

**ЎЗБЕКИСТОН МИЛЛИЙ УНИВЕРСИТЕТИ
ҲУЗУРИДАГИ ИЛМИЙ ДАРАЖАЛАР БЕРУВЧИ
DSc27.06.2017/ФМ 01.02 РАҚАМЛИ ИЛМИЙ КЕНГАШ**

**БИРЛАШГАН АРАБ АМИРЛИҚЛАРИ УНИВЕРСИТЕТИ,
МАТЕМАТИКА ИНСТИТУТИ**

ФАТХАЛЛА АЛИ РИҲАН

**ХОТИРАЛИ ДИФФЕРЕНЦИАЛ ТЕНГЛАМАЛАРНИНГ СИФАТ ВА
МИҚДОРЙ ЖИҲАТЛАРИ ВА УЛАРНИНГ ТАТБИҚЛАРИ**

**05.01.07 – Математик моделлаштириш. Сонли усуллар ва дастурлар мажмуаси
(физика-математика фанлари)**

**ИЛМИЙ МАЪРУЗА ШАКЛИДАГИ
ФИЗИКА-МАТЕМАТИКА ФАНЛАРИ ДОКТОРИ (DSc)
ДИССЕРТАЦИЯСИ**

ТОШКЕНТ – 2019

Докторлик (DSc) диссертацияси автореферати мундарижаси

Оглавление автореферата докторской (DSc) диссертации

Content of the abstract of the doctoral (DSc) dissertation

Фатхалла Али Рихан

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ТОШКЕНТ – 2019

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КИРИШ (докторлик диссертацияси аннотацияси)

Диссертация мавзусининг долзарблиги ва зарурати. Жаҳонда кечикувчи аргументли дифференциал тенгламалар ва каср-тартибли дифференциал тенгламаларни ўз ичига олувчи хотирали дифференциал тенгламалар табиий фанлар ва техникадаги реал жараёнларни математик моделлаштиришда кенг қўлланаётган тушунчалардан биридир. Хотирали дифференциал тенгламалар биологик популяциялар динамикаси, эпидемиология, иммунология ва физиологияда организмдаги биохимик жараёнларни моделлаштириш ва математик моделлаштириш каби соҳалардаги тадқиқотларнинг объектидир. Кечикувчи аргументли дифференциал тенгламалар биологик, кимёвий жараёнларни ва нейрон тўрларини моделлаштиришда ҳамда психологик касалликларни аниқлашда ва даволашда асос сифатида хизмат қилади. Шу сабабли кечикувчи дифференциал тенгламалар ва каср-тартибли дифференциал тенгламалар орқали ифодаланувчи математик моделларни тадқиқ этиш техника, кимё, биология ва экология каби соҳаларнинг муҳим вазифаларидан бири бўлиб қолмоқда.

Ҳозирги кунда жаҳонда тиббиёт, биология ва биотехнология каби соҳаларда учрайдиган кўплаб жараёнларнинг математик моделларини тасвирловчи бутун ва каср тартибли кечикувчи аргументли дифференциал тенгламалар учун қўйилган чегаравий масаларни сонли ечиш ҳамда визуаллаштириш долзарб масалалардан бири ҳисобланади. Математик моделни ифодаловчи дифференциал тенгламада бир ёки бир неча параметр, хусусан, номаълум функциялар ва уларнинг ҳосилалари қийматларининг вақт бўйича кечикиб қатнашиши амалий нуқтаи назардан табиий ҳодиса бўлиб, бу жиҳатни ҳисобга олиш моделни аслиятга яқинлаштиришига имкон бериши билан муҳимдир. Хусусан, кечикувчи аргументли дифференциал тенгламалар ва каср-тартибли дифференциал тенгламалар биологияда популяциялар динамикасини ўрганишда, тиббиётда эпидемиология, иммунология ва физиологияда организмдаги биохимик жараёнларни моделлаштиришда, техникада нейрон тўрлари каби дифференциал тенгламалар билан ифодаланувчи системаларни тадқиқ этишда кенг қўлланилмоқда. Кечикув параметрларини танлаш эвазига моделни табиий жараёнга яқинлаштириш, муайян даражада уни бошқариш мақсадли илмий тадқиқотлардан ҳисобланади.

Бирлашган Араб Амирликлари ва Ўзбекистон Республикасида фундаментал фанларнинг илмий ва амалий татбиқига эга бўлган экологик жараёнларни моделлаштириш ва бошқариш, кўп фазали динамика ва математик моделлаштиришнинг долзарб йўналишларига алоҳида эътибор кучайтирилди. Бу борада биологик ва кимёвий жараёнлар, психологик касалликларни даволаш каби мураккаб объектлар учун кечикишга эга бўлган динамик системалар ва функционал дифференциал тенгламалар шаклидаги математик моделлар қурилиб, уларнинг хоссаларини аниқлашда салмоқли натижаларга эришилди. «Функционал анализ, дифференциал тенгламалар ва

математик физика, амалий математика ва математик моделлаштириш» фанларининг устувор йўналишлари бўйича халқаро стандартлар даражасида илмий тадқиқотлар олиб бориш амалий математика фанининг асосий вазифалар ва фаолият йўналишлари этиб белгиланди¹. Қарор ижросини таъминлашда ғовак-эластик муҳитлардаги жараёнларни математик моделлаштириш ва сонли тадқиқ этишни ривожлантириш муҳим аҳамиятга эга.

Ўзбекистон Республикаси Президентининг 2017 йил 17 февралдаги ПҚ-2789-сон «Фанлар академияси фаолияти, илмий-тадқиқот ишларини ташкил этиш, бошқариш ва молиялаштиришни янада такомиллаштириш чора-тадбирлари тўғрисида»ги Қарори ва 2017 йил 8 февралдаги ПФ-4947-сон «Ўзбекистон Республикасини янада ривожлантириш бўйича ҳаракатлар стратегияси тўғрисида»ги Фармони, Бирлашган Араб Амирликлари Олий таълим ва илмий тадқиқотлар Вазирлигининг 1992 йилдаги №4 Федерал қонуни ҳамда мазкур фаолиятга тегишли бошқа норматив-ҳуқуқий ҳужжатларда белгиланган вазифаларни амалга оширишга ушбу диссертация тадқиқоти муайян даражада хизмат қилади.

Тадқиқотнинг республика фан ва технологиялари ривожланишининг устувор йўналишларига боғлиқлиги. Мазкур тадқиқот Бирлашган Араб Амирликларининг 2010/096 «Аниқ фанлар» ва Ўзбекистон Республикаси фан ва технологияларини ривожлантиришнинг IV. «Математика, механика ва информатика» илмий-техник дастурлари устувор йўналишлари доирасида бажарилган.

Диссертация мавзуси бўйича хорижий илмий-тадқиқотлар шарҳи². Бутун ва қаср тартибли ҳамда хотирага эга дифференциал тенгламалар билан ифодаланувчи математик моделларнинг сифат хоссаларини сонли тадқиқ этиш бўйича илмий изланишлар жаҳоннинг етакчи олий таълим муассасалари ва илмий марказлари, жумладан, Bath, Cambridge, Salford, Manchester, Reding, East Anglia, Norwich, Liverpool, Oxford, Edinburgh, Честер университетлари (Буюк Британия), Orlando, Florida, Las Vegas, Nevada университетлари (АҚШ), БАА университети (Ал-Айн), Султон Қобус университети, Мартин Лотер университети (Германия), American University of Sharjah (БАА), Йилдиз техника университети (Туркия), (Бирлашган Араб Амирликлари Стратегик изланишлар ва тадқиқотлар маркази. (Абу-Даби), Эгей, Карловасси, Самос университетлари (Греция), Zayed University (БАА), Amsterdam University (Нидерландия), Melbourne, Queensland университетлари (Австралия), Россия ФАнинг ҳисоблаш математикаси институти (Москва), Украина ФАнинг В.М.Глушков номидаги кибернетика институти (Киев),

¹ Ўзбекистон Республикаси Вазирлар Маҳкамасининг 2017 йил 18 майдаги «Ўзбекистон Республикаси Фанлар академиясининг янгидан ташкил этилган илмий-тадқиқот муассасалари фаолиятини ташкил этиш тўғрисида»ги 292-сон қарори.

² Диссертация мавзуси бўйича хорижий илмий-тадқиқотлар шарҳи www.eriez.com, docs.lib.purdue.edu, www.cargocaresolutions.com, www.sciencedirect.com, www.link.springer.com, www.iccm-central.org, www.digitimes.com, www.ihs.com, www.webofknowledge.com, www.scholar.google.com ва бошқа манбалар асосида ишлаб чиқилган.

Россия ФАнинг Урал бўлинмасининг Пермь илмий маркази (Пермь), Беларуссия Миллий академиясининг бирлашган информатика муаммолари институти, (Минск), Ўзбекистон ФАнинг В.И.Романовский номидаги Математика институти, Бухоро давлат университетидида олиб борилмоқда.

Кечикишга эга бўлган динамик системалар ва функционал дифференциал тенгламаларни ечиш, улар учун қўйилган чегаравий масалаларни сонли ечиш усуллари ишлаб чиқишга оид жаҳонда олиб борилган тадқиқотлар натижасида қатор, жумладан, қуйидаги илмий натижалар олинган: биологик ва кимёвий жараёнларнинг математик моделлари қурилган (Manchester University, Буюк Британия); йирик тиббий клиникаларда йиғилган маълумотлар базаси асосида даволаш жараёнининг математик моделлари қай даражада аслиятга яқинлиги аниқланган, кечикувчи ва четлашувли дифференциал тенгламаларнинг бу жихатдан адекватлиги ўрганилган (Ҳисоблаш математикаси институти, Россия); инсон танасидаги CD4 лимфоцитларининг иммунитет танқислиги вируси (human immunodeficiency virus) билан зарарланиш жараёнини моделлаштирувчи кечикувчи дифференциал тенгламада Хопф бифуркацияси ҳамда циклик ечимнинг турғунлиги аниқланган ва амалий тавсиялар ишлаб чиқилган (Gandhigram Rural илмий-тадқиқот институти, Deemed University, Ҳиндистон); Эбола (Ebola) гриппи тарқалишининг динамикасини моделлаштирувчи каср тартибли хусусий ҳосилали кечикувчи дифференциал тенглама таклиф қилиниб, унинг ечимлари турғунлиги, Хопф бифуркацияси тадқиқ қилинган ҳамда тақрибий ечиш методлари ёрдамида система параметрларининг баъзи қийматлари учун траекторияларнинг сифат хоссалари аниқланган (Bharathiar University, Ҳиндистон).

Дунёда тиббиёт, экология, биология ва биотехнология соҳаларидаги математик моделларни асосини ташкил этувчи бутун ва каср тартибли кечикувчи дифференциал тенгламалар учун қўйилган Коши ҳамда чегаравий масалаларни ечиш ва амалиётга татбиқ этиш усуллари, воситаларини ишлаб чиқиш бўйича бир қатор устувор йўналишларда илмий тадқиқот ишлари олиб борилмоқда, жумладан: бутун ва каср тартибли кечикувчи дифференциал тенгламалар кўринишидаги математик моделларнинг кенг спектрини тадқиқ қилиш ва уларни хотирасиз дифференциал тенгламалар билан таққослаш; динамик система сифатида мукамалроқ математик структурага эга эканлигини кўрсатиш; кечикувчи ва каср тартибли дифференциал тенгламалар ечимларининг турғунлиги учун янги етарли шартлар олиш ва тақрибий ечиш усуллари ишлаб чиқиш; бутун ва каср тартибли кечикувчи дифференциал тенгламалар учун қўйилган чегаравий масалаларни сонли ечиш алгоритмларини қуриш ва дастурий воситаларини яратиш.

Муаммонинг ўрганилганлик даражаси. Кечикишга эга дифференциал тенгламалар учун ечимнинг мавжудлиги, ягоналиги ва бошланғич қийматлар бўйича узлуксизлиги бу назариянинг асосчилари В.Вольтерра, Ж.Хейл, Н.Н.Красовский томонидан ўрнатилган. Кенгроқ синфлар учун бундай натижалар L.W.Neustadt, А.Н.Тихонов, G.Jones, M.A.Cruz, C.V.Coffman,

J.J.Shaffer, J.A.Yorke, J.Dugundji, A.Halanay томонидан олинган. Чизикли кечикувчи дифференциал тенгламалар айниқса атрофлича ўрганилган. Бу йўналишдаги асосий натижалар А.Д.Мишкис, R.Bellman ва K.L.Cooke, J.Hale ва K.R.Mayer, D.Henry, А.М. Зверькин, Г.А.Каменский, С.Б. Норкин ва Л.Э.Эльсгольц, W.R. Melvin томонидан олинган. Чизикли бўлмаган тенгламаларнинг техникага татбиқларида ечимни давом эттириш масаласи муҳим роль ўйнайди. J.Hale ва Hastings, W.M.Oliva, J.C.Lillo ечим чексиз давом эттирилиши шартларини аниқлаган.

Кечикишга эга дифференциал тенгламалар назариясига оид кўплаб тадқиқотлар олиб борилаётганига қарамай, бу соҳада ҳал этилмай қолиб келаётган масалалар ҳали кўп. Кечикувчи дифференциал тенгламалар ўрганила бошланишига биринчи навбатда амалий эҳтиёжлар туртки берган. Сўнгги бир неча ўн йиллик давомида компьютер технологиялари замонавий биология, кимё, тиббиёт ва бошқа соҳаларга жадал кириб бориши муносабати билан хотирали дифференциал тенгламаларнинг татбиқий аҳамияти ортиб, бу соҳада ҳар ойда юзлаб мақолалар эълон қилина бошлади. Бундай тадқиқотларнинг муайян қисми Н. Smithнинг илмий тадқиқотларида акс эттирилган.

Бугунги кунда кўплаб илмий марказларда каср тартибли кечикувчи дифференциал тенгламалар бўйича илмий тадқиқотлар олиб борилмоқда. Z.Wang ечимни тақрибий куриш учун сонли методлар таклиф этган, S.Bhalekar бундай тенгламалар учун турғунлик ва бифуркация масалаларини ўрганган, F.L.Chen каср тартибли кечикувчи дифференциал тенгламаларнинг дискретлашган схемасини қараган, G.C.Wu ва D.Valeanu Фьюрхолстнинг логистик тенгламасини шу нуқтаи назардан текширган, D.Wang ва J.Yu логистик тенгламалар системасида хаотик траекториялар мавжуд бўлишининг шартини топган. A. Si-Ammour, S. Djennoune ва M. Bettayeb бошқарув масаласини ечиш усулини таклиф этган, S. Bhalekar, V. Daftardar-Gejji, D. Valeanu ва R. Magin каср тартибли Блох тенгламасида кечикувчи аргумент таъсирини тадқиқ этган, Y.Li, Y.Q.Chen, ва I.Podlubny Ляпунов функциялари методини қўллаган.

Диссертация мавзусининг диссертация бажарилаётган илмий тадқиқот институтининг илмий-тадқиқот ишлари режаси билан боғлиқлиги. Диссертация тадқиқоти Ўзбекистон Республикаси Фанлар академияси Математика институтининг ОТ-Ф4-84 рақамли “Полиномиал системалар учун дискрет-сонли метод ҳамда унинг циклик ва бошқарилувчи жараёнларни моделлаштиришга татбиқлари” (2017-2020) ва Бирлашган Араб Амирликлари университетининг SQU/UAEU “Delay Differential Models of Immune Response With Viral and Bacterial Infection in an Organism” (2017-2019), UPAR “Mathematical Models for Kinetics of Coronavirus Infection in Humans” (2015-2017), NRF “Delay Differential Models in Immunology and Infection Diseases in an Individual” (2011-2016) фундаментал лойиҳалари доирасида бажарилган.

Тадқиқотнинг мақсади бутун ва каср тартибли кечикувчи дифференциал тенгламалар кўринишидаги математик моделларни куриш, кечикувчи ва каср тартибли дифференциал тенгламалар ечимларининг турғунлиги учун янги етарли шартлар олиш ва тақрибий сонли ечиш усулларини ишлаб чиқиш ҳамда сонли ечиш дастурий воситаларини яратишдан иборат.

Тадқиқотнинг вазифалари:

кечикувчи дифференциал тенгламалар ечимларининг турғунлигини исботлаш ва бифуркацияларини топиш;

кечикувчи дифференциал тенгламалар учун Коши масаласини тақрибий ечиш усулларини ишлаб чиқиш;

параметрли кечикувчи дифференциал тенгламаларнинг сезгирлигини баҳолаш;

айнан бир жараённинг оддий ва кечикувчи дифференциал тенгламалар орқали моделларини ўзаро таққослаб, кузатувларда олинган маълумотлар асосида баҳолаш;

кечикувчи ва каср тартибли дифференциал тенгламалар ечимларининг турғунлигини тадқиқ этишнинг Ляпунов функционаллари усулини ривожлантириш;

грипп тарқалиши, саратонли ўсимталарнинг ўсиш динамикаси, гепатит касаллигининг кечиши каби жараёнларни кечикувчи ва каср-тартибли дифференциал тенгламалар воситасида математик моделини тузиш ва уларни таҳлил этиш;

кечикувчи ва каср тартибли дифференциал тенгламаларни нейрон тўрларини моделлаштиришга татбиқ этиш.

Тадқиқотнинг объекти хотирали дифференциал тенгламалар орқали моделлаштирилаётган биология, экология, тиббиёт соҳаларидаги жараёнлар ва нейрон тўрларидан иборат.

Тадқиқотнинг предмети кечикувчи ва каср тартибли дифференциал тенгламаларнинг сифат ва миқдорий жиҳатлари ва уларнинг табиий фанлар ва техникага татбиқларидан иборат.

Тадқиқотнинг усуллари. Диссертация ишида замонавий математик моделлаштириш, динамик системаларнинг сифат назарияси, бифуркациялар назарияси, интеграл тенгламалар назарияси, Ляпунов функционаллари методи, Понтрягиннинг максимум принципи, Коши масаласини ечишнинг сонли усуллари, Рунге-Кутта методи учун Батчер жадваллари, каср тартибли ҳосила ва интеграллар, Лотка-Вольтерр туридаги моделлар, компьютерда моделлаштириш каби усулларидан фойдаланилган.

Тадқиқотнинг илмий янгилиги қуйидагилардан иборат:

биосистемалар динамикасини таҳлили учун оддий ва тақсимланган кечикувчи дифференциал тенгламалар, каср тартибли оддий ҳамда каср тартибли хусусий ҳосилалар тенгламалар воситасида янги математик моделлар қурилган;

вақт бўйича кечикиш ҳисобга олинган математик моделлар бу омилни ҳисобга олмаган математик моделларга нисбатан жараёнларни аниқроқ акс эттириши кўрсатилган;

кечикувли дифференциал тенгламаларни ечишнинг самарали усуллари, жумладан, Рунге-Кутта методи учун ноошкор схемалар ишлаб чиқилган; сонли методлар турғунлигини таъминловчи янги критерийлар топилган;

математик моделларнинг параметрлари қўзғатилганда ва "оқ шовқин" тарзидаги тасодифий четлашишларга нисбатан сезгирлигини баҳолаш усули ишлаб чиқилган;

мембранали ВАМ-нейрон тўрлари, комплекс қийматли нейрон тўрлари, Кохен-Кроссберг нейрон тўрлари ҳамда каср ҳосиллали нейрон тўрлари учун синхронлаштириш, турғунлик ва диссипативликни текшириш усуллари ишлаб чиқилган;

Кохен-Кроссберг ВАМ-нейрон тўрларини турғун бўлмаган ҳолда стабиллаштириш алгоритми қурилган.

Тадқиқотнинг амалий натижалари хавfli ўсмаларнинг ўсиш ва юқумли касалликлар эпидемиясининг тарқалиш динамикаси аниқланган, "Йиртқич-ўлжа" популяцион генетик тизим динамикаси каби биологик ва тиббий жараёнларнинг такомиллашган математик моделлари қурилган, вирусли касалликларни даволашнинг оптимал режимлари ҳисобланган.

Тадқиқот натижаларининг ишончилиги. Назарий натижалар теоремалар кўринишида ифодаланиб, қатъий исботлар билан таъминланган, диссертация ҳажмининг асосий қисмини ташкил этувчи хотирали дифференциал тенгламалар воситасида таклиф этилган моделлар амалиётдан олинган маълумотлар асосида таҳлил қилиниб, классик математик моделларга нисбатан аслиятга кўпроқ адекват бўлиши кўрсатилган. Жумладан, касалликларнинг динамикасига оид моделларнинг ишончилиги клиник тадқиқотлар натижалари билан қиёсий таҳлиллар орқали далилланган.

Тадқиқот натижаларининг илмий ва амалий аҳамияти. Тадқиқот натижаларининг илмий аҳамияти икки фазали ғовак муҳитлар оқимларини ва бунда ҳосил бўлувчи иссиқлик-масса алмашинуви жараёнларининг математик моделларини такомиллаштириш ва асослашдан тортиб, конкрет амалий нефт ва газ механикаси масалаларини ечиш ҳамда таҳлил қилишгача бўлган тадқиқ ишларини сақланиш қонунлари, методлари ва кўп фазали муҳитлар механикаси тенгламалари билан биргаликда амалга ошириш билан изоҳланади.

Тадқиқот натижаларининг амалий аҳамияти нефт ва газ саноати масалаларини ечиш, иссиқлик энергетикасида, инновацион тўлқинли методларни ишлаб чиқиш, ёриқ-ғовак коллекторларнинг фильтрацияли-сифимли хусусиятларини башорат қилиш ва уларнинг иш режимларини оптималлаштириш ҳамда углеводородли қонларни ишлатишнинг технологик жараёнлари моделларини такомиллаштириш учун хизмат қилади.

