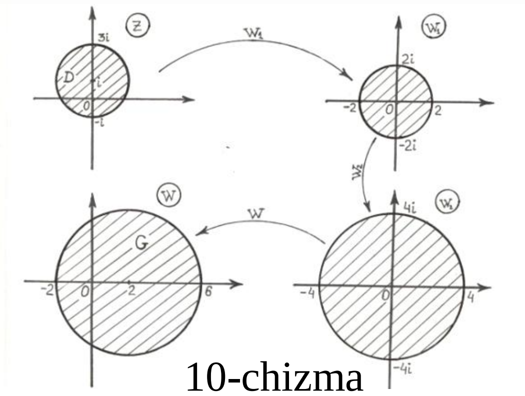
3.30. $w = 3z-2i-3i$.

3.31. $w = 2z+4+i$.

3.32. $w = z+3-2i$.

4.1-misol. Berilgan $D=\{|z-i|<2\}$ doirani $G=\{|w-2|<4\}$ doiraga akslantiruvchi chiziqli funksiyani toping.

Yechish. Ushbu $w_1=z-i$ funksiyani qaraylik. Bu funksiya berilgan D doirani (w_1) tekislikda markazi koordinata boshida bo‘lgan $|w_1|<2$ doiraga akslantiradi. Endi $w_2=2w_1$ *va* $w=w_2+2$ akslantirishlardan ketma-ket foydalansak berilgan doira G doiraga akslanadi (10-chizma).



10-chizma

Demak, $w=w_2+2=2w_1+2=2(z-i)+2=2z+2(1-i)$.

4-misol. Berilgan D doirani G doiraga akslantiruvchi chiziqli funksiyani toping.

4.1. $D=\{|z-1+i|<2\}$, $G=\{|w-i|<4\}$.

4.2. $D=\{|z-1+i|<2\}$, $G=\{|w+i|<4\}$.

4.3. $D=\{|z+1-i|<2\}$, $G=\{|w-i|<4\}$.

4.4. $D=\{|z+1-i|<2\}$, $G=\{|w+i|<4\}$.

4.5. $D=\{|z-1|<2\}$, $G=\{|w+i|<3\}$.

4.6. $D=\{|z-i|<2\}$, $G=\{|w+1|<3\}$.

4.7. $D=\{|z+1|<2\}$, $G=\{|w+i|<3\}$.

4.8. $D=\{|z+i|<2\}$, $G=\{|w+1|<3\}$.

4.9. $D=\{|z-i|<3\}$, $G=\{|w+i|<4\}$.

4.10. $D=\{|z-1|<3\}$, $G=\{|w+i|<4\}$.

4.11. $D=\{|z-1+i|<4\}$, $G=\{|w-i|<3\}$.

4.12. $D=\{|z-1+i|<4\}$, $G=\{|w+i|<2\}$.

4.13. $D=\{|z+1-i|<4\}$, $G=\{|w-i|<2\}$.

4.14. $D=\{|z+1-i|<4\}$, $G=\{|w+i|<5\}$.

4.15. $D=\{|z-1|<4\}$, $G=\{|w+i|<2\}$.

4.16. $D=\{|z-i|<4\}$, $G=\{|w+i|<2\}$.

4.17. $D=\{|z+1|<4\}$, $G=\{|w+i|<2\}$.

4.18. $D=\{|z+i|<4\}$, $G=\{|w+1|<2\}$.

4.19. $D=\{|z-i|<4\}$, $G=\{|w+i|<3\}$.

4.20. $D=\{|z-1|<4\}$, $G=\{|w+1|<2\}$.

4.21. $D=\{|z-i|<2\}$, $G=\{|w-2|<4\}$.

4.22. $D=\{|z-2+i|<4\}$, $G=\{|w-i|<3\}$.

4.23. $D=\{|z-i|<3\}$, $G=\{|w+2+i|<1\}$.

4.24. $D=\{|z+1-2i|<2\}$, $G=\{|w+2i-1|<4\}$.

4.25. $D=\{|z+2i-3|<5\}$, $G=\{|w+2-3i|<2\}$.

4.26. $D=\{|z+3i|<4\}$, $G=\{|w-2i|<3\}$.

4.27. $D=\{|z+i|<3\}$, $G=\{|w-2+i|<3\}$.

4.28. $D=\{|z-4+2i|<2\}$, $G=\{|w+1-2i|<3\}$.

4.29. $D=\{|z-3i|<2\}$, $G=\{|w+5i|<4\}$.

4.30. $D=\{|z+2i-2|<3\}$, $G=\{|w+3+i|<4\}$.

4.31. $D=\{|z|<4\}$, $G=\{|w+3i-2|<2\}$.

4.32. $D=\{|z-3+2i|<2\}$, $G=\{|w-i+1|<4\}$.

5.1-misol. Berilgan $D=\{0<\operatorname{Re} z<1\}$ sohaning $w=\dfrac{z-1}{z}$ akslantirish yordamidagi aksini toping.

Yechish. Bu masalani yechish uchun sohaning saqlanish prinsipi va kasr-chiziqli akslatirishning doiraviylik prinsipidan foydalanamiz. $G=w(D)$ desak, $\partial D=w(\partial D)$ bo‘ladi.

$\partial D=\{\operatorname{Re} z=0\}\cup\{\operatorname{Re} z=1\}$. $z\in\{\operatorname{Re} z=0\}$ va $w(0)=\infty$ bo‘lgani uchun $\{\operatorname{Re} z=0\}$ to‘g‘ri chiziqning aksi to‘g‘ri chiziq bo‘ladi. Uni topish uchun $z_1=i$ *va* $z_2=-i\in\{\operatorname{Re} z=0\}$ nuqtalarni olib, ularning obrazlarini topamiz: $w(i)=\frac{i-1}{i}=1+i,\qquad w(-i)=\frac{-i-1}{-i}=1-i.$ Bu nuqtalardan o‘tuvchi to’g‘ri chiziq $\operatorname{Re}w=1$. $\operatorname{Re}z=1$ to’g‘ri chiziqning aksi esa aylana bo‘ladi, chunki bu chiziqning ustida $w=\frac{z-1}{z}$ funksiyani ∞ ga aylantiradigan nuqta yo‘q. Uni topish uchun $w=\frac{z-1}{z}$ tenglamadan z ni topamiz:

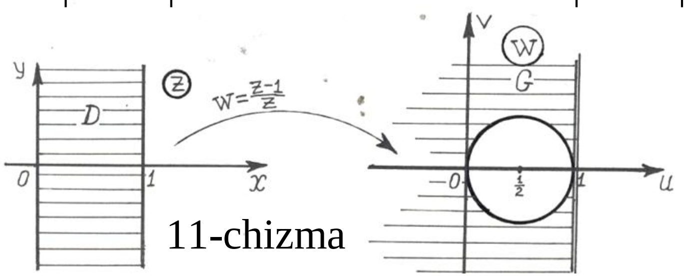
$$z=\frac{-1}{w-1}=\frac{-1}{u+iv-1}=\frac{-1}{u-1+iv}=\frac{-(u-1-iv)}{(u-1)^2+v^2}=$$

$$=\frac{1-u}{(u-1)^2+v^2}+i\frac{v}{(u-1)^2+v^2}.$$

Bu yerdan va $\operatorname{Re}z=1$ dan $\Rightarrow\frac{1-u}{(u-1)^2+v^2}=1\Rightarrow(u-1)^2+v^2=$

$$\Rightarrow(u-\frac{1}{2})^2+v^2=\frac{1}{4}\Rightarrow\left|w-\frac{1}{2}\right|=\frac{1}{2}.$$

Demak $\partial G=\{\operatorname{Re}w=1\}\cup\{\left|w-\frac{1}{2}\right|=\frac{1}{2}\}\Rightarrow G=\{\operatorname{Re}w<1,\ \left|w-\frac{1}{2}\right|>\frac{1}{2}\}$ (11-chizma).



11-chizma

5-misol. Berilgan D sohaning kasr-chiziqli $w=f(z)$ akslantirish yordamida aksini toping.

5.1. $D=\{|z|>1\},\qquad w=\frac{z-1}{z+i}.$

5.2. $D=\{x<0,y<0\},\qquad w=\frac{1}{z}.$

5.3. $D=\{|z|<1\},\qquad w=\frac{z+i}{z+1}.$

5.4. $D=\{\operatorname{Im}z<1\},\qquad w=\frac{z-i}{z}.$

5.5. $D=\{0<\operatorname{Re}z<2\}$, $w=\dfrac{1}{z-2}$.

5.6. $D=\{\dfrac{\pi}{4}<\arg z<\dfrac{\pi}{2}\}$, $w=\dfrac{1}{z}$.

5.7. $D=\{|z|<1, |z-1|<\sqrt{2}\}$, $w=\dfrac{z-i}{z+i}$.

5.8. $D=\{|z|>1, |z-1|<\sqrt{2}\}$, $w=\dfrac{z-i}{z+i}$.

5.9. $D=\{|z-1|>2\}$, $w=\dfrac{2iz}{z+3}$.

5.10. $D=\{|z-1|>2\}$, $w=\dfrac{z+1}{z-2}$.

5.11. $D=\{|z-1|<3\}$, $w=\dfrac{z-1}{2z-6}$.

5.12. $D=\{\operatorname{Re}z>1\}$, $w=\dfrac{z}{z-1+i}$.

5.13. $D=\{\operatorname{Re}z>1\}$, $w=\dfrac{z}{z-2}$.

5.14. $D=\{\operatorname{Re}z>1\}$, $w=\dfrac{z-3+i}{z+1+i}$.

5.15. $D=\{|z|<1, \operatorname{Im}z<0\}$, $w=\dfrac{1-z}{1+z}$.

5.16. $D=\{|z+i|>1, \operatorname{Im}z>1\}$, $w=\dfrac{1}{z}$.

5.17. $D=\{1<|z|<2\}$, $w=\dfrac{1}{z-2}$.

5.18. $D=\{\operatorname{Re}z>0, \operatorname{Im}z<0\}$, $w=\dfrac{z-i}{z+i}$.

5.19. $D=\{|z|<1, \operatorname{Im}z<0\}$, $w=\dfrac{2z-i}{2+iz}$.

5.20. $D=\{\dfrac{3\pi}{4}<\arg z<\pi\}$, $w=\dfrac{z}{z+1}$.

5.21. $D=\{0<\operatorname{Re}z<1\}$, $w=\dfrac{z-1}{z}$.

5.22. $D=\{\operatorname{Re} z>2\},\qquad w=\dfrac{z}{z+2+i}$.

5.23. $D=\{\operatorname{Re} z>1,\ \operatorname{Im} z<0\},\qquad w=\dfrac{z}{z-2}$.

5.24. $D=\{\operatorname{Re} z>1,\ \operatorname{Im} z<2\},\qquad w=\dfrac{z+i}{z+1+i}$.

5.25. $D=\{|z|<1, \operatorname{Im} z<0\},\qquad w=\dfrac{z}{1-z}$.

5.26. $D=\{|z+i|>1, \operatorname{Re} z>1\}\qquad w=\dfrac{1}{z-2}$.

5.27. $D=\{1<|z|<2,\ \mathrm{Imz}>0\},\qquad w=\dfrac{z-1}{z+1}$.

5.28. $D=\{\operatorname{Re} z>0, \operatorname{Im} z<0\},\qquad w=\dfrac{z}{z+i}$.