Тадқиқот натижаларининг жорий қилиниши. Хотирали дифференциал тенгламаларнинг сифат ва миқдорий жиҳатлари ҳамда кечикувчи дифференциал тенгламалар учун қўйилган чегаравий масалаларни сонли ечишга оид олинган илмий натижалар асосида:

чизиқли бўлмаган кечикувчи дифференциал тенгламаларни сонли ечиш усуллари 301 рақамли “Simulation of Radiation Effects in the Central Nervous System” грант лойиҳасида кечикувчи дифференциал тенгламалар ечимларининг турғунлигини исботлашда фойдаланилган (Қоҳира университетининг 2018 йил 1 апрелдаги маълумотномаси). Илмий натижаларнинг қўлланилиши сил билан оғриган беморларда физиологик жараён кечикишини прогноз қилиш усулини яратишга имкон берган;

математик моделларнинг параметрлари кўзғатилганда ва "оқ шовқин" тарзидаги тасодифий четлашишларга нисбатан сезгирлигини баҳолаш усули 2009/2010 рақамли “Epidemiology of Swine flu H1N1 pandemic” грант лойиҳасида С гепатит вируси динамикасини параметр бўйича баҳолашда қўлланилган (Бирлашган Араб Амирликлари университетининг 2018 йил 14 майдаги маълумотномаси). Илмий натижаларнинг қўлланилиши H1N1 типли пандемик юкумли гриппнинг тарқалиши жараёнини прогноз қилиш учун тегишли тавсиялар ишлаб чиқиш имконини берган;

мембранали ВАМ-нейрон тўрлари, комплекс қийматли нейрон тўрлари, Кохен–Кроссберг нейрон тўрлари ҳамда каср ҳосилали нейрон тўрлари учун синхронлаштириш, турғунлик ва диссипативликни текшириш усуллари иммун жараёнларни математик моделлаштиришда фойдаланилган (Россия Фанлар академияси Ҳисоблаш математикаси институтининг 2018 йил 13 апрелдаги 10256/75-сон маълумотномаси). Илмий натижаларнинг қўлланилиши клиник кўрсатувлар натижасида тўпланган маълумотлар базасини таснифлаш имконини берган;

Кохен-Кроссберг ВАМ-нейрон тўрларини турғун бўлмаган ҳолда стабиллаштириш алгоритми хорижий илмий журналларда (Hindawi, Complexity, Volume 2017, Article ID 6875874, 13 pages; Neural Processing Letters, Springer, 2018, pp. 1-19; Journal Neural Networks, vol. 98, pp. 223-235; Numerical Algorithms, vol. 79, issue 1, 2018, pp. 19-40; International Journal of Dynamics and Control, 2018, pp. 1-9) импульсли ва кечикувчи каср тартибли Риман-Лиувилл ВАМ гибрит нейрон тўрлари ечимининг мавжудлигини ва глобаль асимптотик турғунлигини исботлашда фойдаланилган. Илмий натижалардан фойдаланиш ВАМ гибрит нейрон тўрларининг параметрларига боғлиқ мувозанат ечимининг глобаль асимптотик турғунлик критерияларини ишлаб чиқишга хизмат қилган;

биосистемалар динамикасининг таҳлили учун оддий ва тақсимланган кечикувчи дифференциал тенгламалар, каср тартибли оддий ҳамда каср тартибли хусусий ҳосилали тенгламалар воситасида қурилган математик моделлардан хорижий илмий журналларда (Applied Mathematics and Computation, Vol. 293, 2017, pp. 293-310; Journal of Inequalities and Applications, 2014, pp. 1-14; Communications in Nonlinear Science and Numerical Simulation, vol. 39, 2016, pp. 396-410; Applied Mathematics, №8,

2017, 1715-1744) каср тартибли кечикувчи аргументли “йирткич-ўлжа” системасининг бифуркацияларини бошқаришда фойдаланилган. Илмий натижаларнинг қўлланилиши “йирткич-ўлжа” системаси учун Хопф бифуркацияларини самарали бошқариш имконини берган.

Тадқиқот натижаларининг апробацияси. Тадқиқот натижалари 32 та ҳалқаро ва 36 та минтақавий илмий-амалий анжуманларда муҳокамадан ўтказилган. Диссертация натижалари бўйича Ўзбекистон Республикаси Фанлар академияси В.И.Романовский номидаги математика институтининг “Алгебра ва функционал анализ” ва “Динамик системалар назарияси” бўлимларининг бирлашган илмий семинарида ҳамда Ўзбекистон Миллий университетининг “Математик физика” илмий семинарида муҳокама қилинган.

Тадқиқот натижаларининг эълон қилинганлиги. Диссертация мавзуси бўйича жами 102 та илмий иш чоп этилган, шулардан Ўзбекистон Республикаси Олий Аттестация Комиссиясининг докторлик диссертациялари асосий илмий натижаларини чоп этиш тавсия этилган илмий нашрларда 33 та илмий мақола нашр этилган.

Диссертациянинг ҳажми ва тузилиши. Диссертация инглиз тилида илмий маъруза шаклида тайёрланган бўлиб, 76 саҳифадан иборат.

ДИССЕРТАЦИЯНИНГ АСОСИЙ МАЗМУНИ

Хотирали дифференциал тенгламалар деган умумий ном берилган кечикувчи аргументли дифференциал тенгламалар ва каср тартибли дифференциал тенгламалар динамик жараёнларни аслиятга яқинроқ тарзда моделлаштиришнинг замонавий воситаларига айланган. Айниқса, популяцион экология, эпидемиология, иммунология, нейрон тўрлари назарияси каби соҳаларнинг предмети бўлган биологик ва физик жараёнларни математик усуллар билан ўрганишда бундай турдаги дифференциал тенгламалар математик моделларни сезиларли даражада муқобиллаштиришга имкон беради.

Биологик ва муҳандислик тизимларининг мутлақ кўпчилигида системанинг динамикасини белгиловчи муайян параметрларнинг қийматлари жараённинг кечишига кечикиб – маълум вақтдан сўнг таъсир ўтказиши яққол кузатилади. Бу каби тизимларни хотирали, яъни кечикувчи ёки каср тартибли дифференциал тенгламалар воситасида моделлаштириш хотира ҳисобга олинмайдиган моделларга нисбатан бир қатор афзалликларга эга. Хусусан, кўплаб математик моделларда кечикувчи инобатга олиш ёки каср тартибли ҳосилали дифференциал тенглама билан ифодалаш натижасида ечимнинг турғун бўлиши шартлари яхшиланади, модель айрим ижобий хоссалар билан бойийди. Масалан, нотривиал ечимлари чексиз ўсувчи системаларда кечикувчи омили ҳисобга олиниши туфайли ечимнинг чегараланган, турғун ва даврий бўлишига эришиш мумкин. Шунингдек, математик физиканинг аввалдан

Ўрганиб келинган тенгламалари жараён кечадиган фазонинг локал қисми билангина белгиланса, каср тартибли ҳосилали дифференциал тенгламада фазонинг глобал таъсири акс этади.

Диссертация ишининг асосий мазмуни табиий жараёнларнинг математик модели сифатида киритилган кечикувчи ҳамда каср тартибли, хусусан, бир пайтда ҳам кечикувчи, ҳам каср тартибли бўлган дифференциал тенгламаларнинг сифат нуқтаи назаридан хоссаларини ўрганиш ва уларнинг хотирасиз дифференциал тенгламаларга нисбатан бойроқ хоссаларга эга эканлигини очиб беришдан иборат.

Диссертация иши кириш қисми, 5 та боб ва якуний хулосалар қисмидан ташкил топган. 1-бобда кечикувчи оддий дифференциал тенгламалар ва уларнинг татбиқлари қаралган. 2-боб каср тартибли кечикувчи дифференциал тенгламаларга бағишланган. 3-бобда кечикувчи дифференциал тенгламаларнинг нейрон тўрлари назариясига татбиқига оид сифат масалалари тадқиқ этилган. 4-бобда хотирали дифференциал тенгламаларни тақрибий ечиш учун шартсиз турғун ҳисоблаш схемаси берилган. 5-бобда хотирали дифференциал тенгламалар ечимларининг Ляпунов маъносидаги турғунлиги шартлари ўрганилган ва параметрлар ўзгаришига нисбатан сезгирлик учун критерийлар таклиф этилган. Диссертация ниҳоясида хулосалар келтирилган.

Физик ва биологик жараёнлар мутлақ кўпчилик ҳолда чизикли бўлмаган мураккаб динамикага эгаллиги билан ажралиб туради. Шунинг учун уларнинг турғунлиги ва бифуркацияларини ўрганиш реал системаларнинг барқарорлигини кафолатлаш, техник қурилмаларнинг хавфсизлигини таъминлашда долзарб масаладир. Мисол учун биологик жараённинг математик моделини қуришда бундай жараёнга классик турдаги динамик жараён сифатида қараш ҳар доим ҳам мақбул бўлмайди, балки ресурсларнинг тикланиш даври, тирик мавжудотларнинг кўпайиш даври, ташқи муҳитнинг кескин ўзгаришларига адаптация даври каби омилларни эътиборга олиш муҳимдир. Бундай омиллар ночизикли табиатга эга бўлиши баробарида системада параметрларнинг жорий вақтдан аввалги даврдаги қийматлари жорий вақтдаги қийматларига таъсир ўтказиши реал жараёнларнинг табиатига хосдир. Шунингдек, хотирали ва каср тартибли дифференциал тенгламаларда қўшимча параметр мавжудлиги, яъни кечикув катталиги ва каср ҳосиланинг тартиби жараённинг математик моделини муқобиллаштиришга имкон беради.

Хотирали дифференциал тенгламалар асосан

$$y'(t) = f\left(t, y(t), y(\alpha(t, y(t))), \int_{-\infty}^t K(t, s, y(t), y(s)) ds\right), t \geq t_0, \quad (1)$$

кўринишдаги кечикувчи операторли дифференциал тенгламаларнинг хусусий ҳолидир (бунда $\alpha(t, y(t)) \leq t$ – вақт бўйича кечикув шарти, $y(t) = \psi(t)$, $t \leq t_0$ – бошланғич шарт). Айтиш мумкинки, хотирали дифференциал тенгламалар оддий дифференциал тенгламалар билан операторли дифференциал

тенгламалар оралиғида ётади. Шу билан бирга (1) тенгламани чексиз ўлчовли фазода аниқланадиган динамик система сифатида ҳам талқин этиш мумкин.

(1) тенгламалар синфининг хусусий ҳоли бўлган

$$y'(t) = f(t, y(t), y(\alpha(t, y(t))), y'(\beta(t, y(t))), \quad (2)$$

кўринишдаги нейтрал турдаги дифференциал тенгламалар ва

$$y'(t) = f(t, y(t), y(\alpha(t, y(t))), \alpha(t, y(t)) \leq t \quad (3)$$

кўринишдаги кечикувли дифференциал тенгламалар амалий татбиқларда асосий ўрин тутади. Бу ердаги $\alpha(t, y(t))$, $\beta(t, y(t))$ аргументлар кечикиб таъсир этиш омилини ифодалайди. Тайин мисол тариқасида кечикув юқумли касалликнинг инкубацион давридан иборат бўлган моделни қараш мумкин.

Диссертацион ишда хотирага эга бўлган жараёнлар моделларида кечикувчи вақт параметри билан каср тартибли ҳосилаларнинг роли ва ўзаро боғланишига алоҳида эътибор қаратилган, хотирага эга бўлган дифференциал тенгламаларнинг сифат ва миқдорий жихатлари ўрганилган.

Диссертациянинг ўрганиш объекти:

I. Параметрға боғлиқ кечикувли

$$y'(t) = f(t, y(t), y(\alpha(t)), y'(\alpha(t)), u(t), u(\beta(t)), p), \quad t \geq t_0 \quad (4)$$

$$y(t) = \psi(t, p), \quad y'(t) = \psi'(t, p), \quad u(t) = \Phi(t) \quad t \leq t_0.$$

дифференциал тенгламалар учун Коши масаласи ечимининг турғунлиги ва бифуркацияларини тадқиқ этиш.

II. Куйидаги кўринишдаги

$$x'(t) = f(t, x(t), x(t-\tau), v(t)), \quad t \in [0, t_f] \quad (5a)$$

кечикувли дифференциал тенглама учун

$$x(t) = \phi(t), \quad t \in [-\tau, 0], \quad (5b)$$

бошланғич шарт,

$$x(t) \geq c \quad t \in [0, t_f]. \quad (5c)$$

ҳолатлар бўйича шарт ва бошқарув функциясига

$$a \leq v(t) \leq b, \quad t \in [0, t_f] \quad (5d)$$

шартда

$$\max_{x,v} J(x, v) = \Psi(x(t_f)) + \int_0^{t_f} L(t, x(t), v(t)) dt, \quad (5e)$$

функционал билан берилган оптимал бошқарув масаласини ўрганиш, хусусан, Понтрягиннинг максимум принципини қўллаб, оптимал режим учун зарурий шартларни келтириб чиқариш.

III. Каср хусусий ҳосилали вақт бўйича кечикувли

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = F(t, u(x, t), \mathfrak{I}u(x, t), u(x, t-\tau)), \quad x \in \Omega, \quad t \geq 0, \quad 0 < \alpha \leq 1, \quad (6)$$

дифференциал тенгламани тадқиқ этиш (бу тенгламада F – узлуксиз функция, $t-\tau \leq t$; кечикув параметри τ ўзгармас бўлиши, ёки $\tau(t)$ вақт

функцияси, ёки $\tau(t, y)$ кўринишдаги функция бўлиши мумкин, \mathfrak{S} – фазовий координаталар бўйича эллиптик оператор).

(6) тенгламалар оиласининг муҳим хусусий ҳолларидан бири бу – кечикувли диффузия тенгласидир:

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = a \frac{\partial^2 u(x, t)}{\partial x^2} + bu(x, t - \tau), \quad a, b \in \mathbb{R}. \quad (7)$$

Агар (7) тенгламада $\alpha = 1$ бўлса, кечикувли параболик хусусий ҳосилали дифференциал тенглама билан бериладиган диффузион жараёнларнинг қуйидаги моделига эга бўлинади:

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= a \frac{\partial^2 u(x, t)}{\partial x^2} + bu(x, t - \tau), \quad t \geq 0, \\ u(x, t) &= \psi(x, t), \quad x \in [0, L], \quad t \in [-\tau, 0]. \end{aligned} \quad (8)$$

(6) тенглама тоифасига мансуб, чизиқли бўлмаган мисол сифатида Хатсон тенгласи қаралади:

$$\frac{\partial u(x, t)}{\partial t} = \varepsilon \frac{\partial^2 u(x, t)}{\partial x^2} + \gamma u(x, t) [1 - u(x, t - \tau)]. \quad (9)$$

Бу тенглама ҳозирги замон математик экологияда жониворлар турининг муайян муҳитдаги кўпайиши учун эволюцион моделни ифодалайди, ундаги номаълум $u(x, t)$ функция жониворларнинг зичлиги, мусбат ε сони диффузия коэффициенти, τ эса кечикув параметридир. Амалда $u(x, t)$ ечим учун тегишли чегаравий шартлар ва бошланғич шарт қўйилади.

IV. Каср тартибли ва кечикувли нейрон тўрларимодели:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bg(x(t - \tau)) + H, \quad t \geq 0. \quad (10)$$

Бу тенглама учун бошланғич шарт

$$x(s) = \phi(s), \quad s \in [-\tau, 0], \quad (11)$$

бу ерда ҳам $\tau > 0$ кечикув параметри, $B = (b_{ij})_{n \times n}$ – ҳақиқий матрица.

$g(x(t - \tau)) = [g_1(x_1(t - \tau)), \dots, g_n(x_n(t - \tau))]^T$ вектор-функция кечиктирилган ўзаро таъсирни ифодалайди. (10) система учун бошланғич шарт $\phi(s) = [\phi_1(s), \dots, \phi_n(s)]^T$ узлуксиз функция тарзида берилиши лозим. Ечимнинг турғунлигини ўрганганда бошланғич функциялар $\|\phi\| = \sup_{s \in [-\tau, 0]} \sum_{i=1}^n |\phi_i(s)|$ норма билан баҳоланади.

Юқорида таъкидланганидек, ҳам кечикув омили, ҳам каср тартибли ҳосила қатнашган дифференциал тенгламалар математик моделларни муқобиллаштиришда муҳим ўрин тутаяди. Диссертацияда бундай турдаги тенгламалар синфига мансуб

$$D^\alpha x(t) = -Cx(t) + Af(x(t)) + Bg(x(t - \tau)) + H, \quad t \geq 0 \quad (12)$$

нейрон тўрлари модели ўрганилган. Бу ерда D^α – Капуто маъносидаги каср тартибли ҳосила бўлиб, диссертацияда фақат $\alpha \in (0,1]$ бўлган ҳол қаралган.

V.

$$\dot{x}(t) = Dx(t)(-Cx(t) + Af(x(t)) + Bg(x(t-\tau)) + H), t \geq 0 \quad (13)$$

тенглама ва

$$x(s) = \phi(s), s \in [-\tau, 0], \quad (14)$$

бошланғич шарт билан бериладиган Кохен-Кроссберг нейрон тўри модели ўрганилган. Бу системанинг ўлчови n ўзаро таъсирда қараладиган нейронлар сонига тенг, $n \geq 2$. $x(t) = [x_1(t), \dots, x_n(t)]^T$ тўр ҳолатини англатувчи вектор функция, $D(x(t)) = \text{diag}\{d_1(x_1), \dots, d_n(x_n)\}$ – амплификация функцияси деб аталувчи диагонал матрица, $C(x(t)) = (c_1(x_1), \dots, c_n(x_n))^T$ ҳолат функциясини, $A = (a_{ij})_{n \times n}$ нейронларнинг ўзаро боғлиқлик кучини ифодаловчи салмоқ матрицаси, $B = (b_{ij})_{n \times n}$ τ кечикиш параметрига эга тармоқлардаги нейронларнинг ўзаро боғлиқлигини ифодаловчи матрица. Кечикув параметри τ ўзаро таъсирдаги нейронларнинг келиб тушган сигналга реакция оралиғидан иборат.

Каср тартибли Кохен-Кроссберг кечикувли нейрон тизимлари модели синфи куйидаги тенглама билан берилади:

$$D^\alpha x(t) = D(x(t))(-C(x(t)) + Af(x(t)) + Bg(x(t-\tau)) + H), t \geq 0, \quad (15)$$

бу ерда тартибли D^α Капуто маъносидаги каср тартибли ҳосила, диссертацияда фақат $\alpha \in (0,1]$ бўлган ҳол қаралади.

VI. Икки томонлама ассоциатив хотирали (ВАМ) нейрон тўрлари модели:

$$\begin{aligned} \dot{u}(t) &= -C_1 u(t) + A_1 f(v(t)) + B_1 f(v(t-\tau)) + H_1, \\ \dot{v}(t) &= -C_2 v(t) + A_2 f(u(t)) + B_2 f(u(t-\tau)) + H_2, \end{aligned} \quad (16)$$

бошланғич шартда

$$x(s) = \phi(s), y(s) = \Phi(s), s \in [-\tau, 0], \quad (17)$$

бу ерда $u(t) = [u_1(t), \dots, u_n(t)]^T$ ва $v(t) = [v_1(t), \dots, v_n(t)]^T$ орқали нейронларнинг оний ҳолати белгиланган. $C_1 = \text{diag}(c_1^1, \dots, c_n^1)$ ва $C_2 = \text{diag}(c_1^2, \dots, c_n^2)$ – диагонал матрицалар, $c_i^1 > 0$ ва $c_j^2 > 0$, улар ўз-ўзига тескари алоқали вазн матрицалари дейилади. $A_1 = (a_{ij}^1)_{n \times n}$, $A_2 = (a_{ij}^2)_{n \times n}$ ва $B_1 = (b_{ij}^1)_{n \times n}$, $B_2 = (b_{ij}^2)_{n \times n}$ мос равишда ўзаро таъсирдаги нейронларнинг оний ва кечиккан ҳолатларини боғловчи матрицалар, $f(v(t)) = [f_1(v_1(t)), \dots, f_n(v_n(t))]^T$ ва $g(u(t)) = [g_1(u_1(t)), \dots, g_n(u_n(t))]^T$

функциялар нейронларнинг t вақтдаги фаоллигини,
 $f(v(t-\tau)) = [f_1(v_1(t-\tau)), \dots, f_n(v_n(t-\tau))]^T$,
 $g(u(t-\tau)) = [g_1(u_1(t-\tau)), \dots, g_n(u_n(t-\tau))]^T$ функциялар эса $t-\tau$ вақтдаги
 фаоллигини ифодалайди, $H_1 = [h_1^1, \dots, h_m^1]^T$ ва $H_2 = [h_1^2, \dots, h_n^2]^T$ векторлар тўра
 ташқи таъсирни ифодаловчи катталардир.

Тадқиқотнинг мақсади:

Юқорида саналган дифференциал тенгламаларнинг сифат нуқтаи назаридан таҳлили;

кечикувли моделларда кечикув параметрининг ечимга таъсирини ўрганиш;

математик моделларнинг динамик табиатини тадқиқ этиш;

модель хусусиятларининг параметр ўзгаришига таъсирчанлигини ўрганиш ва шу асосда модель муайян маънода энг самарали бўладиган параметрларнинг қийматини аниқлаш;

каср тартибли дифференциал тенгламалар параметрларини тегишли мезонлар бўйича баҳолаш учун самарали ва муқобил ёндошувларни тақдим этиш;

берилган экспериментал маълумотларга мувофиқ системанинг параметрларини аниқлаш учун самарали ва муқобил ёндошувларни тақдим этиш (тескари масала);

саратонни даволаш жараёнининг математик модели сифатида кечикувли ва каср тартибли дифференциал тенгламаларни қўллаш;

каср тартибли ва кечикувли дифференциал тенгламалар воситасида нейрон тўрларидинамикасини тадқиқ этишдан иборат.

Кечикувчи дифференциал тенгламаларнинг биологик тизимларга татбиқлари. Кечикувчи дифференциал тенгламалар хақида умумий тушунчалар келтирилади ва уларнинг татбиқлари тўғрисида тўхталиб ўтамиз. Хусусан, кечикувли дифференциал тенгламаларнинг хотирасиз тенгламаларга нисбатан афзаллигини асословчи хусусиятлари очиб берилади. Бу бобнинг мазмуни 7 та мақолани ўз ичига олади.

1. “Биосферада сонли моделлаштиришда кечикувли дифференциал тенгламалардан фойдаланиш”. Бу ишда хотирали дифференциал тенгламаларни математик моделлаштиришга татбиқи методологияси муҳокама қилинган, кенг татбиқ доирасига эга хулосалар баён қилинган ва улар турли мисоллар воситасида асосланган. Методологияга оид фикр ва мулоҳазалар юқорида баён қилингани учун бу ўринда шу изоҳ билан чекланамиз.

2. «OITV инфекцияси юққан CD4 + T ҳужайралари ўсимтаси ва иммун реакция динамикаси учун кечикувли дифференциал модель». Бу ишда “одам иммун тақчилиги вируси” (OITV) билан хасталанганлар организмдаги динамик жараёнларнинг оддий (яъни хотирасиз) ва кечикувли дифференциал тенгламалар тарзида моделлари ўрганилган:

$$\frac{dT}{dt} = r_1 T(t) - k_1 E(t - \tau) T(t), \quad (18)$$

$$\begin{aligned} \frac{dE}{dt} = r_2 T(t) + \alpha - \mu_1 E(t - \tau) - lk_1 T(t) E(t - \tau) - \\ - k_3 E(t - \tau) V(t) - k'_2 E(t - \tau) I(t), \end{aligned} \quad (19)$$

$$\frac{dI}{dt} = k_3 E(t - \tau) V(t) + k'_2 E(t - \tau) I(t) - \mu_2 I(t), \quad (20)$$

$$\frac{dV}{dt} = N \delta I(t) - c V(t) \quad (21)$$

Моделнинг сифат хоссалари тадқиқ этилган. Жумладан, стационар ҳолатларнинг Ляпунов маъносидаги асимптотик турғунлигини кафолатловчи шартлар келтириб чиқарилган. Амалий татбиқлар эҳтиёжидан келиб чиқиб, реал жараёни рақамли симуляция қилувчи компьютер модели тақдим этилган ва назарий йўл билан олинган натижаларнинг реал жараёнларга етарли аниқликда мутаносиблиги кўрсатилган.

Сўнг моделни чуқурроқ ўрганиш мақсадида қуйидаги соддалаштирилган модель қаралган:

$$\frac{dT}{dt} = r_1 T(t) - k_1 E(t - \tau) T(t), \quad (22)$$

$$\frac{dE}{dt} = r_2 T(t) + \alpha - \mu_1 E(t - \tau) - lk_1 T(t) E(t - \tau). \quad (23)$$

Бу система иккита махсус нуктага эга. Улардан бири – (0, 0) нукта турғун эмас. Иккинчиси S^* деб белгилансин. Бу нуктанинг турғунлигида

$$\tau^* = \frac{1}{\omega_0} \arccos \left(\frac{(r_2 - lr_1) k_1 T^* \omega_0^2}{(\mu_1 + lk_1 T^*)^2 \omega_0^2 + (r_2 - lr_1)^2 k_1^2 (T^*)^2} \right). \quad (24)$$

катталиқ ҳал қилувчи роль ўйнайди унда ω_0 – чизикли яқинлашиш хос сонларининг мавҳум қисми.

Хосса 1. (22-23) системада $r_2 - lr_1 > 0$ бўлсин. У ҳолда кечикув катталиги $0 \leq \tau < \tau^*$ шартни қаноатлантирса, S^* махсус нукта локал асимптотик турғун, акс ҳолда нотурғун бўлади.

Натижа. $\tau = \tau^*$ да Хопф бифуркацияси рўй беради.

Бу натижа воситасида қуйидаги хоссани ўрнатиш мумкин бўлди:

Хосса 2. (18-21) система $r_2 > lr_1$ шартни қаноатлантирсин. У ҳолда

(a) $r_1 k_2 < k_1 \mu_2$ бўлганда $S^* = (T^*, E^*)$ махсус нукта (22-23) система учун турғунлигидан (18-21) система махсус нуктаси турғунлиги келиб чиқади;

(b) $r_1 k_2 > k_1 \mu_2$ бўлганда, аксчинча (18-21) система махсус нуктаси турғун бўлмайди.

3. «Ўсма-иммун тизимидаги ўзаро таъсир жараёнининг кечикувли модели: глобал динамика, параметрларни баҳолаш, сезгирлик таҳлили». Бу

мақолада эффектор ҳужайралар билан ўсма ҳужайралари ўртасидаги ўзаро таъсир динамикаси учун қуйидаги модель таклиф этилган:

$$\begin{aligned}\frac{dE(t)}{dt} &= \sigma + \frac{\rho E(t-\tau)T(t-\tau)}{\eta + T(t-\tau)} - \mu E(t-\tau)T(t-\tau) - \delta E(t), \\ \frac{dT(t)}{dt} &= \alpha T(t)(1 - \beta T(t)) - E(t)T(t).\end{aligned}\quad (25)$$

Модель динамикаси ҳам назарий усуллар билан, ҳам сонли усуллар билан тадқиқ этилган, жумладан, махсус нуқталар аниқланиб, уларнинг локал турғунлиги ва Хопф бифуркациялари учун баҳолар олинган. Енг кичик квадратик яқинлашишлар усулини қўллаб, клиник кузатув маълумотлари асосида система параметрларини аниқлаш масаласи ҳал этилган ҳамда даволаш усулининг даврийлиги ва барқарорлиги кечикув параметрига қай тарзда боғлиқ бўлиши тадқиқ этилган. (25) системани сонли ечиб олинган маълумотлар клиник экспериментлар натижаларига яқин бўлиб чиқиши тасдиқланган.

Бу модель муҳим амалий аҳамиятга эга бўлгани учун батафсилроқ тўхталамиз. (22-23) система тўртта махсус нуқтага эга: E_i^* ($i=0,1,2,3$). Системани бу нуқталар атрофида чизиклаштириш натижаси:

$$\dot{X}(t) = \bar{A}_1 X(t) + \bar{A}_2 X(t-\tau), \quad (26)$$

бу ерда $X = (X_1, X_2)$

$$\begin{aligned}\bar{A}_1 &= \begin{pmatrix} -\delta & 0 \\ -T_i^* & \alpha - 2\alpha\beta T_i^* - E_i^* \end{pmatrix}, \\ \bar{A}_2 &= \begin{pmatrix} \frac{\rho T_i^*}{\eta + T_i^*} - \mu T_i^* & \frac{\rho E_i^*}{\eta + T_i^*} - \frac{\rho E_i^* T_i^*}{(\eta + T_i^*)^2} - \mu E_i^* \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Махсус нуқталарнинг турғунлигини аниқлашга Ляпунов-Красовский функционали татбиқ этилди. Бунинг учун матрицалар учун Шур тўлдирувчиси тушунчасидан фойдаланилади:

Лемма 1 (Шур тўлдирувчиси ҳақида). Мос ўлчовли Ω_1, Ω_2 ва Ω_3 ўзгармас матрицалар берилган ва $\Omega_1^T = \Omega_1$, $\Omega_2^T = \Omega_2 > 0$ бўлсин. У ҳолда $\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$

матрица манфий аниқланган бўлиши учун

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ * & -\Omega_2 \end{bmatrix} < 0 \text{ ёки } \begin{bmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{bmatrix} < 0$$

шарт зарур ва етарли (бу ерда “*” бирор симметрик матрицани белгилайди, $\Omega > 0$, $\Omega < 0$ тенгсизликлар мос равишда Ω матрицанинг мусбат ёки манфий аниқланганини билдиради).

Лемма 2 (Йенсен леммаси). Ҳар қандай мусбат симметрик $M \in \mathbb{R}^{n \times n}$ матрица, $\tau > 0$ ўзгармас ва $X(s) \in \mathbb{R}^n$ вектор функция учун интеграллар аниқланганлик шартида

$$-\int_{t-\tau}^t X^T(s)MX(s)ds \leq -\frac{1}{\tau} \left(\int_{t-\tau}^t X(s)ds \right)^T M \left(\int_{t-\tau}^t X(s)ds \right)$$

тенгсизлик ўринли бўлади.

Теорема 3. Агар симметрик

$$\begin{bmatrix} P\bar{A}_1 + \bar{A}_1^T P + W_1 - W_2 & P\bar{A}_2 + W_2 & \tau \bar{A}_1^T W_2 \\ * & -W_1 - W_2 & \tau \bar{A}_2^T W_2 \\ * & * & -W_2 \end{bmatrix} < 0. \quad (27)$$

тенгсизлик ўринли бўладиган мусбат P , W_1 ва W_2 матрицалар мавжуд бўлса, (26) система глобал асимптотик турғун бўлади.

4. Тиббиёт соҳасида математик моделлаштиришнинг долзарб муаммоларидан бири – оптимал бошқарув масаласидир. Касалликларни даволаш жараёни бошқарилувчи система бўлиши равшан. Шунинг учун ўзидан даволаш жараёнини оптимал бошқарув масаласи юзага келади. Умуман олганда, оптимал бошқарувнинг мавжудлигини исботлаш ва уни излаб топиш учун зарурий шартларни қуришга Понтрягиннинг максимум принципини қўллаш мумкин, аммо бунинг учун, биринчидан, бошқарилувчи моделни имкон қадар адекват қуриш, иккинчидан, мақсад функцияни топиш лозим.

Диссертацияга кирган мақолалардан бир туркуми саратон ва юқумли касалликларни даволаш жараёнининг бошқарилувчи моделини қуриш ва оптимал бошқарувни топиш масаласига бағишланган. Бунда моделга саратонни даволашнинг анъанавий химиотерапия усулини иммун системани кучайтириш билан қўшиб амалга ошириш асос қилиб олинди.

Бу модель

$$\begin{aligned} \frac{dE(t)}{dt} = & \sigma + \frac{\rho E(t-\tau)T(t-\tau)}{\eta + T(t-\tau)} - \mu E(t-\tau)T(t-\tau) - \\ & - \delta E(t) - \alpha_1 (1 - e^{-u(t)}) E(t), \end{aligned} \quad (28)$$

$$\frac{dT(t)}{dt} = r_2 T(t)(1 - \beta T(t)) - nE(t)T(t) - c_1 N(t)T(t) - \alpha_2 (1 - e^{-u}) T(t), \quad (29)$$

$$\frac{dN(t)}{dt} = r_3 N(t)(1 - \beta_2 N(t)) - c_2 T(t)N(t) - \alpha_3 (1 - e^{-u}) N(t), \quad (30)$$

$$\frac{du(t)}{dt} = v(t) - d_1 u(t) \quad (31)$$

кечикувли дифференциал тенгламалар системасидан иборат бўлиб, $v(t)$ – бошқарув параметридир. Чекловлар ҳам бошқарув параметрига, ҳам ҳолатларга қўйилади:

$$0 \leq v(t) \leq v_{\max} < \infty \quad t \in [0, t_f], \quad (32)$$

$$k(N) = N - 0.75 \geq 0, \quad t \in [0, t_f]. \quad (33)$$

Мақсад функцияси сифатида

$$J(v) = \int_0^{t_f} \left(E(t) - T(t) - \frac{B_v}{2} [v(t)]^2 \right) dt \quad (34)$$

кўринишдаги функционал танланган. Унда B_v даволашда химиотерапия ҳиссасини ифодалайди. Бошқарув параметри узлуксиз бўлаккли функциялар синфидан олинади. (32) чекловдаги v_{\max} катталик химиотерапиянинг максимал ҳиссасини белгилайди, $v(t) = 0$ тенглик эса тўлиқ иммунотерапия кўлланишини билдиради. Шундай қилиб, жоиз бошқарувлар синфи

$$V_{ad} = \left\{ v \in L^\infty \left([0, t_f], \mathbb{R} \right), \mid 0 \leq v(t) \leq v_{\max} < \infty, \forall t \in [0, t_f] \right\}.$$

тўпладан иборат.

Теорема 4. а) Жоиз бошқарувлар синфи бўш эмас, кавариқ ва ёпиқ тўплам. б) Оптимал бошқарув мавжуд ва ягона.

Потрягиннинг максимум принципи асосида ҳисобланган оптимал даволаш стратегияси амалиётда қўлланганда ўсма хужайраларининг камайиши ва эффектор хужайраларининг ўсиши кузатилди ва саратонни даволашда химиотерапия билан иммунотерапия қўшиб олиб борилиши самарадор эканлигини илмий асосланди.

5. “Хотирага эга бўлган биологик системаларни кечикувли дифференциал тенгламалар ёрдамида сонли моделлаштириш”. Бу мақолада биологик системалар: хужайра ўсиши, эпидемиология, физиология, иммунологияда қараладиган жараёнларни моделлаштиришда хотирали дифференциал тенгламалар самарадорлиги кўрсатилган. Ўсма хужайралар ва иммунитет реакция хужайралари орасидаги ўзаро таъсирни ифодалаш учун кечикувли системалар асосида бошқарилувчи модель қурилган. Шундай сўнг кечикувли ечимларнинг турғунлиги таҳлил қилинган ва параметрларни баҳолаш масаласи ўрганилган. Қаралаётган моделни таъсирчанлигини текшириш, структуравий ўзгартиришларга нисбатан турғунлик муаммоси амалий татбиқлар учун долзарб бўлгани учун алоҳида эътибор берилган. Бу ўринда биологик системаларда бошқариш масаласи жуда кам ўрганилганини қайд этиб ўтиш ўринли.

$u(t)$ бошқарув параметрига эга m – ўлчамли кечикувли дифференциал тенгламалар системаси

$$y'(t) = f(y(t), y(t - \tau_1), y(t - \tau_2), \dots, y(t - \tau_d), u(t), t) \quad (35)$$

кўринишда бўлади. Умумий ҳолда мақсад функцияси

$$J_0(u) = \Phi_0(y(t)) + \int_0^{t_f} F(y(t), y(t - \tau_1), y(t - \tau_2), \dots, y(t - \tau_d), u(t), t) dt, \quad (36)$$

тарзида олинади. Мақолада кўрилган моделларда бошқарув параметрига $a \leq u(t) \leq b$ чекланиш, ҳолат ўзгарувчисига эса $0 \leq y(t) \leq c$ чекланиш кўйилган (барча тенгсизликлар координаталар бўйича деб тушунилади).

$E(t)$ – эффектор хужайралар популяцияси катталигини, $T(t)$ – ўсмадаги хужайралар популяцияси катталигини, $N(t)$ – нормал хужайралар популяциясини белгиласин. Химиотерапия салмоғи $u(t)$, иммунотерапия салмоғи $w(t)$ бўлсин. Популяциялар бир жинсли тўқима ҳосил қилади деган фаразда модель қуйидаги кўринишда бўлади:

$$\begin{aligned} \frac{dE(t)}{dt} &= \sigma + \frac{\rho E(t-\tau)T(t-\tau)}{\eta + T(t-\tau)} - \mu E(t-\tau)T(t-\tau) - \\ &- \delta E(t) - \alpha_1(1 - e^{-u})E(t) + \omega(t)s_1, \\ \frac{dT(t)}{dt} &= r_2T(t)(1 - \beta T(t)) - nE(t)T(t) - c_1N(t)T(t) - \alpha_2(1 - e^{-u(t)})T(t), \\ \frac{dN(t)}{dt} &= r_3N(t)(1 - \beta_2N(t)) - c_2T(t)N(t) - \alpha_3(1 - e^{-u(t)})N(t), \\ \frac{du(t)}{dt} &= v(t) - d_1u(t) \end{aligned} \quad (37)$$

Оптималь бошқарув масаласини қўйиш учун мақсад функционалини тайинлаш лозим. Қаралаётган моделда бундай функционал даволаш жараёнида ўсма хужайраларини камайтириш, эффектор хужайраларини қўпайтириш каби мулоҳазалар асосида қурилиши табиий. Шундай фаразлар асосида мақсад функционали сифатида

$$J(v, \omega) = \int_0^{t_f} \left(E - T - \left[\frac{B_v}{2} [v(t)]^2 + \frac{B_\omega}{2} [\omega(t)]^2 \right] \right) dt, \quad (38)$$

катталиқ танланди. Яъни, оптималь бошқарув (38) функционалга энг катта қиймат берувчи бошқарув функцияларини танлашдан иборат:

$$J(v^*, \omega^*) = \max \{ J(v, \omega) : (v, \omega) \in W \}, \quad (39)$$

бу ерда

$$W = \left\{ J(v, \omega) : (v, \omega) \ 0 \leq v(t) \leq v_{\max} < \infty, ; \ 0 \leq \omega(t) \leq \omega_{\max} < \infty, \ \forall t \in [0, t_f] \right\}. \quad (40)$$

– бошқарув тўплами. Бу моделда оптималь бошқарув $v^*(t)$ ва $\omega^*(t)$ функцияларининг мавжудлиги оптималь бошқарув назариясининг тегишли теоремаларидан бевосита келиб чиқади. Оптималь бошқарув функциялари Понтрягиннинг максимум принципи асосида топилади.

Теорема 5. Ω соҳада $J(u(t), \omega(t))$ ни минималлаштирувчи $v^*(t)$ ва $\omega^*(t)$ оптималь функциялар жуплиги мавжуд. Оптималь бошқарувлар қуйидаги қўшма системанинг узлуксиз $\lambda_i, i = 1, 2, 3, 4$, ечимлари

$$\begin{aligned}
\lambda_1'(t) &= -1 + \lambda_1(t) \left[\delta + a_1 \left(1 - e^{-u^*} \right) \right] + \\
&\quad + \lambda_2(t) n T^* + \lambda_1(t + \tau) \chi \left[0, t_f - \tau \right] \left[\mu T^* - \frac{\rho T^*}{\eta + T^*} \right], \\
\lambda_2'(t) &= 1 + \lambda_2 \left[-r_2 + 2r_2 \beta T^* + n E^* + c_1 N^* + a_2 \left(1 - e^{-u^*} \right) \right] + \\
&\quad + \lambda_3 c_2 N^* + \chi \left[0, t_f - \tau \right] \lambda_1(t + \tau) \left[\frac{\rho E^* T^*}{(\eta + T^*)^2} - \frac{\rho E^*}{\eta + T^*} + \mu E^* \right], \quad (41) \\
\lambda_3'(t) &= \lambda_2 c_1 T^* - \lambda_3 \left[r_3 - 2r_3 \beta_2 N^* - c_2 T^* - a_3 \left(1 - e^{-u^*} \right) \right] - \gamma, \\
\lambda_4'(t) &= -\lambda_1(t) a_1 e^{-u^*} E^* + \lambda_2(t) a_2 e^{-u^*} T^* + \lambda_3(t) a_3 e^{-u^*} N^* + \lambda_4(t) d_1,
\end{aligned}$$

ва трансверсаллик шартлари

$$\lambda_i(t_f) = 0, \quad i = 1, 2, 3, 4 \quad \text{ва} \quad \chi_{[0, t_f - \tau]} = \begin{cases} 1 & \text{if } t \in [0, t_f - \tau], \\ 0 & \text{otherwise} \end{cases} \quad (42)$$

орқали аниқланади. Бундан ташқари, қуйидаги хоссалар ўринли бўлади:

$$v^* = \min \left(v_{\max}, \frac{\lambda_4}{B_v} \right), \quad \omega^* = \min \left(\omega_{\max}, \frac{\lambda_1 s_1}{B_\omega} \right). \quad (43)$$

Кечикувли каср тартибли дифференциал тенгламалар. Ҳозирги замон математикасида каср тартибли дифференциал-интеграл ҳисоб, айниқса каср тартибли дифференциал тенгламалар жадал ривожланаётган соҳалардан бири бўлиб турибди. Бу унинг техника ва математик моделлаштиришда кўплаб татбиқлар топаётгани билан изоҳланади.

Бутун тартибли ва каср тартибли ҳосилалар орасида асосий фарқ – бири локал табиатли, иккинчиси эса глобал табиатли эканида. Бутун тартибли ҳосила физик катталиқнинг кичик оралиқ ва соҳалардаги ўзгаришини ўзида акс эттирса (яъни локал табиатли бўлса), каср тартибли ҳосилада бу катталиқнинг бутун соҳадаги ўзгаришлари билан боғлиқдир (яъни глобал табиатлидир). Шу сабабли физик, химик ва биологик феноменларни аниқ моделлаштиришда каср тартибли дифференциал тенгламали моделлар кўпроқ муқобил ҳисобланади.

Бу боб мазмунини кечикувли каср-тартиб ҳосилали дифференциал тенгламалар синфи тадқиқ этилган мақолалар ташкил этган.

“II турдаги Ҳоллинг функционал реакцияга эга каср-тартибли кечикувли ўлжа-йиртқич системалар” мақоласида математик биологиянинг асосий объектларидан бўлган “ўлжа-йиртқич” системаси каср-тартибли кечикувли дифференциал тенгламалар орқали умумлаштирилган вариантда ўрганилди. Каср ҳосила тартиби $0 < \alpha \leq 1$ оралиқда бўлганда, стационар ҳолатлар ва уларнинг локал ва глобал турғунликлари кечикиш параметрига ўзгаришига қараб текширилди ва Ҳопф бифуркацияси тадқиқ қилинди ҳамда каср

тартибли кечикувли дифференциал тенгламалар ечимларнинг турғунлик шартларини яхшилаши ва модель динамикасини бойитиши аниқланди.

“Ўлжа-йиртқич” системаси учун мазкур модель

$$\begin{aligned} D^\alpha x(t) &= rx(t) \left(1 - \frac{x(t)}{K} \right) - \frac{\beta x(t)y(t-\tau)}{1 + \sigma x(t)} \\ D^\alpha y(t) &= \frac{\beta x(t)y(t-\tau)}{1 + \sigma x(t)} - ay(t), \quad 0 < \alpha \leq 1, t \geq 0 \end{aligned} \quad (44)$$

тенгламалардан иборат. Бошланғич қийматлар $x(0) > 0$ ва $t \in [-\tau, 0]$ учун $y(t) = \psi(t) > 0$ (бу ерда $\psi(t)$ силлиқ функция). Сўнг, модель динамикасидаги кечикиш параметри τ ва α каср-тартибнинг моделга таъсири тадқиқ қилинди. Ўқувчига қулайлик учун тегишли таърифларни келтирайлик.

$f(t)$ функциянинг $\alpha \in \mathbb{R}^+$ тартибли интегралли

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

формула билан аниқланади.

$f(t)$ функциянинг Капуто маъносидаги α тартибли ҳосиласи

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau) d\tau,$$

формула билан аниқланади, бу ерда $n-1 < \alpha < n \in \mathbb{Z}^+$.

(44) система кўзғалмас нуқталарининг турғунлигини ўрганиш учун даставвал нолдан фарқли кўзғалмас нуқтаси $\bar{x}(t) = x(t) - x^*$, $\bar{y}(t) = y(t) - y^*$ алмаштириш орқали координата бошига олиб келинади, сўнг чизиклаштирилади:

$$\begin{aligned} D^\alpha x(t) &= m_1 x(t) + m_2 y(t-\tau) \\ D^\alpha y(t) &= n_1 x(t) + n_2 y(t) + n_3 y(t-\tau) \quad 0 < \alpha \leq 1, \end{aligned} \quad (45)$$

бу ерда

$$\begin{aligned} m_1 &= r - \frac{2x^*}{K} - \frac{\beta y^*}{1 + \sigma x^*} + \frac{\sigma \beta x^* y^*}{(1 + \sigma x^*)^2}, \\ m_2 &= -\frac{\beta x^*}{1 + \sigma x^*}, \quad n_1 = \frac{\beta y^*}{1 + \sigma x^*} - \frac{\sigma \beta x^* y^*}{(1 + \sigma x^*)^2}, \\ n_2 &= -a, \quad n_3 = \frac{\beta x^*}{1 + \sigma x^*}. \end{aligned}$$

(45) системанинг ҳар икки томонига Лаплас алмаштириши қўлланса,

$$s^\alpha X_1(s) = m_1 X_1(s) + s^{\alpha-1} \varphi_1(0) + m_2 e^{-s\tau} (X_2(s) + \int_{-\tau}^0 e^{-st} \varphi_2(t) dt) \quad (46)$$

$$s^\alpha X_2(s) = n_1 X_1(s) + n_2 X_2(s) + s^{\alpha-1} \varphi_2(0) + n_3 e^{-s\tau} (X_2(s) + \int_{-\tau}^0 e^{-st} \varphi_2(t) dt).$$

алгебраик система ҳосил бўлади, бунда $\bar{x}(t) = \varphi_1(t)$ ва $\bar{y}(t) = \varphi_2(t)$, $t \in [-\tau, 0]$ бошланғич шартлар. (45) система вектор-матрицалар тилида

$$\Delta(s) \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} k_1(s) \\ k_2(s) \end{pmatrix} \quad (47)$$

кўринишда ёзилади, бу ерда

$$\Delta(s) = \begin{pmatrix} s^\alpha - m_1 & -m_2 e^{-s\tau} \\ -n_1 & s^\alpha - n_2 - n_3 e^{-s\tau} \end{pmatrix}$$

ва

$$k_1(s) = s^{\alpha-1} \varphi_1(0) + m_2 e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi_2(t) dt,$$

$$k_2(s) = s^{\alpha-1} \varphi_2(0) + n_3 e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi_2(t) dt.$$

Кечикувли дифференциал тенгламалар назариясига кўра, $\det \Delta(s) = 0$ характеристик тенгламанинг барча илдизлари комплекс текисликнинг чап ярим очиқ соҳасига тегишли, яъни $\operatorname{Re}(s) < 0$ бўлса, ноль ечим турғун бўлиши маълум.