5.29. $D=\{|z|<1, \operatorname{Re} z<0\},\qquad w=\dfrac{i}{2+iz}$.

5.30. $D=\{\dfrac{3\pi}{2}<\arg z<2\pi\},\qquad w=\dfrac{z}{z-1}$.

5.31. $D=\{0<\operatorname{Im} z\cdot\operatorname{Re} z<1\},\qquad w=\dfrac{z-1}{2z+i}$.

5.32. $D=\{2<|z-1|<3\},\qquad w=\dfrac{z+2}{2z-3i}$.

6.1-misol. Quyidagi $w(-1)=i,\quad w(i)=\infty,\quad w(1+i)=1$ shartlarni qanoatlantiruvchi kasr-chiziqli $w(z)$ akslantirishni toping.

Yechish. Bu masalani yechish uchun ushbu

$$\frac{w-w_1}{w-w_2}\cdot\frac{w_3-w_2}{w_3-w_1}=\frac{z-z_1}{z-z_2}\cdot\frac{z_3-z_2}{z_3-z_1}.$$

Angarmonik nisbatdan foydalanamiz. Bizning holda $z_1=-1$, $z_2=i$, $z_3=1+i$ va $w_1=i$, $w_2=\infty$, $w_3=1$. $w_2=\infty$ bo‘lgani uchun angarmonik nisbat quyidagi

$$\frac{w-w_1}{w_3-w_1}=\frac{z-z_1}{z-z_2}\cdot\frac{z_3-z_2}{z_3-z_1}$$

ko‘rinishga keladi. Bu yerdan

$$\frac{w-i}{1-i}=\frac{z+1}{z-i}\cdot\frac{1+i-i}{1+i+1}$$

va $w=\dfrac{(1+2i)z+6-3i}{5(z-i)}$ ekanligini topamiz.

6-misol. Quyidagi shartlarni qanoatlantiruvchi kasr-chiziqli $w(z)$ akslantirishni toping.

6.1. $w(1)=1,\quad w(0)=-1,\quad w(i)=i$.

6.2. $w(\frac{1}{2})=\frac{1}{2},\quad w(2)=2,\quad w(\frac{5}{4}+\frac{3}{4}i)=\infty$.

6.3. $w(0)=2,\quad w(1+i)=2+i,\quad w(2i)=0$.

6.4. $w(4)=0,\quad w(2+2i)=1+i,\quad w(0)=2i$.

6.5. $w(0)=0,\quad w(i)=2,\quad w(2i)=3$.

6.6. $w(0)=0,\quad w(2)=i,\quad w(3)=2i$.

6.7. $w(1)=0,\quad w(1+i)=\infty,\quad w(3i)=3i$.

6.8. $w(0)=1,\quad w(\infty)=1+i,\quad w(3)=4i$.

6.9. $w(i)=2,\quad w(\infty)=2i,\quad w(-i)=0$.

6.10. $w(2)=i,\quad w(2i)=\infty,\quad w(0)=3i$.

6.11. $w(i)=-2,\quad w(\infty)=4i,\quad w(-i)=2$.

6.12. $w(-2)=i,\quad w(4i)=\infty,\quad w(2)=-i$.

6.13. $w(0)=-1,\quad w(2i)=i,\quad w(1+i)=1-i$.

6.14. $w(i)=-1,\quad w(\infty)=i,\quad w(1)=1+i$.

6.15. $w(i)=-1,\quad w(1)=\infty,\quad w(1+i)=i$.

6.16. $w(\infty)=-1,\quad w(i)=\infty,\quad w(i)=i$.

6.17. $w(0)=-1,\quad w(\infty)=\infty,\quad w(1)=i$.

6.18. $w(1)=1,\quad w(\infty)=-1,\quad w(i)=i$.

6.19. $w(\frac{1}{2})=\frac{1}{2},\quad w(2)=2,\quad w(\infty)=\frac{5}{4}+\frac{3}{4}i$.

6.20. $w(2)=0,\quad w(2+i)=1+i,\quad w(\infty)=\infty$.

6.21. $w(-1)=i,\quad w(i)=\infty,\quad w(1+i)=1$.

6.22. $w(2)=i,\quad w(2i)=\infty,\quad w(2)=-3i$.

6.23. $w(0)=-i,\quad w(i)=2i,\quad w(1+2i)=2-i$.

6.24. $w(1-i)=i+1,\quad w(\infty)=3i,\quad w(1)=1-i$.

6.25. $w(i)=-i,\quad w(1+i)=\infty,\quad w(1+i)=2i-1$.

6.26. $w(\infty)=-i,\qquad w(i-1)=\infty,\qquad w(i+2)=i-2$.

6.27. $w(0)=-2,\qquad w(\infty)=\infty,\qquad w(i)=i$.

6.28. $w(1)=1,\qquad w(\infty)=-1,\qquad w(i)=i$.

6.29. $w(i-\frac{1}{2})=\frac{1}{2},\qquad w(2+i)=2i,\qquad w(\infty)=3+2i$.

6.30. $w(2)=i+3,\qquad w(2i)=1-i,\qquad w(\infty)=\infty$.

6.31. $w(-1-i)=i+2,\qquad w(2i)=\infty,\qquad w(1-2i)=1-i$.

6.32. $w(i)=-1,\qquad w(\infty)=i,\qquad w(-i)=1.$

7.1-Misol. $D=\{|z|<2\}$ sohani $G=\{\operatorname{Re} w>0\}$ sohaga akslantiruvchi va $w(0)=1,\ \arg w^1(0)=\frac{\pi}{2}$ shartlarni qanoatlantiruvchi kasr-chiziqli $w(z)$ funksiyani toping.

Yechish. Avval G ni D ga konform akslantiruvchi kasr-chiziqli funksiyaning umumiy ko‘rinishini topib olamiz. Buning uchun ushbu $z_1=iw$, $z_2=e^{i\theta}\dfrac{z_1-a}{z_1-\overline{a}}$ *va* $z=2z_2$ akslantirishlarni ketma-ket bajarish yetarli ekanligini ko‘rish qiyin emas.

Demak,

$$z=2z_2=2e^{i\theta}\frac{z_1-a}{z_1-\overline{a}}=2e^{i\theta}\cdot\frac{iw-a}{iw-\overline{a}}.$$

Bu tenglamani w ga nisbatan yechib, D ni G ga akslantiruvchi funksiyaning umumiy ko‘rinishi

$$w=-i\frac{\overline{a}z-2ae^{i\theta}}{z-2e^{i\theta}}$$

ekanligini hosil qilamiz. Bu yerdagi a va θ noma’lumlarni berilgan shartlardan foydalanib topamiz:

$$w(0)=1\Rightarrow -i\cdot\frac{-2ae^{i\theta}}{-2e^{i\theta}}=-ai=1\Leftarrow a=+i\Rightarrow w=-i\frac{-iz-2ie^{i\theta}}{z-2e^{i\theta}}=-\frac{z+2e^{i\theta}}{z-2e^{i\theta}}$$

$\arg w'(0)=\dfrac{\pi}{2}$ shartdan θ ni topamiz:

$$\arg w'(0)=\arg(-\frac{4e^{i\theta}}{(z-2e^{i\theta})^2})\Big|_{z=0}=\arg\frac{-1}{e^{i\theta}}=\pi-\theta=\frac{\pi}{2}\Rightarrow\theta=\frac{\pi}{2}\Rightarrow$$

$$\Rightarrow e^{i\theta} = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i.$$

Demak,

$$w = -\frac{z+2i}{z-2i}$$

funksiya masala shartini qanoatlantiruvchi funksiya bo‘lar ekan.

7-Misol. D sohani G sohaga akslantiruvchi va quyidagi shartlarni qanoatlantiruvchi kasr-chiziqli $w(z)$ funksiyani toping.

7.1. $D = \{\operatorname{Im} z > 0\}, \quad G = \{\operatorname{Im} w < 0\}, \quad w(i) = i, \quad \arg w'(i) = -\frac{\pi}{2}.$

7.2. $D = \{\operatorname{Im} z > 0\}, \ G = \{\operatorname{Im} w < 0\}, \ w(2i) = 2i, \ \arg w'(2i) = -\frac{\pi}{2}.$

7.3. $D = \{\operatorname{Im} z < 0\}, \quad G = \{\operatorname{Im} w > 0\}, \quad w(-i) = -i, \quad \arg w'(-i) = \frac{\pi}{2}.$

7.4. $D = \{\operatorname{Im} z < 0\}, \ G = \{\operatorname{Im} w > 0\}, \ w(-2i) = -2i, \ \arg w'(-2i) = \frac{\pi}{2}.$

7.5. $D = \{|z| < 1\}, \quad G = \{|w| < 1\}, \quad w(\frac{1}{4}) = 0, \quad \arg w'(\frac{1}{4}) = 0.$

7.6. $D = \{|z| < 1\}, \quad G = \{|w| < 1\}, \quad w(-\frac{1}{2}) = 0, \quad \arg w'(-\frac{1}{2}) = 0.$

7.7. $D = \{|z| < 1\}, \quad G = \{|w| < 1\}, \quad w(\frac{i}{4}) = 0, \quad \arg w'(\frac{i}{4}) = \frac{\pi}{2}.$

7.8. $D = \{|z| < 1\}, \quad G = \{|w| < 1\}, \quad w(-\frac{i}{2}) = 0, \quad \arg w'(-\frac{i}{2}) = \frac{\pi}{2}.$

7.9. $D = \{|z| < 2\}, \quad G = \{|w| < 4\}, \quad w(1) = 2, \quad \arg w'(1) = \frac{\pi}{2}.$

7.10. $D = \{|z| < 1\}, \quad G = \{|w| < 2\}, \quad w(\frac{i}{2}) = 0, \quad \arg w'(\frac{i}{2}) = 0.$

7.11. $D = \{|z| < 1\}, \quad G = \{|w-1| < 1\}, \quad w(0) = \frac{1}{4}, \quad \arg w'(0) = 0.$

7.12. $D = \{\operatorname{Im} z > 0\}, \quad G = \{|w| < 1\}, \quad w(i) = 0, \quad \arg w'(i) = 0.$

7.13. $D = \{\operatorname{Im} z > 0\}, \quad G = \{|w| < 2\}, \quad w(-i) = 0, \quad \arg w'(-i) = 0.$

7.14. $D=\{\operatorname{Im} z>0\},\quad G=\{|w|<1\},\ w(1+i)=0,\ \arg w'(1+i)=\dfrac{\pi}{2}.$

7.15. $D=\{\operatorname{Im} z>0\}, G=\{|w|<1\},\ w(-1+2i)=0, \arg w'(-1+2i)=\dfrac{\pi}{2}.$

7.16. $D=\{\operatorname{Im} z>0\},\quad G=\{|w+1|<1\},\quad w(i)=0,\quad \arg w'(i)=1.$

7.17. $D=\{|z-2i|<1\},\ G=\{\operatorname{Im} w>\operatorname{Re} w\},\ w(2i)=-2,\ w(i)=0.$

7.18. $D=\{\operatorname{Im} z>0\},\quad G=\{\operatorname{Im} w>0\},\quad w(i)=i,\quad \arg w'(i)=\dfrac{\pi}{2}.$