“Монод-Холдейн функционал реакциясига мос каср-тартибли кечикувли “ўлжа-йиртқич” системанинг турғунлиги” мақоласида мазкур моделнинг яна бир варианты тадқиқ этилган. Бу ҳолда тегишли система

$$\begin{aligned} D^\alpha x(t) &= px(t) \left(1 - \frac{x(t)}{k}\right) - \frac{\delta x(t) z(t - \tau)}{\varphi + \beta x^2(t)}, \\ D^\alpha y(t) &= \frac{c \delta x(t) z(t - \tau)}{\varphi + \beta x^2(t)} - (\kappa + \mu_1) y(t), \\ D^\alpha z(t) &= \kappa y(t) - \mu_2 z(t), \end{aligned} \quad (48)$$

кўринишга эга бўлади ($x(0) = x_0 > 0$, $y(0) = y_0 > 0$, $z(s) = \chi(s) > 0$, $s \in [-\tau, 0]$) – бошланғич шартлар, $\chi(s)$ силлиқ функция, D^α Капуто маъносидаги α -каср тартибли ҳосила, $0 < \alpha \leq 1$. Система параметрлари унинг таркибий ва динамик кўрсаткичларини ифодалайди.

Теорема 7. Ҳар бир $(x_0, y_0, z_0) \in \Omega = \{(x, y, z) \in R^3 : \max\{|x|, |y|, |z|\} \leq K\}$ – учун (48) система Ω соҳада ягона ечимга эга.

(48) система кечикувсиз моделдан фарқли равишда учта махсус нуктага эга: тривиал $E_0(0, 0, 0)$ махсус нукта, йиртқичлар сони 0 бўлган ҳолга мос абцисса ўқида ётувчи $E_1(k, 0, 0)$ махсус нукта ва энг қизиқ ҳол бўлган ички

$E_2(x^*, y^*, z^*)$ махсус нукта: $y^* = \frac{\mu_2 z^*}{\kappa}$, x^* ва z^*

$$z^* = \frac{p}{\delta} (\varphi + \beta (x^*)^2) \left(1 - \frac{x^*}{k}\right), \quad \beta (x^*)^2 - \frac{c \kappa \delta}{(\kappa + \mu_1) \mu_2} x^* + \varphi = 0.$$

квадрат тенгламанинг мусбат илдизлари. $E_2(x^*, y^*, z^*)$ махсус нуктанинг асимптотик турғунлиги шартлари келтириб чиқарилган.

Нейрон тўрлари билан боғлиқ кечикувли дифференциал тенгламалар. Ҳозирги вақтда нейрон тўрлари назарияси жадал суръатлар

билан ривожланаётган соҳалардан биридир. Нейронтўрлари оптик электроника, тасвирларни қайта ишлаш, квант қурилмалари, маълумотларни филтрлаш, психологик нейрон системаларини спатиотемпорал анализисоҳаларида кенг қўлланилаётгани сабабли бу соҳага қизиқиш муттасил ортиб бормоқда. Нейронлар ўртасида сигнал алмашилганда кечикиш муҳим факторлардан бири ҳисобланади. Мисол учун, бирор моделга кечикувли таъсир тегишли системада хаос ёки турғун бўлмаган ҳаракатга келтириб чиқариши мумкин. Шу сабабли, кечикувли нейрон тўрларида турғунлик масаласи муҳим аҳамият касб этади. Каср-тартибли моделларни бутун тартибли моделлар билан солиштирганда, уларда чексиз хотиранинг мавжудлиги уларнинг асосий афзаллигидир. Охириги йилларда нейрон тўрларининг каср-тартибли ва кечикувли динамик системаси биргаликда қаралган моделларини ўрганиш муҳим натижалар бермоқда.

Диссертацияда реал физик ва биологик жараёнлар моделларининг турғунлик, синхронлик, диссипативлик ва бошқа муҳим ҳоссалари ўрганилади.

“Кечикувли Кохен-Кросберг ВАМ нейрон тўрларининг турғунлиги” деб номланган бобда чексиз вақт оралиғида турғунликни кафолатловчи натижалар олинган. Бунда Ляпунов функционали ва алгебранинг бир неча теоремасидан фойдаланилган. Кечикувли Кохен-Кросберг ВАМ нейрон тўрлари модели сифатида қуйидаги система қаралган:

$$\begin{aligned} \dot{x}_i(t) &= -p_i(x_i(t)) \{ q_i(x_i(t)) - \sum_{j=1}^m r_{ij} f_j(y_j(t)) - \sum_{j=1}^m s_{ij} f_j(y_j(t - \tau_{ij})) \}, \\ \dot{y}_j(t) &= -P_j(y_j(t)) \{ Q_j(y_j(t)) - \sum_{i=1}^n R_{ji} g_i(x_i(t)) - \sum_{i=1}^n S_{ji} g_i(x_i(t - \sigma_{ji})) \}, \end{aligned} \quad (49)$$

бу ерда $i, j = 1, 2, \dots, n$, $x_i(t)$ ва $y_j(t)$ – мос равишда t вақтдаги F_X соҳанинг i – тўрдаги ва F_Y соҳанинг j – тўрдаги ҳолат функциялари, p_i, P_j – абстракт функциялар ва q_i, Q_j – тезлик функциялари. f_j ва g_i лар мос равишда t вақтдаги F_Y даги j – нейрон ва F_X даги i – нейрон активация функциялари. (49) системада бошланғич қийматлар қуйидагича аниқланади:

$$x_i(s) = \psi_j(s), \quad y_j(s) = \varphi_i(s), \quad s \in [-\gamma, 0].$$

Бу ерда $\gamma = \max_{1 \leq i \leq n, 1 \leq j \leq n} \{ \tau_{ij}, \sigma_{ji} \}$, ва $\psi_j(\cdot), \varphi_i(\cdot)$ лар $[-\gamma, 0]$ оралиқда аниқланган ҳақиқий қийматли узлуксиз функциялар. (49) система ечимларига мос мадел турғунлиги учун етарли шартлар келтириб чиқарилган.

Хотирали дифференциал тенгламаларни сонли ечиш. Хотирали дифференциал тенгламаларда кечикув қатнашгани, шунинг учун бошланғич қиймат функция кўринишида бўлгани сабабли, сонли ечиш ҳали-ҳануз нисбатан жуда кам тадқиқ этилган соҳалардан бири бўлиб қолмоқда. Диссертацияда хотирали дифференциал тенгламалар ечимига сонли яқинлашишларни топиш учун ноошкор Рунге-Кутте методини қўллаш

методологияси ривожлантирилган ҳамда бу методнинг турғунлик хоссалари ўрганилган.

Каср-тартибли дифференциал тенгламаларни сонли ечишга келсак, бу муаммо фақат сўнгги йиллардагина кун тартибига чиқди. Бу йўналишда ҳозирча айрим усулларгина таклиф этилди. Тадқиқотлар шуни кўрсатдики, каср тартибли дифференциал тенгламалар ёки умуман интегро-дифференциал тенгламаларни сонли ечиш компьютердан жуда катта хотира талаб этади. Шу сабабли имкон қадар самарали тақрибий ечиш усулларини ишлаб чиқиш долзарб масаладир. Бу бобда хотирали, яъни каср-тартибли ва кечикувлидифференциал тенгламаларни сонли ечишнинг шартсиз турғун усуллари таклиф этилади. Бу усуллар турғунлик соҳаси чегарасига яқин жойлашган нозик табиатли масалаларга қўллаш учун ҳам қулайдир. Биринчи навбатда Волterra типдаги кечикувли интегро-дифференциал тенгламалар учун сонли ечишнинг янги техникаси киритилади. Бу техника тенгламанинг дифференциал қисми учун ношкор Рунге-Кутте методини қўллаш, интеграл қисми учун эса Буль квадратура қоидасини қўллашга асосланган. Ҳар икки қисм ҳам ўз ҳолича етарлича ўрганилган, жумладан, турғунлик шартлари яхши маълум. Шунга қарамасдан, равшанки, уларнинг комбинацияси алоҳида тадқиқот талаб этади.

Шундай қилиб,

$$\begin{aligned} y'(t) &= f(t, y(t), y(t - \tau), \int_{a(t)}^t g(t, s, y(s)) ds), \quad \text{for } t \geq 0, \\ y(t) &= \phi(t), \quad t \leq 0, \end{aligned} \quad (50)$$

интегро-дифференциал тенглама учун Коши масаласи берилган бўлсин, бу ерда τ ўзгармас мусбат сон. Формуладаги интеграл $a(t) = 0$ бўлган ҳолда (50) тенглама чегараланмаган кечикувли, $a(t) = t - \tau$ бўлганда эса чегараланган кечикувли деб юритилади. f, g функциялар барча аргументлари бўйича етарлича силлиқ ва $\phi(t)$ бошланғич функция узлуксиз деб фараз қилинади. (2) тенгламани сонли ечиш учун нафақат қаралаётган тўр нуқталарида ечимни яқинлаштириш, балки, улардан ташқаридаги нуқталарда $\int_{a(t)}^t g(t, s, y(s)) ds$ интеграл ҳадни ва $y(t - \tau)$ кечикувли ҳадни яқинлаштириш лозим бўлади. (50) масалани сонли ечишнинг бу ерда таклиф этилаётган техникасига кўра даставвал

$$I_n = \int_{a(t_n + c, h_n)}^{t_n + c, h_n} g(t, s, y(s)) ds$$

интегралга квадратура формуласи қўлланади, сўнг $\Delta = \{0 < t_1 < \dots < t_n < \dots < t_i = T\}$ тўрға мос Рунге-Кутта методининг ношкор схемасига мурожаат қилинади:

$$y_{n+1} = y_n + h_n \sum_{r=1}^s b_r K_{n+1}^r, \quad (51)$$

бу ерда K_{n+1}^r коэффициентлар

$$K_{n+1}^r = f(t_n + c_r h_n, (1 - v_r) y_n + v_r y_{n+1} + h_n \sum_{j=1}^{r-1} x_{rj} K_{n+1}^j, y(t_n + c_r h_n - \tau), I_n). \quad (52)$$

формула билан аниыланади.

Амалиётда энг кўп қўлланадиган тўртинчи тартибли ($s = 5$) дискрет ноошкор

Рунге-Кутта схемаси қуйидаги Батчер жадвали билан аниқланади:

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{20} & \frac{29}{4000} & \frac{361}{8000} & \frac{29}{4000} & \frac{361}{8000} & -\frac{19}{8000} \\ \frac{19}{20} & \frac{3971}{4000} & \frac{19}{8000} & \frac{3971}{4000} & \frac{19}{8000} & -\frac{361}{8000} \\ \frac{1}{2} & \frac{11}{16} & \frac{11}{32} & \frac{11}{16} & \frac{1}{32} & \frac{267}{608} \\ & & & & & \frac{25}{684} - \frac{25}{36} \\ & & & & & & -\frac{43}{228} & -\frac{43}{228} & -\frac{43}{228} & \frac{25}{57} & \frac{25}{57} & \frac{1}{2} \end{array}$$

Юқорида таъкидланганидек, (50) масалани сонли ечиш учун дискрет ноошкор Рунге-Кутта схемаси етарли эмас. Мақсадга эришиш учун тўрға кирмайдиган нуқталарни ҳам қараш керак. Бу қуйидаги жадвал билан аниқланадиган

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \frac{1}{20} & \frac{29}{4000} & \frac{29}{4000} & \frac{361}{8000} & -\frac{19}{8000} & 0 \\ \frac{19}{20} & \frac{3971}{4000} & \frac{3971}{4000} & \frac{19}{8000} & -\frac{361}{8000} & 0 \\ \frac{1}{2} & \frac{11}{16} & \frac{11}{16} & \frac{1}{32} & \frac{267}{608} & \frac{25}{684} \\ & & & & & & b_1(\theta) & b_2(\theta) & b_3(\theta) \end{array}$$

схемага мувофиқ амалга оширилади, бу ерда

$$\begin{aligned}
b_1(\theta) &= -\frac{1}{228}\theta(1200\theta^3 - 2714\theta^2 + 1785\theta - 228), \\
b_2(\theta) &= \frac{1}{228}\theta^2(1200\theta^2 - 2086\theta + 843), \\
b_3(\theta) &= \frac{25}{171}\theta^2(40\theta^2 - 86\theta + 49), \\
b_4(\theta) &= -\frac{25}{171}\theta^2(40\theta^2 - 74\theta + 31), \\
b_5(\theta) &= -\frac{1}{2}\theta^2(2\theta - 3).
\end{aligned} \tag{53}$$

Таклиф этилган сонли ечиш методи турғун бўлиши, масала бирк (stiff) бўлмаган ҳолларда ҳам қониқарли натижа бериши бир неча мисолда синаб кўрилган.

“Қаср тартибли кечикувли параболик турдаги хусусий ҳосилали дифференциал тенгламалар учун ҳисоблаш методлари” мақоласида

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = F(t,u(x,t),Lu(x,t),u(x,t-\tau)), x \in \Omega, t \geq 0, \quad 0 < \alpha \leq 1, \tag{54}$$

кўринишдаги тенглама қаралган, бу ерда F узлуксиз функция, $t - \tau \leq t$ ва L фазовий кординаталар бўйича эллиптик оператор, кечикув параметри τ ўзгармас, ё вақт функцияси $\tau(t)$ ёки ҳатто ҳолатга боғлиқ $\tau(t,y)$ функция бўлиши мумкин.

Мисол тарихида ОИД вируси юктирилган CD4+T белгили ҳужайралар билан боғлиқ саратон-иммун системасининг қаср тартибли дифференциал тенглама кўринишидаги модели сонли таҳлил усули билан ўрганилган. Бу усул Капуто маъносидаги α тартибли қаср ҳосила ва ноошкор Эйлер яқинлашишлари воситасида амалга оширилган.

Кечикувли дифференциал тенгламалар учун параметрик баҳолаш ва сезгирлик анализи. Бу бўлим мазмуни худди шу номдаги мақоладан ташкил топган. Бу ерда гап дифференциал тенгламалар назариясининг тескари масаласи устида боради: у дифференциал тенглама ечими тўғрисидаги муайян маълумотлар асосида тенгламадаги параметрларнинг мос қийматини топиш масаласидир. Бунда табиий равишда дифференциал тенгламалар учун сезгирлик хоссаси юзага чиқади: параметрларнинг кичик ўзгариши қай даражада тенглама хоссаларига таъсир қилишини баҳолаш лозим бўлади. Диссертацияда бу масалалар кеикувли дифференциал тенгламалар учун ўрганилган.

Шундай қилиб, параметрларга боғлиқ тайин тенгламалар системаси берилган деб фараз қилайлик. Системани кузатиб, муайян маълумотлар тўпланди. Агар маълумотлар мутлақ аниқликда йиғилган бўлса, масала анча содда бўлади. Аммо ҳар қандай ўлчаш тақрибий бўлиши, ўлчаш жараёнига турли омиллар таъсир қилиши яхши маълум. Кўпинча бу омиллар тасодифий табиатли бўлиб, уларнинг ўлчашга таъсири тақсимот функцияси билангина

берилиши мумкин. Бу омилни ҳисобга олиш стохастик кечикувли дифференциал тенглама тушунчасига олиб келади:

$$\begin{aligned} \mathbf{y}'(t) &= \mathbf{f}(t, \mathbf{y}(t), \mathbf{y}(t - \tau); \mathbf{p}), \quad t \in [0, T], \\ \mathbf{y}(t) &= \boldsymbol{\psi}(t, \mathbf{p}), \quad t \in [-\tau, 0]. \end{aligned} \quad (55)$$

Бу система ечим тўғрисида жамланган $\{t_j; Y_j^i\}_{j=1}^N$ кузатувлар мажмуаси асосида $\mathbf{p} \in R^L$ параметр қийматини баҳолаш масаласи учун модель бўла олади. (55) системада \mathbf{f} вектор функция барча аргументлари бўйича етарлича силлиқ, $\mathbf{y}(t) \in R^M$, $\mathbf{y}(t - \tau) \in R^M$, $\mathbf{p} \in R^L$.

Мавжуд ёндашувлардан бири – ечим қийматини ўлчаш натижалари

$$Y_{ij} = y_j(t_i) + \sigma_j \varepsilon_{ij} \quad (56)$$

кўринишда деб фараз қилишдан иборат, бунда ε_{ij} – Гаусс тақсимотига эга ўзаро боғлиқ бўлмаган тасодифий катталиклар (“Оқ шовқин”). Эҳтимоллий максимум принципига кўра

$$\Phi(\mathbf{P}) = \sum_{i=1}^M \varepsilon_i^T \omega_i(\sigma) \varepsilon_i \equiv \frac{1}{N} \sum_{i=1}^M \sum_{j=1}^N \frac{[y_j(t_i; \mathbf{p}) - Y_{ij}]^2}{2\sigma_j^2}. \quad (57)$$

баҳолаш функциясини \mathbf{p} бўйича минималлаштириш, яъни

$$\Phi(\hat{\mathbf{p}}) =: \min_{\mathbf{p}} \Phi(\mathbf{p}) \equiv \max_{\mathbf{p}} L(\mathbf{p}) \quad (58)$$

масала ҳосил бўлади, бу ерда $L(\mathbf{p}) = [\exp(-\varepsilon_{ij}^2 / 2\sigma_j^2)] / \sqrt{2\pi\sigma_j^2}$.

Яна бир ёндашувга кўра (57)

$$\Phi_L(\mathbf{p}) = \frac{1}{N} \sum_{i=1}^M \sum_{j=1}^N [\log y_j(t_i, \mathbf{p}) - \log Y_{ij}]^2 / 2\sigma_j^2. \quad (59)$$

баҳолаш функцияси олинади (бунда $y^j(t_i, \mathbf{p}) > 0$ деган шарт қўйилади).

$\Phi(\mathbf{p})$ функция минимуми математик оптималлаштиришда қўлланадиган усуллардан бири билан топилиши мумкин. Тегишли минимум нуқтаси қаралаётган масаланинг ечими бўлади.

“Кечикувли динамик системалар учун сезгирлик таҳлили” мақоласида юқоридаги масаланинг давоми – системанинг параметр ўзгаришларига нисбатан қай даражада реакция қилиши ўрганилган. Бунда параметр термини одатдагидан кенгроқ тарзда тушунилиши мумкин: тенгламада қатнашувчи сонли параметрлар, бошқарув параметрлари, кечикув параметри ва бошқалар. “Бундай параметрларнинг кичик ўзгариши системада қандай акс этади?” деган савол катта амалий аҳамитга эга эканлиги равшан. Мақоланинг бош мақсади – кечикув параметрини ўз ичига оладиган математик моделларнинг сезгирлиги тўғрисида умумий назарияни баён қилишдир.

Қуйидаги дифференциал тенглама билан тавсифланадиган модель қаралади:

$$\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t), \mathbf{y}(t - \tau), \mathbf{u}(t), \mathbf{u}(t - \sigma), \mathbf{p}), \quad 0 \leq t \leq T, \quad (60)$$

$$\mathbf{y}(t) = \boldsymbol{\Psi}(t, \mathbf{p}), \quad t \in [-\tau, 0), \quad \mathbf{y}(0) = \mathbf{y}_0 \in R^n \quad (61)$$

$$\mathbf{u}(t) = \Phi(t), \quad t \in [-\sigma, 0), \quad \mathbf{u}(0) = \mathbf{u}_0 \in R^m, \quad (62)$$

Бу ерда \mathbf{f} вектор-функция етарлича силлик, $\mathbf{y}(t) \in R^n$, $\mathbf{y}(t - \tau) \in R^n$, $\mathbf{u}(t) \in R^m$, $\mathbf{u}(t - \sigma) \in R^m$, $\mathbf{p} \in R^r$, τ ва σ мусбат сонлар; $\Psi(t)$ ва $\Phi(t)$ – берилган узлуксиз функциялар. (60) тенгламада $\mathbf{u}(t)$ бошқарув параметри сифатида қатнашиб, $[-\sigma, T]$ ораликда аниқланган узлуксиз қисмли функциялар синфига мансуб. Мақсад функционали

$$J(\mathbf{u}) = F_0(\mathbf{y}(T)) + \int_0^T F_1(t, \mathbf{y}(t), \mathbf{y}(t - \tau), \mathbf{u}(t), \mathbf{u}(t - \sigma), \mathbf{p}) dt, \quad (63)$$

кўринишда танланиб, унинг минимумини излаш масаласи қўйилади (F_0 ва F_1 – узлуксиз функциялар).

Мақолада (60–62) системанинг сезгирлик коэффициенти сифатида

$$S_{ij}(t) = \frac{\partial y_i(t)}{\partial \alpha_j}, \quad (64)$$

катталиқ таклиф этилган, бунда α_j – системанинг p_j параметри ёки кечикув параметри τ_j , ёки $y_j(0)$ бошланғич қийматларни қабул қилади. Яқуний сезгирлик даражаси эса $y_i(t)$ нинг тўлиқ вариациясидан иборат бўлади:

$$\delta y_i(t) = \sum_j \frac{\partial y_i(t)}{\partial \alpha_j} \delta \alpha_j + O(|\alpha|^2). \quad (65)$$

Бу формулада бошқарув параметрларининг ўзгариши ҳисобга олинмаган. Система сезгирлигининг бу қисми қуйидагича баҳоланиши мумкин:

$$\beta_{ij}(t, t^*) = \frac{\partial y_i(t^*)}{\partial u_j}, \quad t < t^*. \quad (66)$$

Яқуний сезгирлик даражаси

$$\delta y_i(t^*) = \sum_j \int_0^{t^*} \frac{\partial y_i(t^*)}{\partial u_j} \delta u_j(t) dt, \quad t < t^*. \quad (67)$$

тўлиқ вариация билан баҳоланади.

ХУЛОСА

Тадқиқотнинг асосий натижалари қуйидагилардан иборат.

1. Биосистемалар динамикасини таҳлили учун оддий ва тақсимланган кечикувчи дифференциал тенгламалар, каср тартибли оддий ҳамда каср тартибли хусусий ҳосилали тенгламалар воситасида янги математик моделлар қурилган.

2. Вақт бўйича кечикиш ҳисобга олинган математик моделлар бу омилни ҳисобга олмаган математик моделларга нисбатан жараёнларни аниқроқ акс эттириши кўрсатилган.

3. Кечикувчи дифференциал тенгламаларни ечишнинг самарали усуллари, жумладан, Рунге-Кутта методи учун ноошкор схемалар ишлаб чиқилган, сонли методлар турғунлигини таъминловчи янги критерийлар топилган.

4. Математик моделларнинг параметрлари кўзғатилганда ва "оқ шовқин" тарзидаги тасодифий четлашишларга нисбатан сезгирлигини баҳолаш усули ишлаб чиқилган.

5. Мембранали ВАМ-нейрон тўрлари, комплекс қийматли нейрон тўрлари, Кохен-Кроссберг нейрон тўрлари ҳамда каср ҳосилали нейрон тўрлари учун синхронлаштириш, турғунлик ва диссипативликни текшириш усуллари ишлаб чиқилган.

6. Кохен-Кроссберг ВАМ-нейрон тўрларини турғун бўлмаган ҳолда стабиллаштириш алгоритми қурилган.

7. Кечикувчи оптимал бошқарув масалалари Понтрягиннинг максимум принципини қўллаш орқали ечилган.

8. Ўсимта-иммун системасининг химио-иммунологик терапия натижасидаги динамикаси учун кечикувчи дифференциал тенглама воситасида математик моделлар қурилган.

9. Каср-тартибли саратон-иммунитет системаси ва *Salmonella* инфекцияси динамикаси учун каср-тартибли дифференциал тенгламалар ноошкор Рунге-Кутте методи ёрдамида сонли ечилган.

10. Гепатит С вируси динамикаси учун параметрик баҳолаш масаласи ечилган.

11. Кечикувчи каср-тартибли "ўлжа-йиртқич" системасининг турғунлиги текширилган.

**SCIENTIFIC COUNCIL AWARDING OF THE SCIENTIFIC DEGREES
DSc.27.06.2017.FM.01.02 AT NATIONAL UNIVERSITY OF UZBEKISTAN**

**UNIVERSITY OF UNITED ARABIC EMIRATES AND INSTITUTE OF
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FATHALLA ALI RIHAN

**Qualitative and Quantitative Features of Differential
Equations with Memory and Their Applications**

**05.01.07 - Mathematical Modeling. Numerical Methods and Package of
Programs
(Physical and Mathematical Sciences)**

**DISSERTATION ABSTRACT OF DOCTORAL DISSERTATION (DSC)
ON PHYSICAL AND MATHEMATICAL SCIENCES**

Tashkent – 2019

Dissertation has been prepared at the Institute of Mathematics of Academy of Sciences of Uzbekistan and University of United Arab Emirates (UAE).

The abstract of the dissertation is posted in three languages (Uzbek, English, Russian (summary)) on the website <http://ik-fizmat.nuu.uz> and on the «ZiyoNET» Information and educational portal <http://www.ziynet.uz/>.
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Defense will take place « ____ » _____ 2019 at ____ at the meeting of Scientific Council number DSc.27.06.2017.FM.01.02 at National University of Uzbekistan. (Address: University str. 4, Almazar district, Tashkent, 100174, Uzbekistan, Ph.: (99871) 227-12-24, fax: (99871) 246-53-21, 246-02-24, e-mail: nauka@nuu.uz).

Doctoral dissertation is possible to review in Information-recourse centre at National University of Uzbekistan (is registration number № __) (Address: University str. 4, Almazar district, Tashkent, 100174, Uzbekistan, Ph.: (99871) 246-02-24).

Abstract of dissertation sent out on « ____ » _____ 2019.

(mailing report № « ____ » on « ____ » _____ 2019).

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INTRODUCTION (abstract of the DSc thesis)

Actuality and demand of the theme of dissertation. In the world differential equations with delay argumentation, including differential equations of fractional-order, are concepts widely used in mathematical modeling of natural sciences and real processes in technology. Differential equations with memory are the object of research in the field of biological populations, modeling and mathematical modeling of biochemical processes in organisms in epidemiology, immunology and physiology. Differential equations with delay arguments serve as the basis for modeling biological, chemical processes and neural networks, as well as identifying and treating psychological illnesses. Therefore, the study of mathematical models, which are expressed by delay differential equations and fractional-order differential equations, remains one of the most important tasks of such fields as technology, chemistry, biology and ecology.

The study of mathematical models in the form of differential equations with all the major processes in the world such as medicine, biology and biotechnology is one of the pressing issues. In a differential equation representing the mathematical model, one or more parameters, in particular, the delay involvement of the unknown functions and their derivatives, are practically natural phenomena, and this is important because it allows the model to be approached to the original. Differential equations with delay argument and fractional differential equations are widely used in the study of the dynamics of populations in biology, in the study of systems representing differential equations such as neural networks in the field of epidemiology, immunology and physiology in modeling of biochemical processes in organisms and techniques. Due to the selection of delay parameters, the model is to be brought closer to natural processes and some extent it is a scientific study aimed at controlling it.

Particular attention was paid to the modeling and management of ecological processes in the Republic of Uzbekistan and United Arab Emirates, which have scientific and practical application of fundamental sciences, actual aspects of multidimensional dynamics and mathematical modeling. In this regard, mathematical models in the form of delayed dynamical systems and functional differential equations for complex objects such as biological and chemical processes, psychological illnesses have been built, and significant results have been achieved in determining their properties. Scientific research at the level of international standards in the areas of "Functional analysis, differential equations and mathematical physics, applied mathematics and mathematical modeling" is defined as the main tasks and areas of practical mathematics. Mathematical modeling and quantitative analysis of processes in porous elastic environments are crucial in the decision-making process.

The Law of the President of the Republic of Uzbekistan dated February 17, 2017 "On Measures for Further Improvement of the Academy of Sciences, Management and Finance of Research Work, the law published February 8, 2017 with No. PF-4947 and the Federal Law No. 4 of the Ministry of Higher Education and Research of the United Arab Emirates in 1992 and other normative-legal

documents relating to these activities. This dissertation research will serve to a certain degree to accomplish the tasks.

Dependence of research to priority directions of development of science and technologies of the Republic. This study was conducted 2010/96 "Actual sciences" in the United Arab Emirates and IV. Scientific-technical programs "Mathematics, Mechanics and Informatics" in the Republic of Uzbekistan.

Review of overseas scientific investigations on topic of thesis. Research on the qualitative properties of mathematical models with differential equations with fractional order and with memory is going in leading world higher education institutions and research centers, including Cambridge, Manchester, Liverpool, Oxford University of Charleston, Edinburgh, Chester Universities (UK), Orlando, Florida, Universities (USA), University of BAA (Al-Ayn), Sultan Qobus University (Oman), Martin Loter University (Germany), American University of Sharjah (UAE) Yildiz Technical University (Turkey), United Arab Emirates University (Abu Dhabi), Aegean, Karlovassi, Samos (Greece), Zayed University (BAA), Amsterdam University (Netherlands), Melbourne, Queensland Universities (Australia), Russian Institute of Computational Mathematics (Moscow), Institute of Cybernetics (Ukraine), Perm Scientific Center of Ural Branch of Russian Academy of Sciences (Perm), Institute of Integrated Informatics Problems of the National Academy of Sciences of Belarus (Minsk), Institute of Mathematics, Bukhara State University (Uzbekistan).

As a result of worldwide research on the solution of delayed dynamical systems and functional differential equations, the development of methods for solving the boundary problems for them, a number of scientific findings have been made: mathematical models of biological and chemical processes (Manchester University, UK) ; The basics of the mathematical models of the treatment process on the basis of the database gathered in the large medical clinics were studied and the adequacy of delayed and discrete differential equations was studied Institute of Computational Mathematics, (Russia), and developed practical recommendations for late bifurcation and cyclic solutions in the latent differential equation that simulates the process of human CD immunodeficiency virus infection (CD4 lymphocytes) Grandhigram Rural Research Institute, Deemed University, (India). A distinctive fractional equation for modeling the dynamics of the distribution of Ebola fluctuation has been proposed, and its solutions have been studied, the Hopf bifurcation has been investigated, and the properties of trajectories have been determined for some values of the system parameters using the alternative solution methods (Bharathiar University, India).

Research is being carried out in a number of priority areas in the field of Differential Equations with memory, which form the basis of mathematical models in the fields of medicine, ecology, biology and biotechnology, in a number of priority areas for the development of methods and means of solving and implementing boundary problems, including:

to investigate a wide range of mathematical models in the form of differential equations with the whole and fractional delay, and to define them unmatched differential equations comparison;

to show that it possesses an excellent mathematical structure as a dynamic system; development of methods for solving new problems and solving new solutions for the stagnation of solutions of delayed and fractional differential equations;

to construct algorithms and software for solving boundary value problems for delay differential equations with integer and fractional-order.

Degree of scrutiny of the problem. The existence, uniqueness, and continuity of the solution for differential equations with delay are investigated by the founders of this theory: V. Walterra, J. Hail, and N. Krasovsky. Such results for larger classes were obtained by L. W. Neustadt, A. Tixonov, G. Jones, M.A.Cruz, C.V.Coffman, J. Shaffer, J.Yorke, J. Dugunji, A.Halanay. Linear differential equations with time delay are especially well-researched. The main results in this direction are investigated by A.Mishkis, R.Bellman and K. L.Cooke, J.Hale and K. R. Mayer, D.Henry, A.M. Zverkin, G.A.Kamensky, S.B. Norkin and L.E.Elsgolts, W.R. Melvin. Continuing the solution in the application of nonlinear equations to the technique is learned by J.Hale and Hastings, W. M. Oliva, J. C.Lillo.

Despite numerous studies on the theory of delayed differential equations, there are still many unsolved problem in this area. The delay in starting the study of differential equations has primarily caused practical needs. Over the past few decades, the application of computer technology in modern biology, chemistry, medicine and other fields has been gaining momentum, and hundreds of articles have been published monthly in this field. Some of these studies are reflected in H. Smith's scientific research.

At present, in many scientific centers are being investigated research on differential equations with memory. Z. Wang proposed numerical methods for construction an approximately solution, S.Bhalekar studied the problem of stability and bifurcation for these equations, which examined the discrete scheme of differential equations with Chen's fractional-order, G.C. Wu and D. Balianu studied Fyurxolst's logistic equation, Wang and J.Yu have found chaotic trajectories in the logistic equations system. A. Si-Ammour, S. Djennoune and M. Bettaib proposed the method of solving the control problem, S. Bhalekar, V. Daftardar-Gejji, D. Balenu and R. Magin, investigated the delay argument in fractional-order Blox equation, Y.Q. Chen, and I. Podlubny were applied Lyapunov function methods.

The connection of the topic of the thesis with research work of the higher educational institution, in which the thesis is carried out. Dissertation research is carried out by the Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan (OT-Φ4-84) "Discrete-method for polynomial systems and its modeling in cyclic and control processes" (2017-2020) and SQU / UAEU "Delay Differential Models of Immune Response With Viral and Bacterial Infection in an Organism "(2017-2019), UPAR" Mathematical Models for Kinetics of Coronavirus Infection in Humans "(2015-2017), NRF" Delay Differential

Models in Immunology and Infection Diseases in an Individual "(2011 -2016) within the framework of fundamental projects.

The aim of the research is to construct mathematical models in the form of delay differential equations with integer and fractional-order, to obtain new adequate conditions for the stability of the solution of delay and fractional differential equations, to develop numerical methods and to design computer programs.

Tasks of the research work:

to prove the stability of the solutions of delay differential equations and to find bifurcations;

to develop numerical methods for solving the Cauchy problem for delay differential equations;

to estimate the sensitivity of differential equations with delay parameter;

to compare the models of a particular process using a simple and delay differential equation, based on observations;

to develop the Lyapunov functional method for analyzing the stability of solutions of delayed and fractional-order differential equations;

to develop and to analysis of the mathematical model of influenza distribution, growth dynamics of cancerous tumors, transmission of hepatitis using delay and fractional-order differential equations;

application of delay and fractional-order differential equations to modeling of neural networks.

The research object consists of processes in biology, ecology, medicine, and neural networks modeled with differential equations of memory.

The research subject of the study is qualitative and quantitative aspects of delay and fractional-order differential equations and their applications to natural sciences and techniques.

Research Methods. In the thesis we have used modern mathematical modelling, theory of dynamic systems, theory of bifurcation, theory of integral equations, methods of Lyapunov functions, Pontryagin's maximum principle, numerical methods for solving the Cauchy problem, Batcher tables for the Runge-Kutta method, the fractional-order derivatives and integrals, Lotka-Volterra models, computer modeling.

The scientific novelty of the research is as follows:

New mathematical models have been constructed using ordinary and distributed delay differential equations, ordinary fractional-order and fractional-order partial differential equations for the analysis of dynamics of biosystems;

It is shown that mathematical models, which are considered with delay parameter, are more accurately reflective than the mathematical models, which do not take an account of this parameter;

effective methods for the solution of delay differential equations, including the implicit schemes for the Runge-Kutta method are constructed;

new criteria to provide the stability of methods are found;

a method of assessing the mathematical models' sensitivity to the perturbation of the parameters and to the "white noise" random errors are developed;

methods of synchronization, stabilization and dissipation analysis for membranous VAM neural networks, neural networks of complex value, Kohsen-Crossberg neural networks and fractional derivative neural networks are developed;

algorithm for stabilization of unstable Kohsen-Krossberg's VAM-neural networks are constructed.

Practical results of the research were used to determine the dynamics of growth of dangerous tumors and the spread of infectious epidemics, the creation of advanced mathematical models of biological and medical processes, such as populations of "predator" populations genetic system, and the calculation of optimal treatment modalities for viral diseases.

The reliability of the results of the research. Theoretical results are presented in the form of theorems, strictly proved and the differential equations with memory, which are the main part of the dissertation volume, are analyzed on the basis of the data obtained from the practice and are more adequate to classical mathematical models. In particular, the reliability of the patterns of disease dynamics has been proven by comparative analysis with the results of clinical trials.

The scientific and practical significance of the results of the research. The scientific value of the research results is based on the improvement and justification of the currents of the two-phase porosity and the mathematical models of the processes of heat-mass metabolism, and the laws, techniques and mechanics equations for the preservation of research works before and after the analysis of specific practical problems of oil and gas mechanics joint implementation. The practical significance of the results of the research will serve to solve the problems of oil and gas industry, development of heat energy, innovative wave technologies, forecasting of filtration properties of hole collectors and optimization of their process, and improving the models of technological processes of hydrocarbon deposits.

Implementations of the research results. On the basis the results of the qualitative and quantitative aspects of differential equations with memory and numerical solutions of boundary value problem for delay differential equation:

Numerical methods for solving nonlinear delay differential equations used to prove the stability of solutions of delay differential equation in the "Simulation of Radiation Effects in the Central Nervous System" under the number 301 (Cairo University Version 1, 2018). The scientific results give an opportunity to construct the method for casting a physiological delayed process of patients diseased with tuberculosis;

mathematical models, and the method of assessing the vulnerability to "randomized" behavior in the form of "white noise" was used for the parametric evaluation of hepatitis virus hepatitis virus in the "Epidemiology of Swine influenza H1N1 pandemic" grant project 2009/2010 (United Arab Emirates, May

14, 2018). The use of scientific findings has allowed the development of recommendations for predicting the spread of H1N1-type pandemic influenza. The synchronization, stability and dissipative methods is used for mathematical modeling of the immune process for VAM-neural networks, complex-valued neural networks, Kohsen-Krossberg's neural networks and fractional-order neural networks (the Institute of Computational Mathematics of the Russian Academy of Sciences, No. 10256/75 of April 13, 2018). The use of scientific results allows classifying the database as a result of clinical trials; algorithm for stabilization of Kohsen-Krossberg VAM-neural networks in foreign scientific journals (Hindawi, Complexity, Volume 2017, Article ID 6875874, 13 pages, Neural Processing Letters, Springer, 2018, pp. 1-19, Journal Neural Networks, vol. International Journal of Dynamics and Control, 2018, pp. 1-9, Numerical Algorithms, Vol. 79, Issue 1, 2018, pp. 1-9) Riman-Liuwill VAM with impulsive and delay fractional-order used to prove the existence and global asymptotic stability of hybrid neural network. The use of scientific results to the development of global asymptotic strenuousness equilibrium solutions based on the parameters of the VAM hybrid neural network; differential equations for simple and distributed delay for the analysis of biosystem dynamics, mathematical models constructed by simple fractional-order ordinary and fractional-order equations in foreign scientific journals (Journal of Inequalities and Applications, 2014, pp. 1-14; Applied Mathematics, No. 8, 2017, 1715-1744), Nonlinear Science and Numerical Simulation, Vol. 39, 2016, pp. 396-410) used to manage the system's bifurcation. The use of scientific results has enabled effective control of Hopf bifurcation for the predator system.

Approbation of the research results. The results of the study were discussed at 32 international and 36 regional scientific and practical conferences. The dissertation was discussed at the joint scientific seminar of “Algebra and Functional Analysis” and “Theory of Dynamical Systems” sections of the Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan and at the scientific seminar "Mathematical Physics" of the National University of Uzbekistan.

Publications of the research results. On the topic of the dissertation 33 articles are published, included in the list of scientific publications proposed by the higher attestation commission of the Republic of Uzbekistan for the defence of the doctoral dissertations.

Volume and structure of thesis. The dissertation is prepared in the form of English scientific lectures and consists of 76 pages.

MAIN CONTENT OF THE DISSERTATION

Differential equations with memory such as delay differential equations (DDEs) and fractional order differential equations (FODEs) is a class of differential equations that have received considerable attention and been shown to

model many real life problems, traditionally formulated by systems of ordinary differential equations (ODEs), more naturally and more accurately. Such classes of DDEs and FODEs are widely used for analysis and predictions of systems with memory such as population dynamics, epidemiology, immunology, physiology, neural networks and other biological and physical systems.

In most of biological and engineering systems, time-lags or time-delays exist intrinsically. Therefore, modeling of such systems by differential equations with memory, represented by time-delay or derivatives with fractional-order, has more advantages than models without memory. The presence of time-lags (time-delays) or/and fractional-order in the differential model improves the stability of the solutions and enrich the dynamics of the model. The fractional-order derivative is related to the whole space for a physical process, while the integer-order derivative describes the local properties of a certain position.

The central aim of this work is to provide a wide range of delay differential equations with integer and fractional-order derivatives, and show that they have a richer mathematical framework (compared with differential equations without memory) for the analysis of dynamical systems. This dissertation consists of 7 chapters, organized as follows: In Chapter I, we give a briefer introduction for the dissertation. In Chapter II, we study qualitative analysis of delay differential equations with applications in biosciences. In Chapter III, we discuss delay differential equations with fractional-order and their applications. In Chapter IV, we study qualitative analysis of delay differential equations with neural networks. In Chapter V, we provide unconditionally stable numerical schemes for differential equations with memory, which are suitable for stiff and non-stiff problems. In Chapter VI, we study parameter estimations and sensitivity analysis of differential equations with memory. In Chapter VII, we conclude.

Originality of the Work. Physical and biological systems have complex nonlinear dynamic behaviors, and studying stability, bifurcation properties, are essential for ensuring safe applications in the real world. The time-delays should be incorporated into biological systems to describe resource regeneration times, maturation periods, reaction times, feeding times, gestation period, etc. However, time-delays and fractional-order gain the model greater degree of freedom and consistency with the reality of the interactions due to its ability to provide an exact description of the nonlinear phenomena. Through mathematical modeling and theoretical investigation of the underlying dynamic behaviors of physical and biological systems, effective control schemes can be designed and validated.

To provide realistic mathematical models for problems with time-lag or after-effect, we should consider retarded functional differential equations (RFDEs), in place of ordinary differential equations (ODEs), such as:

$$y'(t) = f\left(t, y(t), y(\alpha(t, y(t))), \int_{-\infty}^t K(t, s, y(t), y(s)) ds\right), \quad t \geq t_0, \quad (1)$$

where $\alpha(t, y(t)) \leq t$ and $y(t) = \psi(t), t \leq t_0$. Such retarded equations form a class of equations which is, in some sense, between ODEs and time-dependent partial differential equations (PDEs). These equations generate infinite-dimensional

dynamical systems. RFDEs (1) where the integral term is absent are usually called delay differential equations (DDEs) and they assume forms such as

$$y'(t) = f(t, y(t), y(\alpha(t, y(t))), \quad \alpha(t, y(t)) \leq t. \quad (2)$$

Neutral delay differential equations (NDDEs) are defined by equations of the form

$$y'(t) = f(t, y(t), y(\alpha(t, y(t))), y'(\beta(t, y(t))), \quad (3)$$

where $\alpha(t, y(t)), \beta(t, y(t)) \leq t$. The introduction of the “lagging” arguments $\alpha(t, y(t)), \beta(t, y(t))$ is to reflect an “after-effect”; consider (as an example of a time-lag) the gestation period in population modelling.

The concern in this work is to study the role that time-delay and fractional-order of time derivatives play in modelling systems with memory. The main aim is to show that the differential equations with memory such as DDEs have a richer mathematical framework (compared with ODEs) for the analysis of biosystem dynamics, and display better consistency with the nature of the underlying processes.

Problems Addressed:

We study qualitative and quantitative features of the following types of differential equations with memory:

I- Parameterized DDEs:

$$\begin{aligned} y'(t) &= f(t, y(t), y(\alpha(t)), y'(\alpha(t)), u(t), u(\beta(t)), p), t \geq t_0 \\ y(t) &= \psi(t, p), \quad y(t) = \psi'(t, p), u(t) = \Phi(t) \quad t \leq t_0. \end{aligned} \quad (4)$$

Here $y(t) \equiv y(t, p, u)$, $\alpha(t) \leq t$, $\beta(t) \leq t$.

II- Optimal Control Problem Governed by DDEs:

$$\max_{x, v} J(x, v) = \Psi(x(t_f)) + \int_0^{t_f} L(t, x(t), v(t)) dt, \quad (5a)$$

subject to the DDEs

$$x'(t) = f(t, x(t), x(t-\tau), v(t)), \quad t \in [0, t_f] \quad (5b)$$

$$x(t) = \phi(t), \quad t \in [-\tau, 0], \quad (5c)$$

with control constraint(s)

$$a \leq v(t) \leq b \quad t \in [0, t_f]. \quad (5d)$$

and state constraint(s)

$$x(t) \geq c \quad t \in [0, t_f]. \quad (5e)$$

J is called objective functional and the integrand, $L(\cdot)$ is called the Lagrangian of objective functional. Furthermore, a and b are called the lower and upper bounds. The function $v(t)$ is called an admissible control iff it fulfills the inequality constraints (5d).