7.19. $D=\{\operatorname{Im} z>0\},\quad G=\{\operatorname{Im} w>0\},\quad w(2i)=i,\quad \arg w'(2i)=0.$

7.20. $D=\{|z|<3\},\quad G=\{\operatorname{Re} w<0\},\quad w(0)=-1,\quad \arg w'(0)=\dfrac{\pi}{2}.$

7.21. $D=\{|z|<2\},\quad G=\{\operatorname{Re} w>0\},\quad w(0)=1,\quad \arg w'(0)=\dfrac{\pi}{2}.$

7.22. $D=\{\operatorname{Re} z>0\},\quad G=\{|w-i|<1\},\quad w(i)=1,\quad \arg w'(i)=0.$

7.23. $D=\{\operatorname{Im} z>0\},\quad G=\{|w-1|<2\},\quad w(1+i)=0,\quad \arg w'(1+i)=0.$

7.24. $D=\{\operatorname{Re} z>0\},\quad G=\{|w-1-i|<1\},\ w(1-i)=0,\ \arg w'(1-i)=\dfrac{\pi}{2}.$

7.25. $D=\{\operatorname{Im} z>0\}, G=\{|w-2|<1\},\ w(-1-2i)=0, \arg w'(-1-2i)=\dfrac{\pi}{2}.$

7.26. $D=\{\operatorname{Re} z>0\},\quad G=\{|w-2i|<1\},\quad w(3i)=0,\quad \arg w'(3i)=1.$

7.27. $D=\{|z-2-i|<1\},\ G=\{\operatorname{Re} w>\operatorname{Im} w\},\ w(2+i)=-2,\ w(i-2)=0.$

7.28. $D=\{\operatorname{Re} z>0\},\quad G=\{\operatorname{Re} w>0\},\quad w(2i-1)=i,\quad \arg w'(2i+1)=\dfrac{\pi}{2}.$

7.29. $D=\{\operatorname{Re} z<0\},\quad G=\{\operatorname{Re} w>0\},\quad w(2-i)=i,\quad \arg w'(2+i)=0.$

7.30. $D=\{|z+2-i|<2\},\quad G=\{\operatorname{Re} w>0\},\quad w(0)=-1,\quad \arg w'(i)=\dfrac{\pi}{2}.$

7.31. $D=\{|z+2i|<2\},\quad G=\{\operatorname{Re} w\cdot\operatorname{Im} w>0\},\ w(i)=1,\quad \arg w'(i)=\dfrac{\pi}{2}.$

7.32. $D=\{|z-i-2|<1\},\ G=\{|w-1+2i|<1\},\quad w(i-2)=\dfrac{1}{4},\ \arg w'(i-1)=0.$

8.1-Misol. Quyidagi $D=\{|z|=2,\dfrac{\pi}{6}<\arg z<\dfrac{\pi}{3}\}$ to‘plamning $w=z^6$ funksiya yordamidagi aksini toping.

Yechish. Agar $z = re^{i\varphi}$ va $w = \rho e^{i\varphi}$ desak, unda $w = z^6$dan $\rho = r^6$ va $\psi = 6\varphi$ ekanligi kelib chiqadi. Unda $G = w(D) = \{|w| = 64, \ \pi < \arg w < 2\pi\}$ bo‘ladi.

8-Misol. Quyidagi D to‘plamning berilgan akslantirish yordamidagi aksini toping.

8.1. $D = \{\operatorname{Re} z = 2\}, \qquad w = z^2.$

8.2. $D = \{\operatorname{Im} z = 3\}, \qquad w = z^2.$

8.3. $D = \{\arg z = \frac{\pi}{3}\}, \qquad w = z^4.$

8.4. $D = \{|z| = 2, \frac{\pi}{3} < \arg z < \frac{2\pi}{3}\}, \qquad w = z^2.$

8.5. $D = \{\operatorname{Im} z > 1\}, \qquad w = z^2.$

8.6. $D = \{\operatorname{Re} z > 1\}, \qquad w = z^2.$

8.7. $D = \{|z| < 2, \frac{\pi}{2} < \arg z < \pi\}, \qquad w = z^2.$

8.8. $D = \{|z| > 2, \frac{5\pi}{4} < \arg z < \frac{3\pi}{2}\}, \qquad w = z^2.$

8.9. $D = \{\operatorname{Im} z < 0\}, \qquad w = z^2.$

8.10. $D = \{\operatorname{Re} z < -1\}, \qquad w = z^2.$

8.11. $D = \{|z| < 4, \frac{\pi}{4} < \arg z < \frac{3\pi}{4}\}, \qquad w = z^2.$

8.12. $D = \{|z| > 3, \operatorname{Re} z > 0\}, \qquad w = z^2.$

8.13. $D = \{|z| > 2, \arg z = \frac{\pi}{4}\}, \qquad w = z^3.$

8.14. $D = \{|\arg z| < \frac{\pi}{4}, z \notin [0,1]\}, \qquad w = z^4.$

8.15. $D = \{|z| = 4, \frac{\pi}{4} < \arg z < \frac{\pi}{2}\}, \qquad w = z^4.$

8.16. $D = \{|z| > 1, \pi < \arg z < \frac{3\pi}{2}\}, \qquad w = z^2.$

8.17. $D = \{\operatorname{Re} z > 0 \ , \ z \notin [1, +\infty)\}, \qquad w = z^2.$

8.18. $D = \{\operatorname{Im} z < 0, \ z \notin (-\infty, -2]\}, \qquad w = z^2.$

8.19. $D=\{|z|>2, \arg z=\dfrac{\pi}{3}\}, \quad w=z^6.$

8.20. $D=\{|z|<3, \arg z=\dfrac{\pi}{4}\}, \quad w=z^4.$

8.21. $D=\{|z|=2, \dfrac{\pi}{6}<\arg z<\dfrac{\pi}{3}\}, \quad w=z^6.$

8.22. $D=\{|z-i|<2, \quad \operatorname{Im} z>0\}, \quad w=z^3.$

8.23. $D=\{|z|>2, \ 0<\arg z<\pi\}, \quad w=z^2.$

8.24. $D=\{0<\arg z<\dfrac{\pi}{4}, \ z\notin[0,1]\}, \quad w=z^3.$

8.25. $D=\{|z-i|=4, \quad -\dfrac{\pi}{2}<\arg z<\dfrac{\pi}{3}\}, \quad w=z^2.$

8.26. $D=\{|z-i|<2, \quad \dfrac{\pi}{2}<\arg z<\dfrac{3\pi}{2}\}, \quad w=z^3.$

8.27. $D=\{\operatorname{Im} z<0<, \ \operatorname{Re} z>0 \ , z\notin[1,+\infty)\}, \quad w=z^2.$

8.28. $D=\{\operatorname{Im} z>0, \ z\notin(-\infty,-2]\}, \quad w=z^3.$

8.29. $D=\{|z|>2, \ 0<\arg z<\dfrac{\pi}{2}\}, \quad w=z^4.$

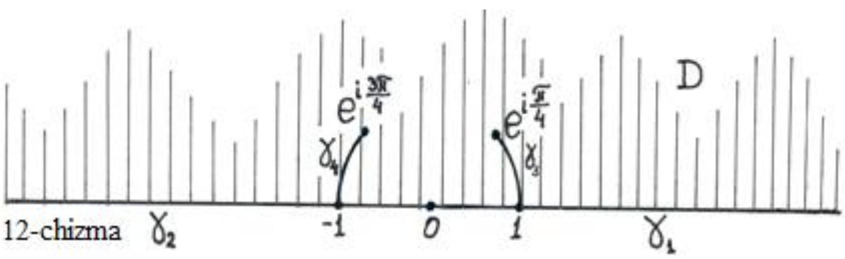
8.30. $D=\{|z-i|<2, \quad 0<\arg z<\pi\}, \quad w=z^6.$

8.31. $D=\{|z+i-2|=3, \quad -\dfrac{\pi}{2}<\arg z<\dfrac{\pi}{2}\}, \quad w=z^4.$

8.32. $D=\{|z-i+1|<4, \quad 0<\arg z<\dfrac{3\pi}{2}\}, \quad w=z^3.$

9.1- misol. Jukovskiy funksiyasidan foydalanib ushbu $D=\{\operatorname{Im} z>0, \ z\notin\{|z|=1, \ 0\le\arg z\le\dfrac{\pi}{4}, \dfrac{3\pi}{4}\le\arg z\le\pi\}$ to‘plamning aksini toping.

Yechish. Bu masalani yechish uchun birinchi navbatda D sohaning chizmasini chizib olamiz (12-chizma)



12-chizma

$$\begin{cases} u = \frac{1}{2}(r + \frac{1}{r})\cos\varphi, \\ v = \frac{1}{2}(r - \frac{1}{r})\sin\varphi. \end{cases}$$

Ma'lumki, sohaning saqlanish prinsipiga ko'ra $w(\partial D) = \partial G$ bo'ladi. Agar $\gamma_1 = [0, +\infty)$, $\gamma_2 = (-\infty, 0]$, $\gamma_3 = \{r = 1,\ 0 \le \varphi \le \frac{\pi}{4}\}$ va $\gamma_4 = \{r = 1, \frac{3\pi}{4} \le \varphi \le \pi\}$ desak, $\partial D = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ bo'ladi va $w(\gamma_1) = [1, +\infty)$, $w(\gamma_2) = (-\infty, -1]$, $w(\gamma_3) = [\frac{\sqrt{2}}{2}, 1]$, $w(\gamma_4) = [-\frac{\sqrt{2}}{2}, -1]$ ekanligini topamiz. Bundan

$$\partial G = w(\gamma_1) \cup w(\gamma_2) \cup w(\gamma_3) \cup w(\gamma_4) = (-\infty, -\frac{\sqrt{2}}{2}] \cup [\frac{\sqrt{2}}{2}, +\infty).$$

Demak, $G = w(D) = \{w \notin (-\infty, -\frac{\sqrt{2}}{2}], w \notin [\frac{\sqrt{2}}{2}, +\infty)\}$.

9-misol. Jukovskiy funksiyasidan foydalanib quyidagi to'plamlarning aksini toping.

9.1. $|z| = \frac{1}{2}$, $\quad \frac{\pi}{4} < \arg z < \frac{3\pi}{4}$.

9.2. $|z| = 2$, $\quad \frac{3\pi}{4} < \arg z < \frac{5\pi}{4}$.

9.3. $|z| > 2$, $\quad z \notin [2, +\infty)$.

9.4. $|z| < \frac{1}{2}$, $\quad z \notin [-\frac{1}{2}; 0]$.

9.5. $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$, $\quad z \notin [i, +i\infty)$.

9.6. $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$, $\quad z \notin [0, 4i]$.

9.7. $|z| < 1$, $\quad z \notin [-1; 0]$.

9.8. $|z| < 1$, $\operatorname{Im} z > 0$ $\quad z \notin [\frac{i}{2}; i]$.

9.9. $|z| < \frac{1}{2}$, $\quad 0 < \arg z < \frac{\pi}{2}]$.

9.10. $|z| < \frac{1}{2}$, $\frac{5\pi}{4} < \arg z < \frac{7\pi}{4}$.

9.11. $|z| > 2$, $0 < \arg z < \frac{\pi}{2}$.

9.12. $|z| > 2$, $\frac{5\pi}{4} < \arg z < \frac{7\pi}{4}$.