III- Time-Fractional Delay Partial Differential Equations:

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = F(t, u(x, t), Lu(x, t), u(x, t-\tau)), x \in \Omega, t \geq 0, \quad 0 < \alpha \leq 1, \quad (6)$$

where F is a continuous function, $t-\tau \leq t$ (the condition on $t-\tau \rightarrow \infty$ as $t \rightarrow \infty$), and L is elliptic in the spatial-direction. τ could be a constant, a function of time, $\tau(t)$, or even state dependent, $\tau(t, y)$.

Specializing the right hand side of (6), gives a time fractional diffusion equation with a delay argument

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a \frac{\partial^2 u(x,t)}{\partial x^2} + bu(x,t-\tau), a, b \in R, \quad (7)$$

Taking $\alpha=1$ in (7), yields the classical model of delay parabolic partial differential equation (DPPDE), with positive constant time-lag τ ,

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= a \frac{\partial^2 u(x,t)}{\partial x^2} + bu(x,t-\tau), t \geq 0, \\ u(x,t) &= \psi(x,t), \quad x \in [0, L], \quad t \in [-\tau, 0]. \end{aligned} \quad (8)$$

This equation can be considered as reaction-diffusion equation with delay in the reaction term. For a nonlinear example, one may consider Hutchinson's equation

$$\frac{\partial u(x,t)}{\partial t} = \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + \gamma u(x,t)[1-u(x,t-\tau)], \quad (9)$$

to model the evolution of a population in mathematical ecology with density $u(x,t)$, where $\varepsilon > 0$ is the diffusion coefficient. In all cases $u(x,t)$ must be subject to appropriate boundary and initial conditions.

IV- Neural Networks Models with Fractional-order and Time-delays:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bg(x(t-\tau)) + H, t \geq 0 \quad (10)$$

with initial condition

$$x(s) = \phi(s), s \in [-\tau, 0], \quad (11)$$

where $\tau > 0$ is the transmission delay. $B = (b_{ij})_{n \times n} \in R^{n \times n}$ represent a delayed interconnection weight matrix. $g(x(t-\tau)) = [g_1(x_1(t-\tau)), \dots, g_n(x_n(t-\tau))]^T \in R^n$ is the delayed neuron activation function at time $t-\tau$. $\phi(s) = [\phi_1(s), \dots, \phi_n(s)]^T \in R^n$ denotes the real-valued continuous functions defined on $[-\tau, 0]$ with the norm $\|\phi\| = \sup_{s \in [-\tau, 0]} \sum_{i=1}^n |\phi_i(s)|$.

The fractional-order neural networks model with time delay is defined as

$$D^\alpha x(t) = -Cx(t) + Af(x(t)) + Bg(x(t-\tau)) + H, t \geq 0 \quad (12)$$

with initial condition

$$x(s) = \phi(s), s \in [-\tau, 0], \quad (13)$$

where D^α denotes the Caputo derivative with fractional order α , $\alpha \in (0, 1]$. The other parameters of (12) have the same meaning as described in (4) and (10).

V- Cohen-Grossberg Neural Networks Models:

$$\dot{x}(t) = D(x(t))(-C(x(t)) + Af(x(t)) + Bg(x(t-\tau)) + H), t \geq 0, \quad (14)$$

with initial condition

$$x(s) = \phi(s), s \in [-\tau, 0]. \quad (15)$$

The number of neurons is $n \geq 2$. $x(t) = [x_1(t), \dots, x_n(t)]^T \in R^n$ denote the neuron state variable. $D(x(t)) = \text{diag}\{d_1(x_1), \dots, d_n(x_n)\}$ is an amplification function, $C(x(t)) = (c_1(x_1), \dots, c_n(x_n))^T \in R^n$ represents an appropriately behaved function. $A = (a_{ij})_{n \times n} \in R^{n \times n}$ indicates the strength of the neuron interconnection weight matrix.

$B = (b_{ij})_{n \times n} \in R^{n \times n}$ indicates the strength of the neuron interconnection within the network with time delay τ . $f(x(t)) = [f_1(x_1(t)), \dots, f_n(x_n(t))]^T \in R^n$ denotes the activation function at time t . $g(x(t-\tau)) = [g_1(x_1(t-\tau)), \dots, g_n(x_n(t-\tau))]^T \in R^n$ represent the activation function at time $t-\tau$. $H = [h_1, \dots, h_n]^T \in R^n$ is a constant input vector from outside of the network.

A class of fractional-order Cohen-Grossberg neural networks model with time delay is defined as follows:

$$D^\alpha x(t) = D(x(t))(-C(x(t)) + Af(x(t)) + Bg(x(t-\tau)) + H), t \geq 0, \quad (16)$$

where D^α represents the Caputo fractional derivative with order α , $\alpha \in (0, 1]$. The other parameters of (16) have the same meaning as described in (14).

VI- Bidirectional Associative Memory (BAM) Neural Networks Models:

$$\begin{aligned} \dot{u}(t) &= -C_1 u(t) + A_1 f(v(t)) + B_1 f(v(t-\tau)) + H_1, \\ \dot{v}(t) &= -C_2 v(t) + A_2 g(u(t)) + B_2 g(v(t-\tau)) + H_2, \end{aligned} \quad (17)$$

with initial conditions

$$x(s) = \phi(s), y(s) = \Phi(s), s \in [-\tau, 0], \quad (18)$$

where $u(t) = [u_1(t), \dots, u_n(t)]^T$ and $v(t) = [v_1(t), \dots, v_n(t)]^T$ denote the neural states. $C_1 = \text{diag}(c_1^1, \dots, c_n^1) \in R^{n \times n}$ and $C_2 = \text{diag}(c_1^2, \dots, c_n^2) \in R^{n \times n}$ are (the positive diagonal matrices ($c_i^1 > 0$, and $c_j^2 > 0$)) self-feedback connection weight matrices. $A_1 = (a_{ij}^1)_{n \times n} \in R^{n \times n}$, $A_2 = (a_{ij}^2)_{n \times n} \in R^{n \times n}$ and $B_1 = (b_{ij}^1)_{n \times n} \in bR^{n \times n}$, $B_2 = (b_{ij}^2)_{n \times n} \in R^{n \times n}$ indicates the strength of the neuron interconnection weight matrices and delayed neuron interconnection weight matrices respectively. $f(v(t)) = [f_1(v_1(t)), \dots, f_n(v_n(t))]^T \in R^n$, and $g(u(t)) = [g_1(u_1(t)), \dots, g_n(u_n(t))]^T \in R^n$ are the activation functions at time t . $f(v(t-\tau)) = [f_1(v_1(t-\tau)), \dots, f_n(v_n(t-\tau))]^T \in R^n$, and $g(u(t-\tau)) = [g_1(u_1(t-\tau)), \dots, g_n(u_n(t-\tau))]^T \in R^n$ represents the activation functions at time $t-\tau$. $H_1 = [h_1^1, \dots, h_n^1]^T \in R^n$ and $H_2 = [h_1^2, \dots, h_n^2]^T \in R^n$ are the external input vector from outside of the network.

Objectives:

Throughout this dissertation we address the following:

Qualitative analysis of the above mentioned differential equations;

Analyze the rôle that delay terms plays in basic time-lag models;

Study the dynamical analysis of biological systems;

Investigate the sensitivity analysis and determine the most effective parameters to improve the predictions and properties of the model and model selection;

Present efficient and suitable approaches for the numerical treatments of stiff and non-stiff delay and fractional-order differential equations;

Present suitable numerical approaches to find best-fit and meaningful parameters when given experimental data (inverse problem);

Apply delay and fractional-order models for cancer-Immune System with external treatment and Optimal Control;

Dynamical analysis of Neural Networks with fractional-order and Time-delays;

We briefly discuss the results of each Chapter:

General Introduction to the Dissertation. In this chapter, we briefly show the actuality and demand of the theme of the dissertation; Review of scientific researches; Originality and aim of the work.

Delay Differential Equations with Applications in Biosciences. This Chapter provides an overview about delay differential equations in bioscience. We remark, therein, how delay differential equations have, prospectively, more interesting dynamics than equations that lack memory effects; in consequence they provide potentially more flexible tools for modelling real phenomena.

This Chapter includes 7 publications, I only give summary of some of them:

1. Numerical modeling in biosciences using delay differential equations. In this work, we investigate models of phenomena in the biosciences that are based on differential equations with memory of the form:

$$y'(t) = f\left(t, y(t), y(\alpha(t, y(t))), \int_{-\infty}^t K(t, s, y(t), y(s)) ds\right), \quad t \geq t_0, \quad (19)$$

wherein $\alpha(t, y(t)) \leq t$ and $y(t) = \psi(t), t \leq t_0$, form a class of equations which is, in some sense, between ordinary differential equations (ODEs) and time-dependent partial differential equations (PDEs). Such retarded equations generate infinite-dimensional dynamical systems. RFDEs (25) where the integral term is absent are usually called delay differential equations (DDEs) and they assume forms such as $y'(t) = f(t, y(t), y(\alpha(t, y(t))))$ with $\alpha(t, y(t)) \leq t$. The introduction of the “lagging” or “retarded” argument $\alpha(t, y(t))$ is to reflect an “after-effect”.

We show that there are prima facie reasons for using such models: (i) they have a richer mathematical framework (compared with ordinary differential equations) for the analysis of biosystem dynamics, (ii) they display better consistency with the nature of certain biological processes and predictive results. We analyze both the qualitative and quantitative role that delays play in basic time-lag models proposed in population dynamics, epidemiology, physiology, immunology, neural networks and cell kinetics.

2. Delay differential model for tumor-immune dynamics with HIV infection of CD4+ T-cells. In this work, we introduce ordinary and delay differential equations to describe the interactions between a malignant tumor and the immune system *in vivo* in the presence of HIV infection of CD4+ T-cells. In the delay model, we take into account the time-lags required by the healthy effector cells components to recognize the pathogens and tumor cells. The models consist of four populations: tumor cells, healthy effector cells (CD4+ T-cells), effector cells infected by HIV viruses and free viral particles. The presence of delay term in the model leads to a notable increase in the complexity of the observed behavior. We investigate the qualitative behavior of the models and find the conditions that

guarantee the asymptotic stability of the steady states. Numerical simulations are provided to illustrate and extend the theoretical results. The obtained results are consistent with the real phenomena and give a better understanding of cancer immunity and viral oncogenesis.

We studied the qualitative behaviour of DDEs of the form

$$\frac{dT}{dt} = r_1 T(t) - k_1 E(t - \tau) T(t), \quad (20a)$$

$$\frac{dE}{dt} = r_2 T(t) + \alpha - \mu_1 E(t - \tau) - \epsilon k_1 T(t) E(t - \tau) - k_3 E(t - \tau) V(t) - k_2 E(t - \tau) I(t), \quad (20b)$$

$$\frac{dI}{dt} = k_3 E(t - \tau) V(t) + k_2 E(t - \tau) I(t) - \mu_2 I(t), \quad (20c)$$

$$\frac{dV}{dt} = N \delta I(t) - c V(t) \quad (20d)$$

In the absence of the HIV viral infection, we also consider the two-variables system

$$\frac{dT}{dt} = r_1 T(t) - k_1 E(t - \tau) T(t), \quad (21a)$$

$$\frac{dE}{dt} = r_2 T(t) + \alpha - \mu_1 E(t - \tau) - \epsilon k_1 T(t) E(t - \tau), \quad (21b)$$

We are interested in studying the stability of the steady state S^* , since the steady state S_0 is unstable. We arrive at the following Proposition.

Proposition 1. For System (20a-20d), if S^* exists and $r_2 - \epsilon r_1 > 0$, then it is locally asymptotically stable for all values of $0 \leq \tau < \tau^*$ (and unstable otherwise), where

$$\tau^* = \frac{1}{\omega_0} \arccos \left(\frac{(r_2 - \epsilon r_1) k_1 T^* \omega_0^2}{(\mu_1 + \epsilon k_1 T^*)^2 \omega_0^2 + (r_2 - \epsilon r_1)^2 k_1^2 (T^*)^2} \right). \quad (22)$$

When $\tau = \tau^*$, a Hopf bifurcation occurs; that is, a family of periodic solutions bifurcates from S^* as τ passes through the critical value τ^* .

Proposition 2. For system (19); if $r_2 > \epsilon r_1$ and if $r_1 k_2 < k_1 \mu_2$, then if $S^* = (T^*, E^*)$ is stable as a steady state in the two-variable system (20a-20d), then S_3 is stable as a steady state for system (19); if $r_1 k_2 > k_1 \mu_2$, then S_3 is unstable.

3. A time delay model of tumor-immune system interactions: global dynamics, parameter estimations, sensitivity analysis. In this work, we present a delay differential model to describe the interactions between the effector and tumour cells. The existence of the possible steady states and their local stability and change of stability via Hopf bifurcation are theoretically and numerically investigated. Parameter estimation problem for given real observations, using least squares approach, is studied. The global stability and sensitivity analysis are also numerically proved for the model. The stability and periodicity of the solutions may depend on the time-lag parameter. The model is qualitatively consistent with the experimental observations of immune-induced tumour dormancy. The model also predicts dormancy as a transient period of growth which necessarily results in either tumour elimination or tumour escape.

Given the model

$$\begin{aligned}\frac{dE(t)}{dt} &= \sigma + \frac{\rho E(t-\tau)T(t-\tau)}{\eta + T(t-\tau)} - \mu E(t-\tau)T(t-\tau) - \delta E(t), \\ \frac{dT(t)}{dt} &= \alpha T(t)(1 - \beta T(t)) - E(t)T(t).\end{aligned}\tag{23}$$

(We consider $\bar{T}_0 = \bar{E}_0 = 10^6$ and the analysis will be performed on model (22).)

To study the global stability of the equilibria E_i^* ($i=0,1,2,3$), we use Lyapunov-Krasovski functional to the linearized system. By letting $X_1 = E - E_i^*$, and $X_2 = T - T_i^*$ where X_1 and X_2 are small perturbations about E_i^* and T_i^* respectively, system (22) is linearized and takes the form

$$\dot{X}(t) = \bar{A}_1 X(t) + \bar{A}_2 X(t-\tau)\tag{24}$$

where $X = (X_1, X_2)$,

$$\bar{A}_1 = \begin{pmatrix} -\delta & 0 \\ -T_i^* & \alpha - 2\alpha\beta T_i^* - E_i^* \end{pmatrix}, \quad \bar{A}_2 = \begin{pmatrix} \frac{\rho T_i^*}{\eta + T_i^*} - \mu T_i^* & \frac{\rho E_i^*}{\eta + T_i^*} - \frac{\rho E_i^* T_i^*}{(\eta + T_i^*)^2} - \mu E_i^* \\ 0 & 0 \end{pmatrix}.$$

We then investigate the delay-dependent stability of the system (24). In order to derive our main results, the following lemmas are introduced.

Lemma 1. (Schur Complement) Given constant matrices Ω_1 , Ω_2 and Ω_3 with appropriate dimensions, where $\Omega_1^T = \Omega_1$ and $\Omega_2^T = \Omega_2 > 0$, then

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ * & -\Omega_2 \end{bmatrix} < 0, \text{ or } \begin{bmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{bmatrix} < 0.$$

where '*' denotes the symmetric matrix and the expression $\Omega > 0$ or $\Omega < 0$ denotes that the matrix Ω is a positive definite or negative definite matrix, respectively.

Lemma 2. (Jensen's Lemma) For any positive symmetric constant matrices $M \in R^{n \times n}$, scalar $\tau > 0$, vector function $X(s) \in R^n$ such that the integrations concerned are well defined, then

$$-\int_{t-\tau}^t X^T(s) M X(s) ds \leq -\frac{1}{\tau} \left(\int_{t-\tau}^t X(s) ds \right)^T M \left(\int_{t-\tau}^t X(s) ds \right).$$

Now, stability analysis of delayed system (24) is given in the following theorem.

Theorem 3. For given scalar $\tau > 0$, system (24) is globally asymptotically stable if there exist symmetric matrices $P > 0$, $W_1 > 0$ and $W_2 > 0$, such that the following LMI holds:

$$\begin{bmatrix} P\bar{A}_1 + \bar{A}_1^T P + W_1 - W_2 & P\bar{A}_2 + W_2 & \tau \bar{A}_1^T W_2 & (10) \\ & -W_1 - W_2 & \tau \bar{A}_2^T W_2 & (11) \\ & * & -W_2 & (12) \end{bmatrix} < 0.\tag{25}$$

4. Delay differential model for tumor-immune response with chemo-immunotherapy and optimal control. In this work, we investigate a tumour-

immune response model using a delay differential dynamical system. By combining chemotherapy and immunotherapy treatment strategies, they invoke optimal control to identify the optimal treatment strategy that can minimize the tumour cells while keeping the number of normal cells above a given rate of their carrying capacity. The delay is incorporated to take into account the time required to stimulate the effector cells. The authors begin by studying the qualitative behaviour of the model without external therapy; they show that the model displays a tumour-free steady state, a dead steady state and a coexisting steady state. They analyze the stability of these equilibriums. On the other hand, they analyze the optimal control problem in two cases: with only the chemotherapy control variable and with a combination of chemotherapy and immunotherapy treatments with two control variables. The existence of the optimal control and optimality conditions are established by invoking Pontryagin's maximum principle. The numerical results show that the optimal treatment strategies reduce the tumour cells and increase the effector cells. We conclude that the performance of the combination therapy protocol is better than that of the standard protocol of chemotherapy alone.

Our main objective is to optimize the functional

$$\max_{v \in V_{ad}} J(v) = \int_0^{t_f} \left(E(t) - T(t) - \frac{B_v}{2} [v(t)]^2 \right) dt \quad (26a)$$

which subject to underlying DDEs

$$\frac{dE(t)}{dt} = \sigma + \frac{\rho E(t-\tau)T(t-\tau)}{\eta + T(t-\tau)} - \mu E(t-\tau)T(t-\tau) - \delta E(t) - a_1(1 - e^{-u(t)})E(t), \quad (26b)$$

$$\frac{dT(t)}{dt} = r_2 T(t)(1 - \beta T(t)) - nE(t)T(t) - c_1 N(t)T(t) - a_2(1 - e^{-u})T(t), \quad (26c)$$

$$\frac{dN(t)}{dt} = r_3 N(t)(1 - \beta_2 N(t)) - c_2 T(t)N(t) - a_3(1 - e^{-u})N(t), \quad (26d)$$

$$\frac{du(t)}{dt} = v(t) - d_1 u(t) \quad (26e)$$

with control constraint

$$0 \leq v(t) \leq v_{max} < \infty \quad t \in [0, t_f], \quad (26f)$$

and state constraint

$$k(N) = N - 0.75 \geq 0, \quad t \in [0, t_f]. \quad (26g)$$

Here, B_v is a weight factor that describes the patient's acceptance level of chemotherapy. We choose the control parameter as a class of piecewise continuous functions defined for all t such that $0 \leq v(t) \leq v_{max} < \infty$, where $v(t) = v_{max}$ represents the maximum chemotherapy, while $v(t) = 0$ represents the case where there is no chemotherapy. Thus, we depict the class of admissible controls as

$$V_{ad} = \left\{ v \in L^\infty([0, t_f], \mathbb{R}), | 0 \leq v(t) \leq v_{max} < \infty, \forall t \in [0, t_f] \right\}.$$

We prove the existence of the optimal solution of the underlying system (25).

Theorem 4. There exists an optimal solution $(x^*, v^*) \in W^{1,\infty}([0, t_f], R^4) \times L^\infty([0, t_f], R)$ for the optimal control problem (14) such that

$$J(v^*) = \max_{v \in V_{ad}} J(v) \quad (27)$$

where $x^* = [E^*, T^*, N^*, u^*]^T$ if the following conditions are satisfied:

The set of admissible state is nonempty.

The admissible set V_{ad} is nonempty, convex and closed.

The right-hand side of the state system is bounded by a linear combination of the state and control variables.

The integrand, $L(E, T, v) = (E(t) - T(t) - \frac{B_v}{2}[v(t)]^2)$, of the objective functional is a concave on V_{ad} .

The exist constants $h_2, h_3 > 0$ and $b > 1$ such that $L(E, T, v) \leq h_2 - h_3(|v|)^b$.

5. Numerical modeling of biological systems with memory using delay differential equations. In this work, show the consistency of delay differential equations with biological systems with memory, in which we present a class of mathematical models with time-lags in immunology, physiology, epidemiology and cell growth. We also incorporate optimal control parameters into a delay model to describe the interactions of the tumor cells and immune response cells with external therapy. We then study parameter estimations and sensitivity analysis with delay differential equations. Sensitivity analysis is an important tool for understanding a particular model, which is considered as an issue of stability with respect to structural perturbations in the model.

We provide many problems in biosciences (such as epidemics, harvesting, chemostat, treatment of diseases, physiological control, vaccination) which can be addressed within an optimal control framework for systems of DDEs. However, the amount of real experience that exists with optimal control problems (OCPs) is still small.

Consider DDE with an m -dimensional control term $u(t)$

$$y'(t) = f(y(t), y(t - \tau_1), y(t - \tau_2), \dots, y(t - \tau_d), u(t), t) \quad (28)$$

and a suitable objective functional (measure): $J_0(u)$

$$\text{Maximize } J_0(u) = \Phi_0(y(T)) + \int_0^{t_f} F(y(t), y(t - \tau_1), y(t - \tau_2), \dots, y(t - \tau_d), u(t), t) dt, \quad (29)$$

and subject to control constraint $a \leq u(t) \leq b$, and state constant $y(t) \leq c$, where a and b are the lower and upper bounds. The integrand, $F(\cdot)$ is called the Lagrangian of objective functional which is continuous in $[0, t_f]$. Additional equality or inequality constraint(s) can be imposed in terms of $J_i(\mathbf{u})$.

OCPs using DDEs were studied in connection with immune responses to infections. Assume that $E(t)$ represents effector cells population, such as $CD8^+$ T cells and $T(t)$ is the tumour cells population. We provide a competing model in terms a system of DDEs, in which we add extra variables namely chemotherapy

variable, $u(t)$, normal cells, $N(t)$ and two control variables $v(t)$ and $w(t)$. We also assume a homogeneity of the tumour cells, then the model takes the form

$$\begin{aligned}\frac{dE(t)}{dt} &= \sigma + \frac{\rho E(t-\tau)T(t-\tau)}{\eta + T(t-\tau)} - \mu E(t-\tau)T(t-\tau) - \delta E(t) - a_1(1-e^{-u})E(t) + w(t)s_1, \\ \frac{dT(t)}{dt} &= r_2T(t)(1-\beta T(t)) - nE(t)T(t) - c_1N(t)T(t) - a_2(1-e^{-u(t)})T(t), \\ \frac{dN(t)}{dt} &= r_3N(t)(1-\beta_2N(t)) - c_2T(t)N(t) - a_3(1-e^{-u(t)})N(t), \\ \frac{du(t)}{dt} &= v(t) - d_1u(t).\end{aligned}\tag{30}$$

The general goal is to keep the patient healthy while killing the tumour. Since our model takes into account the toxicity of the drug to all types of cells, our control problem consists of determining the variables $v(t)$ and $w(t)$ that will maximize the amount of effector cells and minimize the number of tumour cells and the cost of the control with the constraint that we do not kill too many normal cells. Therefore, our objective is to maximize the functional

$$J(v, w) = \int_0^{t_f} \left(E - T - \left[\frac{B_v}{2}[v(t)]^2 + \frac{B_w}{2}[w(t)]^2 \right] \right) dt,\tag{31}$$

where B_u , B_w are, respectively, the weight factors that describe the patient's acceptance level of chemotherapy and immunotherapy with a constraint

$$k(E, T, N, u, E_\tau, T_\tau, v) = N - 0.75 \geq 0, \quad 0 \leq t \leq t_f.\tag{32}$$

We are seeking optimal control pair (v^*, w^*) such that

$$J(v^*, w^*) = \max\{J(v, w) : (v, w) \in W\},\tag{33}$$

where W is the control set defined by

$$\begin{aligned}W &= \{(v, w) : (v, w) \text{ piecewise continuous, such that} \\ &0 \leq v(t) \leq v_{\max} < \infty, ; 0 \leq w(t) \leq w_{\max} < \infty, \forall t \in [0, t_f]\}.\end{aligned}\tag{34}$$

The existence of optimal controls $v^*(t)$ and $w^*(t)$ for this model is guaranteed by standard results in Optimal Control Theory. Necessary conditions that the controls must satisfy are derived via Pontryagin's Maximum Principle. The optimal control problem given by expressions (29)-(33) is equivalent to that of minimizing the Hamiltonian H :

$$\begin{aligned}H(t, E, T, E_\tau, T_\tau, u, v, w, \lambda) &= E - T - \frac{B_v}{2}[v(t)]^2 - \\ &\frac{B_w}{2}[w(t)]^2 + \lambda_1 \frac{dE}{dt} + \lambda_2 \frac{dT}{dt} + \lambda_3 \frac{dN}{dt} + \lambda_4 \frac{du}{dt} + \gamma k\end{aligned}\tag{35}$$

and $\gamma \geq 0$ with $\gamma(t)k(t) = 0$, where

$$\gamma = \begin{cases} 1 & \text{if } N(t) \leq 0.75, \\ 0 & \text{otherwise} \end{cases}$$

A standard application of Pontryagin's Maximum Principle leads to the following result:

Theorem 5. There exists an optimal pair $v^*(t)$ and $w^*(t)$ and corresponding solutions E^* , T^* , N^* and u^* and that minimizes $J(u(t), w(t))$ over Ω . The explicit optimal controls are connected to the existence of continuous specific functions λ_i for $i=1,2,3,4$ satisfying the adjoint system

$$\begin{aligned}
\lambda_1'(t) &= -1 + \lambda_1(t) \left[\delta + a_1(1 - e^{-u^*}) \right] + \\
&\quad + \lambda_2(t)nT^* + \lambda_1(t + \tau) \chi_{[0, t_f - \tau]} \left[\mu T^* - \frac{\rho T^*}{\eta + T^*} \right], \\
\lambda_2'(t) &= 1 + \lambda_2 \left[-r_2 + 2r_2\beta T^* + nE^* + c_1N^* + a_2(1 - e^{-u^*}) \right] + \\
&\quad + \lambda_3c_2N^* + \chi_{[0, t_f - \tau]} \lambda_1(t + \tau) \left[\frac{\rho E^* T^*}{(\eta + T^*)^2} - \frac{\rho E^*}{\eta + T^*} + \mu E^* \right], \\
\lambda_3'(t) &= \lambda_2c_1T^* - \lambda_3 \left(r_3 - 2r_3\beta_2N^* - c_2T^* - a_3(1 - e^{-u^*}) \right) - \gamma, \\
\lambda_4'(t) &= -\lambda_1(t)a_1e^{-u^*} E^* + \lambda_2(t)a_2e^{-u^*} T^* + \lambda_3(t)a_3e^{-u^*} N^* + \lambda_4(t)d_1,
\end{aligned} \tag{36}$$

with transversality conditions

$$\lambda_i(t_f) = 0, i = \{1, 2, 3, 4\} \text{ and } \chi_{[0, t_f - \tau]} = \begin{cases} 1 & \text{if } t \in [0, t_f - \tau], \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Furthermore, the following properties hold

$$v^* = \min \left(v_{max}, \frac{\lambda_4}{B_v} \right), \quad w^* = \min \left(w_{max}, \frac{\lambda_1 s_1}{B_w} \right). \tag{38}$$

Delay Differential Equations with Fractional-Order. The subject of fractional calculus (that is, calculus of integral and derivatives of arbitrary order) has gained popularity and importance, mainly due to its demonstrated applications in numerous diverse and widespread fields of science and engineering. There are two main differences between integer-order derivation and fractional-order derivation. Firstly, the integer-order derivative indicates a variation or certain attribute at particular time for a physical or mechanical process, while the fractional-order derivative is concerned with the whole time domain. Secondly, the integer-order derivative describes the local properties of a certain position, while the fractional-order derivative is related to the whole space for a physical process. Accordingly, accurate modelling of many physical, chemical and biological phenomena may require fractional-order differential equations (FODEs) models.

In this Chapter, we provide a class of delay differential equations with derivative of fractional-order and applications.

As an example in ‘‘Fractional order delayed ‘‘predator-prey’’ systems with Holling type-II functional response’’ we provide a fractional dynamical system of predator-prey with Holling type-II functional response and time delay is studied. Local and global stability of existence steady states and Hopf bifurcation with

respect to the delay is investigated, with fractional-order $0 < \alpha \leq 1$. It is found that Hopf bifurcation occurs when the delay passes through a sequence of critical values. Unconditionally, stable implicit scheme for the numerical simulations of the fractional-order delay differential model is introduced. The numerical simulations show the effectiveness of the numerical method and confirm the theoretical results. The presence of fractional order in the delayed differential model improves the stability of the solutions and enrich the dynamics of the model.

In this paper, we extend the derivatives of to an arbitrary order to investigate the combination of both fractional-order formulation and time-delay, so that

$$\begin{aligned} D^\alpha x(t) &= rx(t) \left(1 - \frac{x(t)}{K}\right) - \frac{\beta x(t)y(t-\tau)}{1 + \sigma x(t)} \\ D^\alpha y(t) &= \frac{\beta x(t)y(t-\tau)}{1 + \sigma x(t)} - ay(t), \quad 0 < \alpha \leq 1, t \geq 0 \end{aligned} \quad (39)$$

with initial conditions $x(0) > 0$ and $y(t) = \psi(t) > 0$, when $t \in [-\tau, 0]$, where $\psi(t)$ is a smooth function. We next study the impact of the fractional-order and time delay τ in the dynamics of the model.

Definition 1. The fractional integral of order $\alpha \in R^+$ for a function $f(t)$ is described as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Definition 2. The Caputo derivative of fractional-order α for the function $f(t)$ is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau) d\tau,$$

where $n-1 < \alpha < n \in Z^+$.

To study the global stability of the equilibrium points of (38), we linearize the system into the form

$$\begin{aligned} D^\alpha x(t) &= m_1 x(t) + m_2 y(t-\tau) \\ D^\alpha y(t) &= n_1 x(t) + n_2 y(t) + n_3 y(t-\tau) \end{aligned} \quad 0 < \alpha \leq 1. \quad (40)$$

where

$$\begin{aligned} m_1 &= r - \frac{2x^*}{K} - \frac{\beta y^*}{1 + \sigma x^*} + \frac{\sigma \beta x^* y^*}{(1 + \sigma x^*)^2}, \\ m_2 &= -\frac{\beta x^*}{1 + \sigma x^*}, \quad n_1 = \frac{\beta y^*}{1 + \sigma x^*} - \frac{\sigma \beta x^* y^*}{(1 + \sigma x^*)^2}, \\ n_2 &= -a, \quad n_3 = \frac{\beta x^*}{1 + \sigma x^*}. \end{aligned}$$

If the linear fractional differential equation has non-zero equilibrium point, we can shift equilibrium point to the origin. Put $\bar{x}(t) = x(t) - x^*$, $\bar{y}(t) = y(t) - y^*$, then the equations (39) becomes

$$\begin{aligned} D^\alpha \bar{x}(t) &= m_1 \bar{x}(t) + m_2 \bar{y}(t-\tau) \\ D^\alpha \bar{y}(t) &= n_1 \bar{x}(t) + n_2 \bar{y}(t) + n_3 \bar{y}(t-\tau) \end{aligned} \quad 0 < \alpha \leq 1. \quad (41)$$

In order to study the stability of system (38), we take Laplace transform on both sides of (40). Then we have

$$\begin{aligned} s^\alpha X_1(s) &= m_1 X_1(s) + s^{\alpha-1} \varphi_1(0) + m_2 e^{-s\tau} (X_2(s) + \int_{-\tau}^0 e^{-st} \varphi_2(t) dt) \\ s^\alpha X_2(s) &= n_1 X_1(s) + n_2 X_2(s) + s^{\alpha-1} \varphi_2(0) + n_3 e^{-s\tau} (X_2(s) + \int_{-\tau}^0 e^{-st} \varphi_2(t) dt). \end{aligned} \quad (42)$$

Here, it should be mentioned that the initial values $\bar{x}(t) = \varphi_1(t)$ and $\bar{y}(t) = \varphi_2(t)$ with $t \in [-\tau, 0]$. Also, $X_1(s)$ and $X_2(s)$ are Laplace transform of $\bar{x}(t)$ and $\bar{y}(t)$ with $X_1(s) = L(\bar{x}(t))$ and $X_2(s) = L(\bar{y}(t))$. The system (41) can be rewritten as follows

$$\Delta(s) \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} k_1(s) \\ k_2(s) \end{pmatrix} \quad (43)$$

in which

$$\Delta(s) = \begin{pmatrix} s^\alpha - m_1 & -m_2 e^{-s\tau} \\ -n_1 & s^\alpha - n_2 - n_3 e^{-s\tau} \end{pmatrix}$$

and

$$\begin{aligned} k_1(s) &= s^{\alpha-1} \varphi_1(0) + m_2 e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi_2(t) dt \\ k_2(s) &= s^{\alpha-1} \varphi_2(0) + n_3 e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi_2(t) dt. \end{aligned}$$

$\Delta(s)$ is considered as characteristic matrix of system (38) and $\det \Delta(s)$ as its characteristic polynomial. Therefore the distribution of the eigenvalues of the characteristic polynomial determines the stability of system (38). In other words, if all roots of the characteristic equation have negative parts, then the equilibrium of the above fractional-order PP system is Lyapunov globally asymptotical stable if the equilibrium exists. If we multiply both sides of (42) by s , we have

$$\Delta(s) \begin{pmatrix} sX_1(s) \\ sX_2(s) \end{pmatrix} = \begin{pmatrix} sk_1(s) \\ sk_2(s) \end{pmatrix} \quad (44)$$

Therefore, if all roots of the transcendental equation $\det \Delta(s) = 0$ lie in open left complex plane, i.e., $\text{Re}(s) < 0$, then we consider (43) in $\text{Re}(s) \geq 0$. In this restricted area, system (43) has a unique solution $(sX_1(s), sX_2(s))$, so that

$$\lim_{s \rightarrow 0, \text{Re}(s) \geq 0} sX_i(s) = 0, \quad i = 1, 2.$$

From the assumption of all roots of the characteristic equation $\det \Delta(s) = 0$ and the final-value theorem of Laplace transform, we get

$$\begin{aligned} \lim_{t \rightarrow +\infty} \bar{x}(t) &\equiv \lim_{s \rightarrow 0, \text{Re}(s) \geq 0} sX_1(s) = 0, \\ \text{and } \lim_{t \rightarrow +\infty} \bar{y}(t) &\equiv \lim_{s \rightarrow 0, \text{Re}(s) \geq 0} sX_2(s) = 0. \end{aligned}$$

It implies that the zero solution of the fractional-order PP system is Lyapunov globally asymptotically. Therefore, we arrive at the following result.

Theorem 6. If all the roots of the characteristic equation $\det \Delta(s) = 0$ have negative real parts, then the positive equilibrium points (x^*, y^*) of system (38) is Lyapunov globally asymptotically stable.

In ‘‘Stability of fractional-order prey-predator system with time-delay and Monod-Haldane functional response’’ we study the dynamics of a fractional-order

delayed prey-predator system with Monod-Haldane response function. The model describes the interaction between prey and two populations of predator species: immature or juvenile and mature or adult predator. A single time delay is incorporated in the model to justify the gestation period of adult predator. The existence of solutions, steady states, and sufficient conditions to ensure the asymptotic stability of the steady states are investigated. Global stability around the interior equilibrium point by constructing the suitable Lyapunov functional is also investigated. The system displays Hopf bifurcation depending on the conversion coefficient (prey to immature predator) and time delay. The fractional-order derivatives in the model develop the stability results of solutions and improve the model behaviors.

We consider a fractional-order prey-predator model with delays and Monod-Haldane functional response of the form

$$\begin{aligned} D^\alpha x(t) &= px(t)\left(1 - \frac{x(t)}{k}\right) - \frac{\delta x(t)z(t-\tau)}{\varphi + \beta x^2(t)}, \\ D^\alpha y(t) &= \frac{c\delta x(t)z(t-\tau)}{\varphi + \beta x^2(t)} - (\kappa + \mu_1)y(t), \\ D^\alpha z(t) &= \kappa y(t) - \mu_2 z(t), \end{aligned} \quad (45)$$

with initial conditions $x(0) = x_0 > 0$, $y(0) = y_0 > 0$, $z(s) = \chi(s) > 0$, $s \in [-\tau, 0]$, and $\chi(s)$ is smooth function. Here, D^α is the Caputo fractional derivative of order $0 < \alpha \leq 1$. The parameter p is the biotic potential that the prey species grow logistically without any predator species. k represents the environmental carrying capacity. δ is maximal prey population uptake of mature predators. β denotes the inverse measure of inhibitory effect and φ is half saturation constant. c defines the conversion co-efficient (conversion of prey to juvenile predators) and κ represents the transition rate on the evolution of juvenile and adult predators. μ_1 and μ_2 denote the natural death rate of juvenile and adult predator populations, respectively.

Theorem 7. For every $(x_0, y_0, z_0) \in \Omega$, then there exists a unique solution $X = (x, y, z) \in \Omega$ of system (45), where $\Omega = \{(x, y, z) \in R^3 : \max\{|x|, |y|, |z|\} \leq K\}$.

System (45) has a trivial equilibrium point $E_0(0,0,0)$, axial equilibrium point or predator-free equilibrium point $E_1(k,0,0)$ and an interior equilibrium point

$E_2(x^*, y^*, z^*)$, where $y^* = \frac{\mu_2 z^*}{\kappa}$, and x^* and z^* satisfy the quadratic equations

$$z^* = \frac{p}{\delta} (\varphi + \beta (x^*)^2) \left(1 - \frac{x^*}{k}\right), \quad \beta (x^*)^2 - \frac{c\kappa\delta}{(\kappa + \mu_1)\mu_2} x^* + \varphi = 0.$$

Herein, we investigate the local and global asymptotic stabilities of fractional-order delayed prey-predator system (45).

Delay Differential Equations with Neural Networks. In this Chapter, we study and analyze delay differential equations with different types time-lags and fractional orders for modeling neural networks.

Neural network is a growing field of research and has attracted interest due to its widespread applications in various fields of optoelectronics, imaging, quantum devices, computer version, filtering, speech synthesis, remote sensing, and spatiotemporal analysis of physiological neural systems. Time delay is an unavoidable factor during the signal transmission between the neurons because of the finite switching speed of neurons and amplifiers. The impact of time delay in the model may lead to some complex dynamical behaviors of the whole system in terms of instability, chaos, periodic, and poor performance. Especially, stability research on delayed neural networks has gained significant progress. An advantage of using fractional-order models compared with classical integer-order models is that the fractional-order systems have infinite memory. Therefore, it is necessary to investigate the impact of fractional-order on the dynamics of neural networks. Recently, a combination of fractional-order and time-delays in the dynamic systems of neural networks has received much more attention to many researchers.

We also study stability analysis of such dynamic behaviors, which include steady state conditions, synchronization, dissipativity and stabilization, is essential for ensuring safe application of physical and engineering systems in the real world. The synchronization studies for the neural network model based on suitable control techniques are important in both theoretical and applications sense. Stabilization results dealing with chaotic system is to control the states of the neural network model to some equilibrium points or periodic orbits.

In “Stabilization of delayed Gohen-Grossberg BAM neural networks” we deal with finite-time stabilization results of delayed Cohen-Grossberg BAM neural networks under suitable control schemes. We propose a state-feedback controller together with an adaptive-feedback controller to stabilize the system of delayed Cohen-Grossberg BAM neural networks. Stabilization conditions are derived by using Lyapunov function and some algebraic conditions. We also estimate the upper bound of settling time functional for the stabilization, which depends on the controller schemes and system parameters. Consider the delayed CGBAMNNs of the form

$$\begin{aligned}\dot{x}_i(t) &= -p_i(x_i(t))\{q_i(x_i(t)) - \sum_{j=1}^m r_{ij}f_j(y_j(t)) - \sum_{j=1}^m s_{ij}f_j(y_j(t-\tau_{ij}))\}, \\ \dot{y}_j(t) &= -P_j(y_j(t))\{Q_j(y_j(t)) - \sum_{i=1}^n R_{ji}g_i(x_i(t)) - \sum_{i=1}^n S_{ji}g_i(x_i(t-\sigma_{ji}))\},\end{aligned}\tag{46}$$

for $i, j = 1, 2, \dots, n$, and $x_i(t)$ and $y_j(t)$ are the state variables in the i^{th} neuron field F_X and j^{th} neuron field F_Y at time t , respectively. The functions p_i, P_j denote the abstract amplification functions, while, q_i, Q_j denote the behaved functions (self-excitation rate functions). $r_{ij}, R_{ji}, s_{ij}, S_{ji}$ are the connection weights, which denote the strengths of connectivity between the cell j in F_Y and cell i in F_X at time $t, t-\tau_{ij}, t-\sigma_{ji}$, respectively. f_j represents the activation functions of the j^{th} neuron

from F_Y and g_i represents the activation functions of the i^{th} neuron from F_X at time t . The initial values of system (45) are

$$x_i(s) = \psi_j(s), \quad y_j(s) = \varphi_i(s), \quad s \in [-\gamma, 0].$$

Here, $\gamma = \max_{1 \leq i \leq n, 1 \leq j \leq n} \{ \tau_{ij}, \sigma_{ji} \}$, and $\psi_j(\cdot), \varphi_i(\cdot)$ denote real valued continuous functions defined on $[-\gamma, 0]$.

The main goal of this paper is to stabilize the CGBAMNNs (45) in finite-time based on state and adaptive-feedback control techniques, where controlled system of CGBAMNNs(45) takes the form

$$\begin{aligned} \dot{x}_i(t) &= -p_i(x_i(t)) \{ q_i(x_i(t)) - \sum_{j=1}^m r_{ij} f_j(y_j(t)) - \sum_{j=1}^m s_{ij} f_j(y_j(t - \tau_{ij})) \} + u_i(t), \\ \dot{y}_j(t) &= -P_j(y_j(t)) \{ Q_j(y_j(t)) - \sum_{i=1}^n R_{ji} g_i(x_i(t)) - \sum_{i=1}^n S_{ji} g_i(x_i(t - \sigma_{ji})) \} + v_j(t). \end{aligned} \quad (47)$$

The controllers $u_i(t)$ and $v_j(t)$ will be defined, later on, according to the concepts of state-feedback and adaptive control techniques.

Numerical Treatments of Differential Equations With Memory. The numerical treatments of delay differential equations and related issues due to the presence of the time-delays still remain a relatively unexplored area of research. Herein, we adapt Mono-Implicit Runge-Kutta schemes for numerical approximations of delay differential equations. The schemes are developed to reduce the computational cost of the fully implicit method which combine the accuracy of implicit method and efficient implementation. Numerical stability properties of the schemes are investigated.

There are also some important challenges in numerical approximations of the fractional-order differential equations. Fractional differential equations are integro-differential equations, and the numerical solution requires large computer memory and long runs of numerical simulations which makes it very difficult to investigate general properties of fractional dynamical system. As a consequence, accurate approximation and suitable numerical technique are playing important role in identifying the solution behavior of such fractional equations and exploring their applications and the references therein). In this Chapter, we provide unconditionally stable numerical schemes for differential equations with memory (delay differential equations without and with fractional-order). The methods are suitable for stiff and non-stiff problems.

In “Numerical treatments for Volterra delay integro-differential equations” we present a new technique for numerical treatments of delay differential equations and Volterra delay integro-differential equations that have many applications in biological and physical sciences. The technique is based on the mono-implicit Runge-Kutta method for treating the differential part and the collocation method (using Boole’s quadrature rule) for treating the integral part. The efficiency and stability properties of this technique have been studied. Numerical results are presented to demonstrate the effectiveness of the methodology.

In this paper, we consider a scalar VDIDE of the form

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau), \int_{a(t)}^t g(t, s, y(s)) ds), \quad \text{for } t \geq 0, \\ y(t) &= \phi(t), \quad t \leq 0, \end{aligned} \quad (48)$$

with fixed $\tau > 0$ and the integral term in (47) introduces either unbounded time-lag *i.e.* $a(t) = 0$ or bounded time-lag *i.e.* $a(t) = t - \tau$. The functions f, g are assumed to be sufficiently smooth with respect to their arguments, and $\phi(t)$ is an initial function which is assumed to be continuous. To solve (47) numerically we need not only approximation to the solution at the proposed mesh-points but also to create the solution at the non-mesh points to produce dense-output of the solution in order to approximate the delay term $y(t - \tau)$ and the integral term $\int_{a(t)}^t g(t, s, y(s)) ds$. The proposed technique is based on Continuous Mono-Implicit Runge-Kutta (CMIRK) method to treat the differential part, and Boole's quadrature rule to treat the integral part.

Mono-implicit Runge-Kutta scheme is used to solve the VDIDE (47) on the defined mesh $\Delta = \{0 < t_1 < \dots < t_n < \dots < t_i = T\}$ as follows

$$y_{n+1} = y_n + h_n \sum_{r=1}^s b_r K_{n+1}^r, \quad (49)$$

where stages are defined by

$$K_{n+1}^r = f(t_n + c_r h_n, (1 - v_r) y_n + v_r y_{n+1} + h_n \sum_{j=1}^{r-1} x_{rj} K_{n+1}^j, y(t_n + c_r h_n - \tau), I_n). \quad (50)$$

Here $I_n = \int_{a(t_n + c_r h_n)}^{t_n + c_r h_n} g(t, s, y(s)) ds$ is evaluated numerically by Boole's quadrature rule.