9.13. $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$.

9.14. $\operatorname{Re} z < 0$, $\operatorname{Im} z < 0$.

9.15. $|z| < \frac{1}{2}$, $\operatorname{Im} z > 0$.

9.16. $|z| < \frac{1}{2}$, $\operatorname{Im} z < 0$.

9.17. $|z| > 2$, $\operatorname{Im} z > 0$.

9.18. $|z| < 2$, $\operatorname{Im} z < 0$.

9.19. $1 < |z| < 2$, $\operatorname{Im} z > 0$.

9.20. $\frac{1}{2} < |z| < 2$, $\operatorname{Im} z > 0$ $\operatorname{Re} z > 0$.

9.21. $\operatorname{Im} z > 0,$ $z \notin \{|z| = 1,\ 0 < \arg z \le \frac{\pi}{4},\ \frac{3\pi}{4} \le \arg z \le \pi\}$.

9.22. $|z| > 1$, $z \notin [2, +\infty)$.

9.23. $|z| < \frac{1}{3}$, $z \notin [-1; 0]$.

9.24. $|z| > \frac{1}{3}$, $z \in [0; \frac{1}{2}]$.

9.25. $|z| > -1$, $z \notin [0; \frac{1}{2}]$.

9.26. $|z| < \frac{1}{2}$, $z \notin [-\frac{1}{3}; 0]$.

9.27. $|z| < \frac{2}{3}$, $z \in [\frac{1}{2}; 0]$.

9.28. $|z - 1| < \frac{1}{3}$, $z \notin [-\frac{1}{3}; 0]$.

9.29. $|z - i| < \frac{1}{3}$, $z \notin [0; \frac{1}{3}]$.

9.30. $|z+2i|<\frac{1}{2}$, $z\notin[0;\frac{1}{3}]$.

9.31. $|z-2i|<\frac{1}{3}$, $z\in[0;\frac{1}{3}]$.

9.32. $|z|>\frac{1}{3}$, $z\in[-\frac{1}{3};0]$.

8. KOMPLEKS O'ZGARUVCHILI FUNKSIYA INTEGRALI. BOSHLANG'ICH FUNKSIYA VA ANIQMAS INTEGRAL.

10.1-misol. Ushbu

$$\int_\gamma dz$$

integralni hisoblang, bunda γ chiziq boshi $a(a\epsilon C)$ nuqtada, oxiri $b(b\epsilon C)$ nuqtada bo'lgan egri chiziq.

Yechish. $f(z)\equiv 1$ funksiyaning integral yig'indisi

$$\sigma=\sum_{k=1}^{n} f(\xi_k)\,(z_k-z_{k-1})=\sum_{k=1}^{n}(z_k-z_{k-1})=$$
$$z_1-z_0-z_2-z_1+\cdots+z_n-z_{n-1}=z_n-z_0$$

bo'ladi. Agar

$$\int_\gamma dz=\lim_{\lambda\to 0}\sigma$$

va $z_0=a,\ z_n=b$ ekanligini e'tiborga olsak, unda

$$\int_\gamma dz=b-a$$

bo' lishini topamiz.

10.2-misol. Boshi $a=2+2i$ oxiri $b=i$ nuqtada bo'lgan to'g'ri chiziq kesmasi bo'yicha quyidagi

$$\int_\gamma (2z-1)dz$$

integralni ta'rif yordamida hisoblang.

Yechish. γ chiziqni a dan b ga qarab $z_0, z_1, ... z_n$ nuqtalar yordamida n ta $\gamma_1, \gamma_2, ..., \gamma_n$ yoylarga ajratamiz. $\forall\ \xi_k \in \gamma_k$ nuqta olib, quyidagi

$$\sigma = \sum_{k=1}^{n} f(\xi_k)\cdot(z_k - z_{k-1})$$

integral yig‘indini tuzamiz. Unda ta’rifga ko‘ra

$$\int_{\gamma}(2z-1)dz = \lim_{\lambda\to 0}\sigma = \lim_{\lambda\to 0}\sum_{k=1}^{n} f(\xi_k)\cdot(z_k - z_{k-1})$$

bo‘ladi. $f(z) = 2z-1 \in C(\gamma)$ bo‘lgani uchun yuqoridagi limit mavjud va bu limitning qiymati γ ning bo‘linish usuliga va ξ_k nuqtalarning tanlanishiga bog‘liq emas. Agar $\xi_k = \dfrac{z_{k-1}+z_k}{2}$ deb olsak,

$$\sigma = \sum_{k=1}^{n}(2\xi_k - 1)\cdot(z_k - z_{k-1}) = \sum_{k=1}^{n}\left[2\cdot\frac{z_k + z_{k-1}}{2} - 1\right]\cdot(z_k - z_{k-1}) =$$

$$\sum_{k=1}^{n}[(z_k + z_{k-1})\cdot(z_k - z_{k-1}) - (z_k - z_{k-1})] = \sum_{k=1}^{n}(z_k^2 - z_{k-1}^2) -$$

$$-\sum_{k=1}^{n}(z_k - z_{k-1}) = z_n^2 - z_0^2 - (z_n - z_0) = b^2 - a^2 - (b-a) =$$

$$= i^2 - (2+2i)^2 - (i - 2 - 2i) = -1 - 4\cdot 2i + i + 2 = 1 - 7i.$$

Demak,

$$\int_{\gamma}(2z-1)dz = \lim_{\lambda\to 0}\sigma = 1 - 7i.$$

10- misol. Boshi $a(a \in C)$ oxiri $b(b \in C)$ nuqtada bo‘lgan γ to‘g‘ri chiziq kesmasi bo‘yicha quyidagi integrallarni ta’rif yordamida hisoblang.

10.1. $\int_{\gamma}(3z+1)dz, \quad a = 1+i, \quad b = 1-i.$

10.2. $\int_{\gamma}(z-i)dz, \quad a = 1+i, \quad b = 1+2i.$

10.3. $\int_{\gamma}(z+i)dz, \quad a = 1+i, \quad b = i.$

10.4. $\int_{\gamma}(3z-i)dz, \quad a = 1+i, \quad b = -1-i.$

10.5. $\int\limits_{\gamma}(3z+i)dz,\quad a=2i,\quad b=1-i.$

10.6. $\int\limits_{\gamma}(z+2i)dz,\quad a=2i,\quad b=1+i.$

10.7. $\int\limits_{\gamma}(z-2i)dz,\quad a=2i,\quad b=-1-i.$

10.8. $\int\limits_{\gamma}(z-2)dz,\quad a=2,\quad b=1+i.$

10.9. $\int\limits_{\gamma}(z+2)dz,\quad a=2,\quad b=1-i.$

10.10. $\int\limits_{\gamma}(3z-1)dz,\quad a=2,\quad b=-1+i.$

10.11. $\int\limits_{\gamma}(3z-2)dz,\quad a=2,\quad b=-1-i.$

10.12. $\int\limits_{\gamma}(z+3)dz,\quad a=1+i,\quad b=i.$

10.13. $\int\limits_{\gamma}(z-3)dz,\quad a=1+i,\quad b=-i.$

10.14. $\int\limits_{\gamma}(z-3i)dz,\quad a=1-i,\quad b=i.$

10.15. $\int\limits_{\gamma}(z+3i)dz,\quad a=1-i,\quad b=-i.$

10.16. $\int\limits_{\gamma}(2z-3)dz,\quad a=-1+i,\quad b=i.$

10.17. $\int\limits_{\gamma}(2z+3)dz,\quad a=-1+i,\quad b=-i.$

10.18. $\int\limits_{\gamma}(2z-3i)dz,\quad a=-1-i,\quad b=i.$

10.19. $\int\limits_{\gamma}(2z+3i)dz,\quad a=-1-i,\quad b=-i.$

10.20. $\int\limits_{\gamma}(3z-i)dz,\quad a=2+i,\quad b=2-i.$

10.21. $\int\limits_{\gamma}(2z-1)dz,\quad a=2+2i,\quad b=i.$

10.22. $\int\limits_{\gamma}(4z+3i)dz, \quad a=1-i, \ \ b=2+i.$

10.23. $\int\limits_{\gamma}(z-4)dz, \quad a=2i, \ \ b=3+i.$

10.24. $\int\limits_{\gamma}(2z+3)dz, \quad a=i+2, \ \ b=1-i.$

10.25. $\int\limits_{\gamma}(z+i)dz, \quad a=2, \ \ b=3+i.$

10.26. $\int\limits_{\gamma}(-z+3)dz, \quad a=i-2, \ \ b=i+2.$

10.27. $\int\limits_{\gamma}(2z-5)dz, \quad a=3i, \ \ b=5i.$

10.28. $\int\limits_{\gamma}(z+2i)dz, \quad a=1+2i, \ \ b=-2-i.$

10.29. $\int\limits_{\gamma}(3z-i)dz, \quad a=3-2i, \ \ b=-i.$

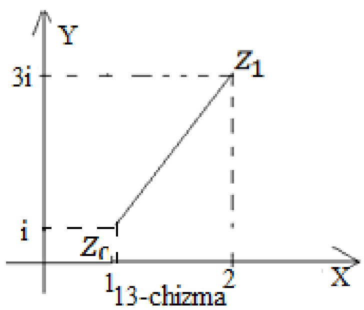
10.30. $\int\limits_{\gamma}(z+3)dz, \quad a=2-i, \ \ b=i.$

10.31. $\int\limits_{\gamma}(4z-i)dz, \quad a=3-i, \ \ b=5i.$

10.32. $\int\limits_{\gamma}(2z-5)dz, \quad a=i, \ \ b=4i.$

11.1–misol. Quyidagi $\int\limits_{\gamma}(x^2+iy^2)dz$ integralni $z_0=1+i, \quad z_1=2+3i$ nuqtalarni tutashtiruvchi γ to‘g‘ri chiziq bo‘yicha hisoblang.

Yechish. Birinchi navbatda γ to‘g‘ri chiziqning tenglamasini topamiz. $\gamma: y = 2x - 1, \ \le 1x$ ekanligini ko‘rish qiyin emas, (13-chizma).



13-chizma

Bu tenglama, $z=x+iy$, $dz=dx+idy$ va γ da $dy=2dx$ ekanligidan foydalanib, berilgan integralni hisoblaymiz:

$$\int_\gamma (x^2+iy^2)dz=\int_\gamma (x^2+iy^2)\cdot(dx+idy)=\int_\gamma (x^2dx-y^2dy)+$$

$$+i\int_\gamma (x^2dy+y^2dx)=\int_1^2[x^2-(2x-1)^2\cdot 2]dx+i\int_1^2[x^2\cdot 2+(2x-1)^2]dx=$$

$$=\int_1^2(-7x^2+8x-2)dx+i\int_1^2(6x^2-4x+1)dx=-\frac{19}{3}+9i.$$

11.1–misol. Ushbu

$$\mathrm{I}_n=\int_\gamma (z-a)^n\ dz \quad (n\text{-butun son})$$

integralni hisoblang, bunda $\gamma=\{z\epsilon C: |z-a|=\rho, \rho>0\}$ aylanadan iborat (yo'nalish soat strelkasiga qarama-qarshi olingan.

Yechish. γ aylananing tenglamasini quyidagi $z=z(t)=a+\rho e^{it},\ 0\le t\le 2\pi)$ ko'rinishida yozib olamiz. Unda

$$dz=d(a+\ \rho eit)=i\,\rho e^{it}dt$$

bo'lib,

$$\mathrm{I}_n=\int_\gamma (z-a)^n\,dz\ =i\rho^{n+1}\int_0^{2\pi} e^{it(n+1)}dt$$

bo'ladi. Agar $n\ne -1$ bo'lsa,

$$\mathrm{I}_n=i\rho^{n+1}\int_0^{2\pi} e^{it(n+1)}dt=i\rho n+1\frac{e^{it(n+1)}}{i(n+1)}\ |_0^{2\pi}=0$$

bo'ladi.

Agar $n=-1$ bo'lsa,

$$I-1=i\int_0^{2\pi} e^{it*0}\,dt=2\pi i$$

bo'ladi. Demak

$$\int_\gamma (z-a)^n\,dz=\int_{|z-a|=\rho}(z-a)^n dz=\begin{cases}0, & \text{agar } n\ne -1 \text{ bo'lsa}\\ 2\pi i, & \text{agar } n=-1 \text{bo'lsa}\end{cases}$$

11-misol. Quyidagi integrallarni berilgan z_0 va z_1 nuqtalarni tutashtiruvchi γ to'g'ri chiziq bo'yicha hisoblang.

11.1. $\int_\gamma (x+iy^2)dz,\quad z_0=1+i,\ \ z_1=2+3i.$

11.2. $\int\limits_{\gamma}(x^2+iy^2)dz,\quad z_0=2+2i,\ z_1=3+4i.$

11.3. $\int\limits_{\gamma}(x^2+iy)dz,\quad z_0=1+i,\ z_1=2+3i.$

11.4. $\int\limits_{\gamma}(x+iy^2)dz,\quad z_0=2+2i,\ z_1=3+4i.$

11.5. $\int\limits_{\gamma}(x^2-iy^2)dz,\quad z_0=1+i,\ z_1=2+3i.$

11.6. $\int\limits_{\gamma}(x^2+iy^2)dz,\quad z_0=2+2i,\ z_1=3+4i.$

11.7. $\int\limits_{\gamma}zdz,\quad z_0=1+i,\ z_1=2+3i.$

11.8. $\int\limits_{\gamma}(x^2-iy^2)dz,\quad z_0=2+2i,\ z_1=3+4i.$

11.9. $\int\limits_{\gamma}\overline{z}dz,\quad z_0=1+i,\ z_1=2+3i.$

11.10. $\int\limits_{\gamma}\overline{z}dz,\quad z_0=2+2i,\ z_1=3+4i.$

11.11. $\int\limits_{\gamma}(x^2+iy^2)dz,\quad z_0=1+i,\ z_1=3+2i.$

11.12. $\int\limits_{\gamma}(x^2+iy^2)dz,\quad z_0=2+2i,\ z_1=4+3i.$

11.13. $\int\limits_{\gamma}(x+iy^2)dz,\quad z_0=1+i,\ z_1=3+2i.$

11.14. $\int\limits_{\gamma}(x^2-iy^2)dz,\quad z_0=2+2i,\ z_1=4+3i.$

11.15. $\int\limits_{\gamma}(x^2+iy)dz,\quad z_0=1+i,\ z_1=3+2i.$

11.16. $\int\limits_{\gamma}(x+iy^2)dz,\quad z_0=1+2i,\ z_1=3+4i.$

11.17. $\int\limits_{\gamma}(x^2-iy^2)dz,\quad z_0=1+i,\ z_1=3+2i.$

11.18. $\int\limits_{\gamma}(x^2+iy)dz,\quad z_0=1+2i,\ z_1=3+4i.$

11.19. $\int_{\gamma} zdz, \quad z_0 = 1+i, \; z_1 = 3+2i.$

11.20. $\int_{\gamma} (x^2 - iy)dz, \quad z_0 = 1+2i, \; z_1 = 3+4i.$

11.21. $\int_{\gamma} (x^2 + iy^2)dz, \quad z_0 = 1+i, \; z_1 = 2+3i.$

11.22. $\int_{\gamma} (x^2 + iy^2)dz, \; z_0 = 1+i, \; z = 2+3i.$

11.23. $\int_{\gamma} (x - iy)^2(1-\mathrm{i})dz, \; z_0 = 1-i, \; z = 2i.$

11.24. $\int_{\gamma} (ix^2y^2 - 2)dz, \; z_0 = 3+i, \; z = 1-2i.$

11.25. $\int_{\gamma} (x + i(y-2x))dz, \; z_0 = 2i, \; z = 4+3i.$

11.26. $\int_{\gamma} (x^2 + ixy)dz, \; z_0 = 1-i, \; z = 1-3i.$

11.27. $\int_{\gamma} z\sin z dz, \; z_0 = 0, \; z_1 = i.$

11.28. $\int_{\gamma} (x + 2ixy^2)dz, \; z_0 = -1+i, \; z = 2.$

11.29. $\int_{\gamma} z^2\, dz, \quad z_0 = 1, \; z_1 = i.$

11.30. $\int_{\gamma} z^2\, dz, \quad z_0 = -i,, \; z_1 = 1+i.$

11.31. $\int_{\gamma} z^2\, dz, \quad z_0 = i-2, \; z_1 = i+2.$

11.32. $\int_{\gamma} z^2\, dz, \quad z_0 = i-1, \; z_1 = 2i-1.$

12.1- Misol. Agar γ egri chiziq yuzasi S ga teng bo’lgan sohani chegaralovchi yopiq chiziq bo’lsa, u holda

$$\frac{1}{i}\oint_{\gamma} x\, dz = S$$

tenglikning o’rinli bo’lishini isbotlang.

Yechish. Bundan buyon $\oint -$ belgi γ yopiq kontur bo'yicha olingan integralni bildiradi.

$f(z) = u(x,y) + iv(x,y) = x$ uchun $u(x,y) = x$ $v(x,y) = 0$ bo'ladi.

$$\oint_\gamma xdz = \oint_\gamma xdx + i\oint_\gamma xdy.$$

Bu tenglikning o'ng tomonidagi har bir egri chiziqli integralga Grin formulasini qo'llasak, natijada

$$\oint_\gamma xdz = \iint_{(S)} 0dxdy + i\iint_{(S)} dxdy = i\iint_{(S)} dxdy = iS$$

bo'lishi kelib chiqadi.

12.2- Misol. Ushbu $\int_\gamma x\,dz$ integralni hisoblang, bunda γ egri chiziq {z∊C: |z|=1, 0≤ $\arg z \le \pi\}$ dan iborat (chiziqning boshi $z = 1$ nuqtada).

Yechish. γ egri chiziqni quyidagicha $z = e^{it}$, $0 \le t \le \pi$ parametrik ko'rinishda yozib olamiz. U holda

$$\int_\gamma xdz = \int_0^\pi \cos t\, d(e^{it})$$

bo'ladi. Bu tenglikning o'ng tomonidagi aniq integralni hisoblaymiz:

$$\int_0^\pi \cos t\, d(e^{it}) = \int_0^\pi \cos t\, d(cost + isint) = \int_0^\pi cost\ d(cost) +$$

$$+\mathrm{i}\int_0^\pi cost\ d(sint) \ = \frac{(cost)^2}{2}|_0^\pi + \mathrm{i}[\mathrm{cost\,sint}|_0^\pi -$$

$$-\int_0^\pi (-(sint)^2 dt)] = i\int_0^\pi \sin^2 t\, dt = i\int_0^\pi \frac{(1-cos2t)}{2}dt =$$

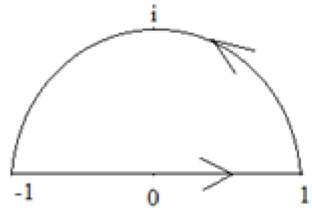
$$= \frac{i}{2}\left(t - \frac{1}{2}sin2t\right)|_0^\pi = \frac{i\pi}{2}$$

Demak,

$$\int_\gamma x dz = \frac{i\pi}{2} \, .$$

12.3- Misol. Ushbu

$$\oint_\gamma |z|\bar{z}\, dz$$



14 *chizma*

integralni hisoblang, bunda γ egri chiziq $\{z\epsilon C: |z| = 1, Imz > 0\}$ yuqori yarim aylana hamda $[-1,1]$ kesmadan iborat bo’lgan yopiq chiziq (14 chizma).

Yechish. γ_1 deb $\{z = x + iy\epsilon C: -1 \leq x \leq 1,\ y = 0\}$ ni, γ_2 deb $\{z\epsilon C: |z| = 1,\ Imz < 0\}$ ni belgilash olsak, unda $\gamma = \gamma_1 \cup \gamma_2$ bo’lib, integralash xossasiga ko’ra

$$\oint_\gamma |z|\bar{z}\, dz = \oint_{\gamma_1} |z|\bar{z}\, dz + \oint_{\gamma_2} |z|\bar{z}\, dz$$

bo’ladi. Bu tenglikning o’ng tomonidagi integralni alohida-alohida hisoblaymiz:

$$\oint_{\gamma_1} |z|\bar{z}\, dz = \int_{-1}^{1} x|x| dx = \int_{-1}^{0} x(-x) dx + \int_0^1 x^2 dx = 0.$$

Ikkinchi

$$\oint_{\gamma_2} |z|\bar{z}\, dz$$

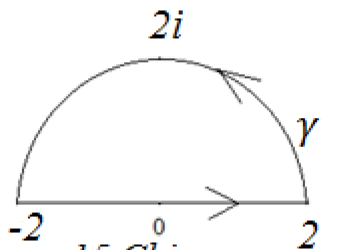
integralni hisoblash uchun $z = e^{it},\ (0 \leq t \leq \pi)$ deymiz. Unda

$$\oint_{\gamma_2} |z|\bar{z}\, dz = \int_0^\pi |e^{it}|\, e^{-it} \mathrm{d}(e^{it}) = \int_0^\pi 1 \cdot e^{-it} \cdot ie^{it}\, dt = i \int_0^\pi dt = i\pi$$

bo’ladi. Demak

$$\oint_\gamma |z|\, \bar{z}\, dz = i\pi.$$

12-Misol. Chizmada tasvirlangan γ chiziq bo‘yicha olingan quyidagi integrallarni hisoblang (15 chizma).



15 Chizma

12.1. $\oint\limits_{\gamma} \frac{\bar{z}}{z} dz.$

12.2. $\oint\limits_{\gamma} \frac{2z - \bar{z}}{\bar{z}} dz.$

12.3. $\oint\limits_{\gamma} \frac{2\bar{z} + z}{z} dz.$

12.4. $\oint\limits_{\gamma} \frac{3z - \bar{z}}{\bar{z}} dz.$

12.5. $\oint\limits_{\gamma} \frac{3\bar{z} + z}{z} dz.$

12.6. $\oint\limits_{\gamma} \frac{3\bar{z} - 2z}{z} dz.$

12.7. $\oint\limits_{\gamma} \frac{2z - 3\bar{z}}{\bar{z}} dz.$

12.8. $\oint\limits_{\gamma} \frac{3\bar{z} - z}{z} dz.$

12.9. $\oint\limits_{\gamma} \frac{2\bar{z} + 3z}{z} dz.$

12.10. $\oint\limits_{\gamma} \frac{5z - 6\bar{z}}{\bar{z}} dz.$

12.11. $\oint\limits_{\gamma} \frac{6\bar{z} - 5z}{z} dz.$

12.12. $\oint\limits_{\gamma} \frac{4z - 5\bar{z}}{\bar{z}} dz.$

12.13. $\oint\limits_{\gamma} \frac{4\bar{z} + 5z}{z} dz.$

12.14. $\oint\limits_{\gamma} \frac{5z - 4\bar{z}}{\bar{z}} dz.$

12.15. $\oint\limits_{\gamma} \frac{5\bar{z} + 4}{z} dz.$

12.16. $\oint\limits_{\gamma} \frac{7z - 8\bar{z}}{\bar{z}} dz.$

12.17. $\oint\limits_{\gamma} \frac{7\bar{z} + 8z}{z} dz.$

12.18. $\oint\limits_{\gamma} \frac{8z - 5\bar{z}}{\bar{z}} dz.$

12.19. $\oint\limits_{\gamma} \frac{6\bar{z} - 7z}{z} dz.$

12.20. $\oint\limits_{\gamma} \frac{6z + 7\bar{z}}{\bar{z}} dz.$

12.21. $\oint\limits_{\gamma} \frac{7z + 6\bar{z}}{\bar{z}} dz.$

12.22. $\oint\limits_{|\gamma} \frac{e^z dz}{z^2 - 1}.$

12.23. $\oint\limits_{|z|=2} \frac{\sin z^2 dz}{z^2 (z+5)^2}.$

12.24. $\oint\limits_{|z|=2} \frac{chz dz}{(z-1)^4}.$

12.25. $\oint\limits_{|z|=1} \frac{z \sin z dz}{1 - e^z}.$

12.26. $\oint\limits_{|z|=1} \frac{\mathrm{ctg} z dz}{z}.$

12.27. $\oint\limits_{\gamma} \frac{\bar{z}-5z}{z^2}dz.$ **12.30.** $\oint\limits_{\gamma} \frac{3\bar{z}-4z}{z}dz.$

12.28. $\oint\limits_{\gamma} \frac{6z-3\bar{z}}{z}dz.$ **12.31.** $\oint\limits_{\gamma} \frac{3+4\bar{z}}{\bar{z}}dz.$

12.29. $\oint\limits_{\gamma} \frac{2z+5\bar{z}}{\bar{z}}dz.$ **12.32.** $\oint\limits_{\gamma} \frac{6\bar{z}-7}{z}dz.$

13.1-Misol. Agar $f(z)$ funksiya $z = 0$ nuqtaning biror atrofida uzluksiz bo'lsa, u holda

$$\lim_{r\to 0}\int_{\gamma_r} \frac{f(z)}{z}dz = 2\pi i f\ (0)$$

tenglikning o'rinli bo'lishini isbotlang. Bu yerda $\gamma_r := \{z\epsilon C: |z| = r\}$ aylana.

Yechish. $f(z)$ funksiya z=0 nuqtada uzluksiz. Ta'rifga binoan $\forall\varepsilon > 0$ olinganda ham shunday $\delta > 0$ son topiladiki, $|z| < \delta$ tengsizlikni qanoatlantiruvchi barcha $z \in C$ lar uchun

$$|f(z) - f(0)| < \frac{\varepsilon}{2\pi}$$

tengsizlik o'rinli bo'ladi. Haqiqatdan ham, $r < \delta$ tengsizlikni qanoatlantiruvchi barcha r lar uchun

$$|f(re^{i\varphi}) - f(0)| < \frac{\varepsilon}{2\pi} \qquad (0 \le \varphi \le 2\pi)$$

tengsizlik bajariladi. γ_r yopiq chiziqni z=r$e^{i\varphi}$, $0 \le \varphi \le 2\pi$ shaklida ifodalasak,

$$\int\limits_{\gamma_r} \frac{f(z)}{z}dz = i\int_0^{2\pi} f\big(re^{i\varphi}\big)d\varphi$$

bo'lib, bundan

$$\left|\int\limits_{\gamma_r} \frac{f(z)}{z}dz - 2\pi i f(0)\right| = \left|i\int_0^{2\pi} f\big(re^{i\varphi}\big)d\varphi - 2\pi i f(0)\right| =$$

$$= \left|\int_0^{2\pi} f\big(re^{i\varphi}\big)d\varphi - \int_0^{2\pi} f(0)d\varphi\right| \le$$

$$\leq \left|\int_0^{2\pi}\left[f\left(re^{i\varphi}\right)-f(0)\right]d\varphi\right| \leq \int_0^{2\pi}\left|f\left(re^{i\varphi}\right)-f(0)\right| d\varphi.$$

$|f(re^{i\varphi})-f(0)| < \frac{\varepsilon}{2\pi}$ tengsizlikga ko'ra oxirgi integral ε dan katta emas. Demak, $r < \delta$ lar uchun

$$\left|\int_{\gamma_r}\frac{f(z)}{z}dz - 2\pi i f(0)\right| < \varepsilon$$

bo'lib, bu

$$\lim_{r\to 0}\int_{\gamma_r}\frac{f(z)}{z}dz = 2\pi i f(0)$$

bo'lishini ko'rsatadi.

13-Misol. Agar $\gamma: x = a\cos t,\ y = b\sin t,\ 0 < t \leq 2\pi,$ ellips bo'lsa, quyidagi integrallar hisoblansin.

13.1. $\int\limits_\gamma y dz, \quad a=2, \ b=3.$

13.2. $\int\limits_\gamma z dz, \quad a=2, \ b=3.$

13.3. $\int\limits_\gamma \bar{z} dz, \quad a=2, \ b=3.$

13.4. $\int\limits_\gamma (2x-iy) dz, \quad a=2, \ b=3.$

13.5. $\int\limits_\gamma (x-iy) dz, \quad a=2, \ b=3.$

13.6. $\int\limits_\gamma x^2 dz, \quad a=3, \ b=2.$

13.7. $\int\limits_\gamma y^2 dz, \quad a=3, \ b=2.$

13.8. $\int\limits_\gamma (x^2-iy) dz, \quad a=3, \ b=2.$

13.9. $\int\limits_\gamma (x-iy^2) dz, \quad a=3, \ b=2.$

13.10. $\int_{\gamma}(x+i\cdot 2y)dz,\ a=3, b=2.$

13.11. $\int_{\gamma}(2x+iy)dz,\quad a=3,\quad b=2.$

13.12. $\int_{\gamma}(x-i\cdot 2y)dz,\ a=2,\ b=3.$

13.13. $\int_{\gamma}(3x-iy)dz,\quad a=3,\quad b=2.$

13.14. $\int_{\gamma}(x-i3y)dz,\quad a=3,\ b=2.$

13.15. $\int_{\gamma}(3x+iy)dz,\quad a=2,\quad b=3.$

13.16. $\int_{\gamma}(x+i3y)dz,\quad a=2,\ b=3.$

13.17. $\int_{\gamma}(3x-2iy)dz,\ a=3, b=2.$

13.18. $\int_{\gamma}(2x-3iy)dz,\quad a=3,\ b=2.$

13.19. $\int_{\gamma}(3x+2iy)dz,\ a=2,b=3.$

13.20. $\int_{\gamma}(2x+3iy)dz,\ a=2, b=3.$

13.21. $\int_{\gamma}(4x+3iy)dz, a=3, b=2.$

13.22. $\int_{\gamma}(\mathrm{x}+\mathrm{yi})^2 dz,\quad a=-2,\quad b=2$

13.23. $\int_{\gamma}(2x-i3y)dz,\quad a=1,\quad b=2.$

13.24. $\int_{\gamma}(4x-i3y)dz,\quad a=3,\quad b=5.$

13.25. $\int_{\gamma}(x-iy)dz,\quad a=2,\quad b=3.$

13.26. $\int_{\gamma}(2x-iy)dz,\quad a=2,\quad b=3.$

13.27. $\int\limits_{\gamma}(2x-iy)dz,\quad a=2,\ \ b=3.$

13.28. $\int\limits_{\gamma}(2x-iy)dz,\quad a=2,\ \ b=3.$

13.29. $\int\limits_{\gamma}(2x-iy)dz,\quad a=2,\ \ b=3.$

13.30. $\int\limits_{\gamma}(2x-iy)dz,\quad a=2,\ \ b=3.$

13.31. $\int\limits_{\gamma}(2x-iy)dz,\quad a=2,\ \ b=3.$

13.32. $\int\limits_{\gamma}(2x-iy)dz,\quad a=2,\ \ b=3.$

14.1- misol. Quyidagi $\int\limits_0^{\pi}\frac{\cos^2 x}{2-\sin^2 x}dx$ aniq integralni hisoblang.

Yechish. Bu integralda $e^{2ix}=z$ almashtirishni bajarsak, $x\in[0,\pi]\Rightarrow z\in\{z\in C:|z|=1\}$,

$$dx=\frac{1}{2iz}dz,$$

$$\cos^2 x=\frac{1+\cos 2x}{2}=\frac{1+\frac{1}{2}(z+\frac{1}{z})}{2},$$

$$\sin^2 x=\frac{1-\cos 2x}{2}=\frac{1-\frac{1}{2}(z+\frac{1}{z})}{2}$$

bo‘lib,

$$\int\limits_0^{\pi}\frac{\cos^2 x dx}{2-\sin^2 x}=\frac{1}{2i}\oint\limits_{|z|=1}\frac{1}{z}\cdot\frac{\frac{1+\frac{1}{2}(z+\frac{1}{z})}{2}}{2-\frac{1-\frac{1}{2}(z+\frac{1}{z})}{2}}dz=$$

$$=\frac{1}{2i}\oint\limits_{|z|=1}\frac{1}{z}\cdot\frac{(z+1)^2}{z^2+6z+1}dz$$

tenglik o‘rinlidir.

Integral ostidagi

$$f(z) = \frac{(z+1)^2}{z(z^2+6z+1)} = \frac{(z+1)^2}{z\cdot[z-(-3+2\sqrt{2})]\cdot[z-(-3-2\sqrt{2})]}$$

funksiyaning $z_0 = 0$, $z_1 = -3+2\sqrt{2}$, $z_2 = -3-2\sqrt{2}$ maxsus nuqtalari bo‘lib, ulardan $z_0 = 0$ *va* $z_1 = -3+2\sqrt{2}$ lar $\{|z|<1\}$ sohaga tegishli bo‘lgan qutb nuqtalaridir.

Koshi teoremasini qo‘llab, topamiz:

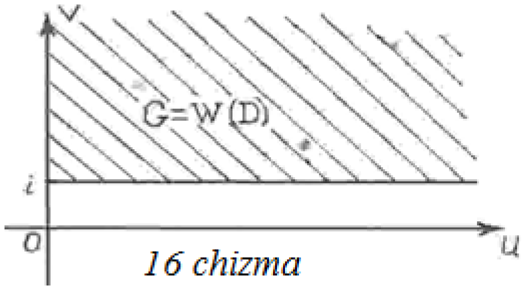
$$\oint\limits_{|z|=1} f(z)dz = 2\pi i[f(z_0)+f(z_1)] = 2\pi i[\frac{1}{z_1 z_2}+$$

$$+\frac{1}{z_1}\cdot\frac{(z_1+1)^2}{z_1-z_2}] = 2\pi i[1+\frac{1}{-3+2\sqrt{2}}\cdot\frac{(-3+2\sqrt{2}+1)^2}{4\sqrt{2}}] =$$

$$= 2\pi i(1-\frac{1}{\sqrt{2}}).$$

Demak,

$$\int_0^{\pi}\frac{\cos^2 x}{2-\sin^2 x}dx = \pi\cdot(1-\frac{1}{\sqrt{2}}).$$



16 chizma

14 -Misol. Quyidagi integrallarni hisoblang.

14.1. $\int_{-3}^{-3+i} zdz$.

14.2. $\int_{-3}^{-3+i} (e^z+1)dz$.

14.3. $\int_{1}^{1+i} zdz$.

14.4. $\int_{3i}^{1+3i} zdz$.

14.5. $\int_{-3}^{-3+i} (e^z-1)dz$.

14.6. $\int_{-2i}^{1-2i} zdz$.

14.7. $\int_{3i}^{1+3i} z\cos zdz$.

14.8. $\int_{3i}^{1-3i} (1+z^2)z\cos zdz$.

14.9. $\int_{2}^{2+i} z^2dz$.

14.10. $\int_{-\pi}^{-\pi+i} tg2zdz$.

14.11. $\int_{-4-i}^{-2+i} (z^3-8)dz$

14.12. $\int_{-2}^{-2+3i} zdz$.

14.13. $\int\limits_{-2i}^{1-2i} zctgzdz$.

14.14. $\int\limits_{-3i}^{2-i} z(z^3+1)\,\mathrm{d}\,z$.

14.15. $\int\limits_{-2+i}^{1+i} zdz$.

14.16. $\int\limits_{-2+i}^{1+i} zdz$.

14.17. $\int\limits_{-5\pi i}^{-\pi i} (tgz + ctgz)dz$.

14.18. $\int\limits_{-5\pi i}^{-\pi i} z^2 \cdot ctgzdz.$

14.19. $\int\limits_{-3-i}^{4-i} zdz$.

14.20. $\int\limits_{-5\pi i}^{-\pi i} z^2 \cdot tgzdz \cdot$

14.21. $\int\limits_{-2+i}^{1+i} z^2dz$.

14.22. $\int\limits_{3}^{3+i} zdz$.

14.23. $\int\limits_{2}^{2+i} zdz.$

14.24. $\int\limits_{-1+2i}^{2+2i} z^2dz.$

14.25. $\int\limits_{3-2i}^{3+i} zdz$.

14.26. $\int\limits_{-3-i}^{4-i} z^2dz.$

14.27. $\int\limits_{i}^{2+i} z^2dz.$

14.28. $\int\limits_{-2}^{1-2i} z^2dz.$

14.29. $\int\limits_{1+2i}^{2+i} zdz.$

14.30. $\int\limits_{1+i}^{2+i} z^2dz$.

14.31. $\int\limits_{1+i}^{2+i} z^2dz$.

14.32. $\int\limits_{1+i}^{2+i} z^2dz$.

9. KOSHINING INTEGRAL FORMULASI. KOSHI TURIDAGI INTEGRAL

15.1- misol. Ushbu

$$\int\limits_{\gamma} \frac{z^2}{z-2i}dz$$

integralni hisoblang, bunda $\gamma = \{z\epsilon C: |z| = 1\}$.

Yechish. Agar D $(D \subset C)$ soha deb quyidagi $D = \left\{z\epsilon C: \ |z| < \frac{3}{2}\right\}$ soha olinsa, unda birinchidan $\mathrm{f}(z) = \frac{z^2}{z-2i}$ funksiya golomorf bo'ladi, ikkinchidan qaralayotgan γ yopiq chiziq shu sohaga tegishli bo'ladi: $\gamma \subset D, \ 2i \notin D$. U holda Koshining integral teoremasiga ko'ra

$$\int_{\gamma} f(z)\, dz = \int_{\gamma} \frac{z^2}{z-2i} dz = 0$$

bo'ladi.

15.2-misol. 5 $I = \oint\limits_{|z|=4} \frac{e^{iz} dz}{z-\pi}$ integralni hisoblang

Yechish. Avval integral ostidagi funksiyaning maxsus nuqtalarini topamiz, ya'ni $z-\pi=0, \quad z=\pi.$ Bu nuqta $|z|=4$ aylana ichidagi sohada yotadi. Demak $z=\pi$ maxsus nuqta. Koshining $2\pi i f\left(z_0\right) = \oint\limits_{|z|=\rho} \frac{f\left(z\right)dz}{z-z_0}$ integral farmulasiga ko'ra

$$\oint\limits_{|z|=4} \frac{e^{iz} dz}{z-\pi} = 2\pi i f\left(z_0\right) = 2\pi i e^{iz_0} = 2\pi i e^{i\pi} = -2\pi i.$$

15.3-Misol. 6 $I = \oint\limits_{|z|=1} \frac{z^2 dz}{\sin z \cos z}$ integralni hisoblang.

Yechish. Integral ostidagi funksiyaning maxsus nuqtalarini aniqlaymiz:

$$\sin z \cdot \cos z = 0, \quad z = \pi k, \quad z = \frac{\pi}{2} + \pi k, \quad k \in \mathrm{Z}.$$

Bundan faqat $z=0$ nuqta $|z|=1$ bilan chegaralangan sohada joylashgan. Demak, $z=0$ maxsus nuqta. Koshining $2\pi i f\left(z_0\right) = \oint\limits_{|z|=\rho} \frac{f\left(z\right)dz}{z-z_0}$ integral formulasiga ko'ra

$$I = \oint\limits_{|z|=1} \frac{z^2 dz}{\sin z \cos z} = \oint\limits_{|z|=1} \frac{\frac{z^2}{\cos z} dz}{\sin z} = 2\pi i f\left(0\right) = 2\pi i \frac{0}{\cos 0} = 0.$$

15.4-Misol. 7 Ushbu

$$\int\limits_{\gamma} \frac{dz}{z^2+9}$$

integralni hisoblang, bunda γ egri chiziq C tekislikning $\pm 3i$ nuqtalaridan o'tmaydigan ixtiyoriy yopiq chiziq.

Yechish. γ yopiq chiziq bilan chegaralangan to'plam D bo'lsin.

a) $\pm 3i$ nuqtalar D sohaga tegishli bo'lmasin :$\pm 3i \,\overline{\in}\, D$. U holda $\varphi(z) = \frac{1}{z^2+9} \in 0(\overline{D}$bo'lib, Koshi teoremasiga ko'ra

$$\oint\limits_{\gamma} \frac{dz}{z^2+9} = 0$$

bo'ladi.

b) +3i$\in D, -3i \,\overline{\in}\, \overline{D}$ bo'lsin. U holda, avvalo integral ostidagi funksiyani

$$\frac{1}{z^2+9} = \frac{1}{(z+3i)(z-3i)} = \frac{\frac{1}{z+3i}}{z-3i}$$

ko'rinishida yozib olamiz. Unda $f(z) = \frac{1}{z+3i}$, a = 3i lar uchun Koshi teoremasiga asosan

$$\oint\limits_{\gamma} \frac{dz}{z^2+9} = \oint\limits_{\gamma} \frac{\frac{1}{z+3i}dz}{z-3i} = \oint\limits_{\gamma} \frac{f(z)dz}{z-3i} = 2\pi i f(3i) = \frac{2\pi i}{3i+3i} = \frac{\pi}{3}$$

bo'ladi.

c) -3$i \in D, 3i \,\overline{\in}\, \overline{D}$ bo'lsin. Bunda, yuqoridagi b) holdagiga o'xshash mulohaza yuritish bilan topamiz:

$$\oint\limits_{\gamma} \frac{dz}{z^2+9} = \oint\limits_{\gamma} \frac{\frac{1}{z-3i}dz}{z+3i} = \oint\limits_{\gamma} \frac{f(z)dz}{z+3i} = 2\pi i f(-3i) = \frac{2\pi i}{-3i-3i} = -\frac{\pi}{3}.$$

d) $3i \in D,\ -3i \in D$ bo'lsin. Bu holda, avvalo integral ostidagi funksiyani sodda kasrlarga ajratamiz:

$$\frac{1}{z^2+9} = \frac{1}{(z+3i)(z-3i)} = \frac{1}{6i}\left(\frac{1}{(z-3i)} - \frac{1}{(z+i)}\right).$$

U holda

$$\oint_\gamma \frac{dz}{z^2+9} = \oint_\gamma \frac{dz}{(z+3i)(z-3i)} = \frac{1}{6i}\left[\oint_\gamma \frac{dz}{z-3i} - \oint_\gamma \frac{dz}{z+3i}\right] =$$

$$\frac{1}{6i} 2\pi i(1-1) = 0$$

bo'lishini topamiz.

15- misol. Koshining integral formulasidan foydalanib quyidagi integrallarni hisoblang.

15.1. $\int\limits_{|z-1|=3} \frac{e^z dz}{(z-1)(z+3)(z+i)}$.

15.2. $\int\limits_{|z+1|=3} \frac{e^z dz}{(z-3)(z+3)(z+i)}$.

15.3. $\int\limits_{|z-1|=2} \frac{\sin z}{(z^2+1)(z-2i)} dz$.

15.4. $\int\limits_{|z-1|=2} \frac{e^z}{(z-1)(z-2)(z+2i)} dz$.

15.5. $\int\limits_{|z|=2,5} \frac{\sin z}{(z-3i)(z^2-5z+6)} dz$.

15.6. $\int\limits_{|z-1|=2} \frac{e^z}{(z+i)(z+2)(z+2i)} dz$

15.7. $\int\limits_{|z|=2} \frac{\cos z}{(z-i)(z+1)(z+3)} dz$.

15.8. $\int\limits_{|z|=3} \frac{\sin z}{(z-2)(z+i)(z+4i)} dz$.

15.9. $\int\limits_{|z|=2,5} \frac{e^z dz}{(z-3i)(z^2+3z+1)}$.

15.10. $\int\limits_{|z-i|=2} \frac{e^z}{z(z-2i)(z+2i)} dz$.

15.11. $\int\limits_{|z-i|=2} \frac{\sin z}{(z+i)(z-2i)(z+3)}dz$.

15.12. $\int\limits_{|z-i|=2} \frac{\cos z}{z(z+i)(z+2i)}dz$.

15.13. $\int\limits_{|z-i|=2} \frac{e^z}{(z-1)(z-2i)(z+2i)}dz$.

15.14. $\int\limits_{|z-i|=2} \frac{e^z}{z(z+1)(z+2)}dz$.

15.15. $\int\limits_{|z-i|=2} \frac{\sin z}{(z-1)(z-2i)(z+3i)}dz$.

15.16. $\int\limits_{|z-i|=2} \frac{\cos z}{(z+1)(z-i)(z-2)}dz$.

15.17. $\int\limits_{|z|=3} \frac{e^z}{(z-4i)(z-2i)(z-i)}dz$.

15.18. $\int\limits_{|z|=3} \frac{e^z}{(z^2+4)(z-5i)}dz$.

15.19. $\int\limits_{|z|=3} \frac{\sin z}{(z^2-4)(z+4)}dz$.

15.20. $\int\limits_{|z|=3} \frac{\cos z}{(z-2)(z+2i)(z+4i)}dz$.

15.21. $\oint\limits_{|z|=\frac{1}{2}} \frac{\cos\frac{\pi}{z+1}}{z^2}dz.$

15.22. $\oint\limits_{|z|=2} \frac{ze^z dz}{(z-1)^2}$.

15.23. $\oint\limits_{|z-i|=1} \frac{z\cos z dz}{(z-1)^3}$

15.24. $\oint\limits_{|z+1|=2} \frac{(z-1)e^z dz}{(z^3-1)^2}.$

15.25. $\oint\limits_{|z|=2} \frac{\sin z dz}{z^2+1}.$

15.26. $\oint\limits_{|z-1-i|=1} \frac{\sin(\pi z-\pi) dz}{z^2-2z+2}.$

15.27. $\oint\limits_{|z|=4} \frac{\cos z dz}{z^2-\pi^2}.$

15.28. $\oint\limits_{|z|=\frac{1}{2}} \frac{e^z dz}{z^4+z^2}.$

15.29. $\oint\limits_{|z|=2} \frac{\sin z \sin(z-1)}{z^2-z} dz.$

15.30. $\oint\limits_{|z|=2} \frac{\sin z^2 dz}{z^2(z+5)^2}.$

15.31. $\oint\limits_{|z-i|=1} \frac{\sin 2(z-1) dz}{(z-i)^2(z+6)}.$

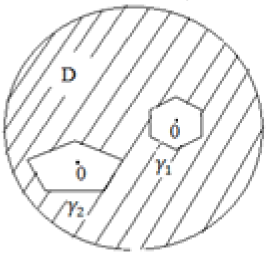
15.32. $\oint\limits_{|z+i|=1} \frac{\cos(2z+1) dz}{(z+i)^3(z-1)}.$

16.1 - misol. Ushbu

$$\oint\limits_{|z|=2} \frac{z+1}{z(z-1)^2(z-3)} dz$$

integralni hisoblang.

Yechish. $z_0 = 0$, $z_1 = 1$ nuqtalar $\{z \in C: |z| = 2\}$ aylana bilan chegaralangan $\{z \in C: |z| < 2\}$ doiraga tegishli bo'lib, $z_2 = 3$ nuqta esa shu doiraga tegishli emas. $z_0 = 0$ va $z_1 = 1$ nuqtalarni $\{z \in C: |z| < 2\}$ doiraga tegishli va o'zaro kesishmaydigan γ_1 va γ_2 yopiq chiziqlar bilan o'raymiz. Bu γ_1, γ_2 chiziqlar hamda



17 chizma

$\{z \in C: |z| = 2\}$ aylana bilan chegaralangan uch bog'lamli sohani *D* bilan belgilaymiz (17 chizma).
Qaralayotgan integralda integral ostidagi

$$\frac{z+1}{z(z-1)^2(z-3)}$$

funksiya *D* sohada golomorf bo'ladi. Ko'p bog'lamli soha uchun Koshi teoremasidan foydalanib topamiz:

$$\oint_{|z|=2} \frac{z+1}{z(z-1)^2(z-3)} dz$$

$$= \oint_{\gamma_1} \frac{(z+1)dz}{z(z-1)^2(z-3)} + \oint_{\gamma_2} \frac{(z+1)dz}{z(z-1)^2(z-3)} = I_1 + I_2$$

Agar

$$I_1 = \oint_{\gamma_1} \frac{(z+1)dz}{z(z-1)^2(z-3)}$$

integralda

$$\mathrm{f(z)} = \frac{z+1}{(z-1)^2(z-3)}$$

deb olib, Koshining integral formulasi qo'llasak

$$I_1 = \oint_{\gamma_1} \frac{f(z)}{z} dz$$

$$= \oint_{\gamma_1} \frac{(z+1)dz}{z(z-1)^2(z-3)} = 2\pi i f(0) = 2\pi i(-\frac{1}{3}) = -\frac{2}{3}\pi i$$

bo'lishi kelib chiqadi. Ko'p bog'lamli soha uchun Koshi integral formulasi

$$\oint_{\partial D} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi = \frac{2\pi i f^{(n)}(z)}{n!}$$

dan foydalanib ikkinchi integralni hisoblaymiz:

$$I_2 = \oint\limits_{\gamma_2} \frac{(z+1)dz}{z(z-1)^2(z-3)} = \oint\limits_{\gamma_2} \frac{\frac{z+3}{z(z-3)}}{(z-1)^2} dz = 2\pi i \left(\frac{z+1}{z(z-3)}\right)'_{z=1} =$$

$$2\pi i \left(\frac{-z^2-2z+3}{(z^2-3z)^2}\right)_{z=1} = 0.$$

Shunday qilib,

$$\oint\limits_{|z|=2} \frac{z+1}{z(z-1)^2(z-3)} dz = I_1 + I_2 = \frac{-2}{3}\pi i$$

bo'ladi.

16.2 - misol. Agar $f(z)$ funksiya kompleks tekislik C da golomorf va chegaralangan bo'lsa, u holda $f(z)$ funksiyaning C da o'zgarmas bo'lishini isbotlang.

Yechish. Ushbu

$$\oint\limits_{|z|=r} \frac{f(z)dz}{(z-a)(z-b)}$$

($|a|<r, |b|<r$, $a \neq b$) integralni qaraymiz. Bu integralni Koshining integral formulasidan foydalanib hisoblaymiz:

$$\oint\limits_{|z|=r} \frac{f(z)dz}{(z-a)(z-b)} = \frac{1}{a-b}\left[\oint\limits_{|z|=r} \frac{f(z)dz}{(z-a)} - \oint\limits_{|z|=r} \frac{f(z)dz}{(z-b)}\right] =$$

$$\frac{2\pi i}{a-b}[f(a)-f(b)]. \qquad (1)$$

Shartga ko'ra $f(z)$ chegaralangan funksiya $|f(z)| < M$.

Unda

$$\left|\oint\limits_{|z|=r} \frac{f(z)dz}{(z-a)(z-b)}\right| \le \oint\limits_{|z|=r} \frac{|f(z)||dz|}{\big||z|-|a|\big|\,\big||z|-|b|\big|} \le$$

$$\le \frac{M}{(r-|a|)\,(r-|b|)} \oint\limits_{|z|=r} |dz| = \frac{2\pi M r}{(r-|a|)\,(r-|b|)}$$

bo'ladi. Demak,

$$0 \le \left| \oint\limits_{|z|=r} \frac{f(z)dz}{(z-a)(z-b)} \right| \le \frac{2\pi M r}{(r-|a|)\,(r-|b|)}. \qquad (2)$$

Ravshanki,

$$\lim_{r\to\infty} \frac{2M\pi r}{(r-|a|)(r-|b|)} = 0 \qquad (3)$$

(1), (2) va (3) munosabatlardan

$$\frac{2\pi i}{a-b}[f(a)-f(b)] = 0,$$

ya'ni

$$f(a) = f(b)$$

bo'lishi kelib chiqadi. Bu esa $f(z)$ funksiyaning C da o'zgarmas, ya'ni $f(z) = c, c = const$ bo'lishini bildiradi.

16-Misol. Koshining integral formulasidan foydalanib quyidagi integrallarni hisoblang.

16.1. $\displaystyle\int\limits_{|z-1|=3} \frac{z+1}{(z-1)^3\cdot(z+2)^2}dz$.

16.2. $\displaystyle\int\limits_{|z+1|=3} \frac{z-1}{(z-3)^2\cdot(z+i)^3}dz$.

16.3. $\displaystyle\int\limits_{|z-1|=2} \frac{z+2}{z^2\cdot(z^2+1)}dz$.

16.4. $\displaystyle\int\limits_{|z-1|=2} \frac{z-2}{(z+i)^3\cdot(z+2)^2}dz$.

16.5. $\displaystyle\int\limits_{|z|=2,5} \frac{z-1}{(z-2)^3\cdot(z-3)}dz$.

16.6. $\displaystyle\int\limits_{|z-1|=2} \frac{z+2}{z(z-1)^3\cdot(z-2)^2}dz$.

16.7. $\displaystyle\int\limits_{|z|=2} \frac{z-1}{(z-i)^3\cdot(z+1)^2}dz$.

16.8. $\displaystyle\oint_{|z-1|=2} \frac{z+1}{(z-2)^2\cdot(z+i)^3}\,dz$.

16.9. $\displaystyle\oint_{|z|=2,5} \frac{z-1}{(z+2)^2\cdot(z+1)^3}\,dz$.

16.10. $\displaystyle\oint_{|z-i|=2} \frac{z+1}{z^3\cdot(z-2i)^2}\,dz$.

16.11. $\displaystyle\oint_{|z-i|=2} \frac{z+1}{(z+i)^3\cdot(z-2i)^2}\,dz$.

16.12. $\displaystyle\oint_{|z-i|=2} \frac{z-1}{z^3\cdot(z+i)^2}\,dz$.

16.13. $\displaystyle\oint_{|z-i|=2} \frac{z+1}{(z-1)^3\cdot(z-2i)^2}\,dz$.

16.14. $\displaystyle\oint_{|z-i|=2} \frac{z-1}{(z+1)^3\cdot z^2}\,dz$.

16.15. $\displaystyle\oint_{|z-i|=2} \frac{z-1}{(z+1)^3\cdot(z-2i)^2}\,dz$.

16.16. $\displaystyle\oint_{|z-i|=2} \frac{z-1}{(z+1)^3\cdot(z-i)}\,dz$.

16.17. $\displaystyle\oint_{|z|=3} \frac{z+1}{(z-2i)\cdot(z-i)^3}\,dz$.

16.18. $\displaystyle\oint_{|z|=3} \frac{z+1}{(z+2)^3\cdot(z-2)^2}\,dz$.

16.19. $\displaystyle\oint_{|z|=3} \frac{z+1}{(z-2i)^3\cdot(z+i)^2}\,dz$.

16.20. $\displaystyle\oint_{|z|=3} \frac{z+1}{(z-2)^3\cdot(z+2i)^2}\,dz$.

16.21. $\displaystyle\oint_{|z|=2} \frac{z+1}{z(z-1)^3\cdot(z-3)}\,dz$.

16.22. $\oint\limits_{|z|=2} \frac{e^z dz}{z^2-1}$.

16.23. $\int\limits_{|z|=3} \frac{z-1}{(z+2)^3 \cdot (z-2i)^2} dz$.

16.24. $\oint\limits_{|z|=\frac{1}{2}} \frac{e^z dz}{z^6 - z^4}$.

16.25. $\oint\limits_{|z|=1} \frac{ch^2 z dz}{z^2}$.

16.26. $\oint\limits_{|z|=t} \frac{dz}{(z-a)^n (z-b)}, \quad (|a| < t < |b|)$.

16.27. $\oint\limits_{|z|=2} \frac{\cos z^2 dz}{z^2 (z+5)^2}$.

16.28. $\oint\limits_{|z|=\frac{1}{2}} \frac{\cos \frac{\pi}{z+1}}{z^2} dz$.

16.29. $\oint\limits_{|z-i|=1} \frac{\cos 2(z-1) dz}{(z-i)^2 (z+6)}$.

16.30. $\oint\limits_{|z+i|=1} \frac{\sin(2z+1) dz}{(z+i)^3 (z-1)}$.

16.31. $\oint\limits_{|z|=2} \frac{chz dz}{(z-1)^4}$.

16.32. $\oint\limits_{|z|=2} \frac{\cos z \cos(z-1)}{z^2 - z} dz$.

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