The fourth order ($s = 5$) discrete MIRK scheme is defined by the tableau

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{20} & \frac{29}{4000} & \frac{361}{8000} & -\frac{19}{8000} & 0 & 0 \\ \frac{19}{20} & \frac{3971}{4000} & \frac{19}{8000} & -\frac{361}{8000} & 0 & 0 \\ \frac{1}{2} & \frac{11}{16} & \frac{1}{32} & \frac{267}{608} & \frac{25}{684} & -\frac{25}{36} \\ & & -\frac{43}{228} & -\frac{43}{228} & \frac{25}{57} & \frac{25}{57} \end{array}$$

The discrete solution at the mesh points obtained using discrete MIRK scheme is not enough to solve the VDIDEs. Dense output solution is needed at non-mesh points as explained earlier, using fourth order CMIRK scheme (with $s^* = 5$), such that

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{20} & \frac{29}{4000} & \frac{361}{8000} & -\frac{19}{8000} & 0 & 0 \\ \frac{19}{20} & \frac{3971}{4000} & \frac{19}{8000} & -\frac{361}{8000} & 0 & 0 \\ \frac{1}{2} & \frac{11}{16} & \frac{1}{32} & \frac{267}{608} & \frac{25}{684} & -\frac{25}{36} \end{array}$$

$$b_1(\theta) \ b_2(\theta) \ b_3(\theta) \ b_4(\theta) \ b_5(\theta)$$

where

$$b_1(\theta) = -\frac{1}{228}\theta(1200\theta^3 - 2714\theta^2 + 1785\theta - 228), (6)$$

$$b_2(\theta) = \frac{1}{228}\theta^2(1200\theta^2 - 2086\theta + 843), (7)$$

$$b_3(\theta) = \frac{25}{171}\theta^2(40\theta^2 - 86\theta + 49), (8) \tag{51}$$

$$b_4(\theta) = -\frac{25}{171}\theta^2(40\theta^2 - 74\theta + 31), (9)$$

$$b_5(\theta) = -\frac{1}{2}\theta^2(2\theta - 3).$$

This numerical method is unconditionally stable and suitable for stiff and non-stiff problems.

In ‘‘Computational methods for delay parabolic and time-fractional partial differential equations’’ we also provided unconditionally stable numerical implicit schemes for solving time-fractional partial differential equations with time-delays of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = F(t, u(x,t), Lu(x,t), u(x,t-\tau)), x \in \Omega, t \geq 0, \quad 0 < \alpha \leq 1, \tag{52}$$

where F is a continuous function, $t-\tau \leq t$ (the condition on $t-\tau \rightarrow \infty$ as $t \rightarrow \infty$), and L is elliptic in the spatial-direction. τ could be a constant, a function of time, $\tau(t)$, or even state dependent, $\tau(t, y)$.

The numerical technique have been used for a class of fractional-order differential models of biological systems with memory, such as dynamics of tumor-immune system and dynamics of HIV infection of CD4+ T cells. The method is based on Caputo fractional derivative of order α and implicit Euler’s approximation. The numerical simulations confirm the advantages of the numerical technique and using fractional-order differential models in biological systems over the differential equations with integer order.

Parameter Estimations and Sensitivity Analysis for Delay Differential Equations.

In this Chapter, we present suitable numerical approaches to find best-fit and meaningful parameters when given experimental data (inverse problem). We also investigate the sensitivity analysis of delay differential equations.

Parameter identification problem, to estimate the values of the parameters which appear in model equations, is considered as an inverse problem. We assume that we have observed our system and collected data; we then wish to determine the unknown parameters by fitting the model equations to the data. Consider a general form of predictive DDE model

$$\begin{aligned} \mathbf{y}'(t) &= \mathbf{f}(t, \mathbf{y}(t), \mathbf{y}(t-\tau); \mathbf{p}), \quad t \in [0, T], \\ \mathbf{y}(t) &= \psi(t, \mathbf{p}), \quad t \in [-\tau, 0]. \end{aligned} \tag{53}$$

This model is parameterized by $\mathbf{p} \in R^L$ which are estimated using a given set of observations, $\{t_j; Y_j^i\}_{j=1}^N$. We assume that, in (52), the vector function \mathbf{f} is sufficiently smooth with respect to each arguments; $\mathbf{y}(t) \in R^M$, $\mathbf{y}(t-\tau) \in R^{M'}$, $\mathbf{p} \in R^L$, and $\tau \in R^L$ is positive constant lag, which may have to be identified as a parameter

($L' \leq L$, $M' \leq M$). $\psi(t)$ is given continuous function. Our concern is to fit the given data to the system. The model-fitting problem is then select a value or a set values for \mathbf{p} for which the function $\mathbf{y}(t; \mathbf{p})$ provides a ‘best’ fit, at arguments $t = t_j$, to the given set $\{Y_{ij}\}_{j=1}^N$ ($1 \leq i \leq M$). The key part in fitting a model to data is the formulation of the objective function to be optimized that depends on the stochastic features of the errors in the data.

We assume that the data \mathbf{Y}_i satisfy the following observation equation

$$Y_{ij} = y_j(t_i) + \sigma_j \varepsilon_{ij} \quad (54)$$

where $\sigma_j > 0$ measures the variance of the noise associated with the j th component and is related to the scale of the expected magnitude of the j th component, $|y_j(t)|$. The ε_{ij} are independent and standard Gaussian distributed random variables. The principle of maximum-likelihood yields an appropriate cost function which should be minimized with respect to the parameters \mathbf{p} to yield an approximation, $\hat{\mathbf{p}}$ to the true value. We define the cost function or objective function by

$$\Phi(\mathbf{P}) = \sum_{i=1}^M \varepsilon_i^T \omega_i(\sigma) \varepsilon_i \equiv \frac{1}{N} \sum_{i=1}^M \sum_{j=1}^N \frac{[y_j(t_i; \mathbf{p}) - Y_{ij}]^2}{2\sigma_j^2}. \quad (55)$$

We seek $\hat{\mathbf{p}}$ that satisfies

$$\Phi(\hat{\mathbf{p}}) =: \min_{\mathbf{p}} \Phi(\mathbf{p}) \equiv \max_{\mathbf{p}} L(\mathbf{p}). \quad (56)$$

where $L(\mathbf{p}) = [\exp(-\varepsilon_{ij}^2 / 2\sigma_j^2)] / \sqrt{2\pi\sigma_j^2}$ is the likelihood function.

If we adapt the Log Least Squares (LLS) approach, the objective function may take the form

$$\Phi_L(\mathbf{p}) = \frac{1}{N} \sum_{i=1}^M \sum_{j=1}^N [\log y_j(t_i, \mathbf{p}) - \log Y_{ij}]^2 / 2\sigma_j^2. \quad (57)$$

The choice of LLS in model-fitting problem may decrease the exponential nonlinearity of model predictions with respect to \mathbf{p} . (It is assumed that $y^j(t_i, \mathbf{p}) > 0$.) Another significant feature of the LLS approach is that small relative changes in large data values can be unduly weighted.

The methods for minimizing $\Phi(\mathbf{p})$ are iterative in nature. We start with a given point \mathbf{p}_1 known as the initial guess, and proceed to generate a sequence of points $\mathbf{p}_2, \mathbf{p}_3, \dots$ which we hope that they converge to the point \mathbf{p} at which $\Phi(\mathbf{p})$ is minimum. In practice, one terminates the sequence after a finite number k of iterations, and one accepts \mathbf{p}_k as an approximation to \mathbf{p} . The vector

$$\delta_i = \mathbf{p}_{i+1} - \mathbf{p}_i \quad (58)$$

is called the i th step. We wish each step to bring us closer to the minimum. Since we do not know where the minimum is, we cannot test for this condition directly. In a sense, however, we may consider the i th step to have ‘‘improved’’ our situation if

$$\Phi_{i+1} < \Phi_i, \quad (59)$$

where $\Phi_j = \Phi(\mathbf{p}_j)$ ($j=1,2,\dots$). We call the i th step *acceptable* if equation (58) holds. An iterative method is acceptable if all the steps of its procedures are acceptable. We shall only consider acceptable methods. The methods we consider are then based on following scheme:

Set $i=1$. An initial guess \mathbf{p}_1 must be provided.

The model solution values $\{\mathbf{y}(t_j, \mathbf{p}_1)\}$ are obtained numerically.

Determine a vector \mathbf{v}_i in the direction of the proposed i th step.

Determine a scalar ρ_i such that the step

$$\delta_i = \rho_i \mathbf{v}_i$$

is acceptable. That is, we take

$$\mathbf{p}_{i+1} = \mathbf{p}_i + \rho_i \mathbf{v}_i \quad (60)$$

and require that ρ_i be chosen so that equation (58) holds.

Test whether the termination criterion

$$|\mathbf{p}_{i+1,j} - \mathbf{p}_{i,j}| \leq \epsilon_j \quad (j=1,2,\dots,L),$$

where $\mathbf{p}_{i,j}$ is the j th component of \mathbf{p}_i . If not, increase i by one and return to step 3. Otherwise, accept \mathbf{p}_{i+1} as the value of \mathbf{p} .

In ‘‘Sensitivity analysis for dynamic systems with time-lags’’ we study the sensitivity analysis for dynamic systems with time-lag. Many problems in bioscience for which observations are reported in the literature can be modelled by suitable functional differential equations incorporating time-lags (other terminology: delays) or memory effects, parameterized by scientifically meaningful constant parameters \mathbf{p} or/and variable parameters (for example, control functions) $u(t)$. It is often desirable to have information about the effect on the solution of the dynamic system of perturbing the initial data, control functions, time-lags and other parameters appearing in the model. The main purpose of this paper is to derive a general theory for sensitivity analysis of mathematical models that contain time-lags. We use adjoint equations and direct methods to estimate the sensitivity functions when the parameters appearing in the model are not only constants but also variables of time.

We consider a class of systems modelled by delay differential equations (DDEs) of the form:

$$\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t), \mathbf{y}(t-\tau), \mathbf{u}(t), \mathbf{u}(t-\sigma), \mathbf{p}), \quad 0 \leq t \leq T, \quad (61a)$$

$$\mathbf{y}(t) = \Psi(t, \mathbf{p}), \quad t \in [-\tau, 0), \quad \mathbf{y}(0) = \mathbf{y}_0 \in R^n \quad (61b)$$

$$\mathbf{u}(t) = \Phi(t), \quad t \in [-\sigma, 0), \quad \mathbf{u}(0) = \mathbf{u}_0 \in R^m, \quad (61c)$$

where the vector function \mathbf{f} in the right-hand side is sufficiently smooth with respect to each arguments; $\mathbf{y}(t) \in R^n$, $\mathbf{y}(t-\tau) \in R^{n'}$, $\mathbf{u}(t) \in R^m$, $\mathbf{u}(t-\sigma) \in R^{m'}$, $\mathbf{p} \in R^r$, and $\tau \in R^{r'}$ and $\sigma \in R^{r''}$ are positive constant lags ($r', r'' \leq r$, $n' \leq n$, $m' \leq m$). $\Psi(t)$ and $\Phi(t)$ are given continuous functions. We note that $\mathbf{u}(t)$ in (61a) can be viewed as a control variable, defined on $[-\sigma, T]$, that gives a minimum to the objective functional

$$J(\mathbf{u}) = F_0(\mathbf{y}(T)) + \int_0^T F_1(t, \mathbf{y}(t), \mathbf{y}(t-\tau), \mathbf{u}(t), \mathbf{u}(t-\sigma), \mathbf{p}) dt, \quad (62)$$

where F_0 and F_1 are continuous functionals.

Definition 3. For the given DDEs (61a)-(61c):

The sensitivity coefficients, when the parameters are constants, are defined by the partial derivatives

$$S_{ij}(t) = \frac{\partial y_i(t)}{\partial \alpha_j}, \quad (63)$$

where α_j represent the parameters p_j , the constant lags τ_j or the initial values $y_j(0)$. Then the total variation in $y_i(t)$ due to small variations in the parameters α_j is such that

$$\delta y_i(t) = \sum_j \frac{\partial y_i(t)}{\partial \alpha_j} \delta \alpha_j + O(|\alpha|^2). \quad (64)$$

Thus Eq. (62) estimates the sensitivity of the state variable to small variations in the parameters α_j .

The functional derivative sensitivity coefficients, when the parameters are functions of time, are defined by

$$\beta_{ij}(t, t^*) = \frac{\partial y_i(t^*)}{\partial u_j(t)}, \quad t < t^*. \quad (65)$$

Then the total variation in $y_i(t^*)$ due to any perturbation in the parameters $u_j(t)$ is, denoted by $\delta y_i(t^*)$, such that

$$\delta y_i(t^*) = \int_0^{t^*} \frac{\partial y_i(t^*)}{\partial u_j(t)} \delta u_j(t) dt, \quad t < t^*. \quad (66)$$

Thus the functional derivative sensitivity density function $\frac{\partial y_i(t^*)}{\partial u_j(t)}$ measures the sensitivity of $y_i(t)$ at location t^* to variation in $u_j(t)$ at any location $t < t^*$. It is then noted that the sensitivity density functions inherently contain and provide more information than the sensitivity coefficients.

Conclusion

The main focus of this work was in studying Qualitative and Quantitative Features of Delay Differential Equations (Models with Memory) with Applications. We have conducted research in the following areas:

- (i) Qualitative analysis of differential equations with memory (time-delays);
- (ii) Dynamical analysis of Neural Networks with Time-delay;
- (iii) Dynamical analysis of biological systems.

We established strong research groups in the following points of study:

Numerical mathematical modeling using differential equations with memory;
Inverse problems, parameter estimations, sensitivity analysis with DDEs;

Mathematical analysis of an SIS model with imperfect vaccination and backward bifurcation;

Numerical method based on extended one-step schemes for optimal control problem with time-lags;

Delay Differential Model for Tumour-Immune Response with Chemo-immunotherapy and Optimal Control. Pontryagin's maximum (or minimum) principle and Incorporate optimal control parameters into a delay differential model to describe the interactions of the disease response cells with external therapy;

Dynamics of Tumor-Immune System with Fractional-Order; Fractional SIRC Model with Salmonella Bacterial Infection;

A fractional-order delay differential model for Ebola infection and CD8+ T-cells response: Stability analysis and Hopf bifurcation;

Fractional-order delayed predator-prey systems with Holling type-II functional response;

Dynamics of Hepatitis C Virus Infection: Mathematical Modeling and Parameter Estimation; Jacobian matrix, Eigenvalues method, Routh-Hurwitz criterion, center manifold theorem, norm form method, and Lyapunov functional;

Stability of fractional-order prey-predator system with time-delays and monod-haldane functional response. Fractional calculus, linearized techniques and stability theory.

**НАУЧНЫЙ СОВЕТ DSc 27.06.2017.FM.01.02 ПО ПРИСУЖДЕНИЮ
УЧЕНЫХ СТЕПЕНЕЙ ПРИ НАЦИОНАЛЬНОМ УНИВЕРСИТЕТЕ
УЗБЕКИСТАНА**

**УНИВЕРСИТЕТ ОБЪЕДИНЕННЫХ АРАБСКИХ ЭМИРАТОВ И ИНСТИТУТ
МАТЕМАТИКИ АКАДЕМИИ НАУК РЕСПУБЛИКИ УЗБЕКИСТАН**

ФАТХАЛЛА АЛИ РИХАН

**КАЧЕСТВЕННЫЕ И КОЛИЧЕСТВЕННЫЕ АСПЕКТЫ ДИФФЕРЕНЦИАЛЬНЫХ
УРАВНЕНИЙ С ПАМЯТЬЮ И ИХ ПРИЛОЖЕНИЯ**

**05.01.07 – Математическое моделирование. Численные методы и программные
комплексы**

**АВТОРЕФЕРАТ ДИССЕРТАЦИИ ДОКТОРА (DSc)
ФИЗИКО-МАТЕМАТИЧЕСКИХ НАУК**

Ташкент – 2019

Докторская диссертация выполнена в Университете Объединенного Арабского Эмирата и Институте математики имени В.И. Романовского.

Автореферат диссертации на трех языках (узбекский, английский, русский (резюме)) размещен на веб-странице по адресу <http://fti-kengash.uz/> и на Информационно-образовательном портале "ZiyoNet" по адресу <http://www.ziyo.net.uz/>.

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Защита диссертации состоится « ____ » _____ 2019 года в ____ часов на заседании Научного совета 27.06.2017.FM.01.02 при Национальном университете Узбекистана. (Адрес: 100174, г. Ташкент, Алмазарский район, ул. Университетская, 4. Тел.: (99871)227-12-24, факс: (99871) 246-53-21, 246-02-24, e-mail: nauka@nuu.uz).

С диссертацией можно ознакомиться в Информационно-ресурсном центре Национального университета Узбекистана (зарегистрировано за № ____) (Адрес: 100174, г. Ташкент, Алмазарский район, ул. Университетская, 4. Тел.: (99871) 246-02-24).

Автореферат диссертации разослан « ____ » _____ 2019 года.
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ВВЕДЕНИЕ (аннотация докторской диссертации (DSc))

Актуальность и востребованность темы диссертации. По мере все большего внедрения современных компьютерных технологий, в технике и естествознании особую актуальность начало приобретать методы математического моделирования. При этом важное значение имеет адекватность модели с действительным моделируемым процессом. Оказалось, что математические модели посредством классических динамических систем не всегда отвечают этому требованию. Один из путей обеспечения адекватности составляет учет воздействия на текущий процесс значения параметров в прошлом. Этот фактор является существенным в приложениях математики в медицине, биологии и теории нейронных сетей. Диссертация посвящена математическому моделированию таких процессов посредством дифференциальных уравнений с запаздывающим аргументом и дифференциальных уравнений дробного порядка, которых в настоящее время принято называть дифференциальными уравнениями с памятью.

Целью исследования являются разработка, изучение и приложения математических моделей в биологии и медицине в форме дифференциальных уравнений с памятью.

Задачи исследования:

Построение математических моделей процессов терапии раковых опухолей, распространения инфекционных заболеваний, процесс передачи сигналов в нейронных сетях нахождение критериев устойчивости соответствующих систем, разработка методов численного решения дифференциальных и интегро-дифференциальных уравнений с запаздыванием, критериев чувствительности моделей к изменению параметров.

Объект исследования. Дифференциальные уравнения с запаздывающим аргументом, дифференциальные уравнения дробного порядка, математические модели динамики раковых опухолей, математические модели распространения вирусного гепатита и гриппа Эбола, нейронные сети.

Научная новизна исследования. Предложены новые математические модели, использующие обыкновенные дифференциальные уравнения с запаздыванием, с распределенным запаздыванием, а также дифференциальные уравнения дробного порядка и дробные уравнения в частных производных с запаздыванием для анализа динамики биосистем.

Показано, что учет запаздывания приводит к моделям, более адекватно описывающим реальные системы, чем без учета запаздывания. Разработаны эффективные численные методы для решения уравнений с запаздыванием, в том числе неявные схемы метода Рунге-Кутты.

Найдены новые критерии устойчивости численных методов.

Разработан метод исследования чувствительности математических моделей к малым возмущениям параметров и при случайных возмущениях в виде «белого шума».

Построены методы синхронизации, а также анализа устойчивости и диссипативности для различных моделей нейронных сетей, таких как мембранных нейронных сетей, комплекснозначных нейронных сетей, нейронных сетей Коэна-Гроссберга и нейронных сетей дробных порядка.

Дано описание обратной связи состояния и предложен алгоритм управления с целью стабилизации неустойчивых нейронных сетей Коэна-Гроссберга.

Разработан инновационный подход, основанный на теории матричных измерений для изучения диссипативности комплекснозначных нейронных сетей с временной задержкой.

Выводы диссертации.

1. Математические модели в виде дифференциальных уравнений с памятью или/и дробного порядка позволяют за счет выбора значения параметров запаздывания и порядка дифференцирования созданию более адекватных математических моделей.
2. Учет запаздывания информации при математическом моделировании динамических процессов в биологии, медицине и теории нейронных сетей не только обеспечивает близость модели к реальности, но также способствует улучшению критериев устойчивости.
3. Математические модели динамики раковых опухолей с учетом запаздывания окажутся более адекватными к экспериментальным данным, снятым с клинических наблюдений.
4. Математическая модель процесса лечения рака сочетанием химиотерапии и иммунотерапии позволяет вычислить оптимальный режим сочетания этих методов лечения.
5. Численные методы решения дифференциальных уравнений с рассредоточенным запаздыванием требует особый подход, при котором помимо схемы численного интегрирования по основной сетке необходимо еще и подсетка или привлечение неявных непрерывных схем метода Рунге-Кутты.
6. Математические модели нейронных сетей с учетом запаздывания позволяют улучшить критерии устойчивости.

ЗАКЛЮЧЕНИЕ

Основное внимание в этой работе было уделено изучению качественных и количественных характеристик дифференциальных уравнений с запаздыванием (модели с памятью) с приложениями. Мы провели исследования в следующих областях: (i) Качественный анализ дифференциальных уравнений с памятью (временные задержки); (ii) Динамический анализ нейронных сетей с задержкой по времени; (iii) Динамический анализ биологических систем. Мы видели, что модели с задержкой реальных явлений имеют более интересную динамику, чем уравнения, в которых отсутствуют эффекты памяти.

Следует отметить некоторые моменты:

- Физические и биологические системы имеют сложное нелинейное динамическое поведение. Изучение качественного поведения в условиях стабильности и бифуркационных свойств имеет важное значение для обеспечения безопасного применения в реальном мире.
- Дифференциальные модели задержки явлений реальной жизни имеют потенциально более интересную динамику, чем уравнения, в которых отсутствуют эффекты памяти: они более качественно и количественно согласуются с явлениями пролиферации клеток, чем сопоставимые модели ОДУ.
- Задержки времени были включены в биологические системы для описания времени регенерации ресурса, периодов созревания, времени реакции, времени кормления, периода беременности и т. д.
- Наличие дробного порядка и временной задержки в моделях улучшает стабильность решений и обогащает динамику модели.
- Задержки во времени считаются важным фактором, который напрямую влияет на производительность системы, и бифуркация Хопфа также происходит, когда задержка проходит через последовательность критических значений.
- Анализ чувствительности является важным инструментом для понимания конкретной модели, которая рассматривается как проблема устойчивости относительно структурных возмущений в модели.
- Оптимальные результаты контроля представляют эффективность медикаментозного лечения в подавлении вирусной продукции и предотвращении новых инфекций.
- Жесткие проблемы могут возникать в области применения биоматематики, и неявные или полуявные численные методы больше подходят для этого типа моделей, чем явные методы.

- Нелинейность, чувствительность к малым возмущениям в параметрах (или шумовые данные), идентифицируемость и выбор модели являются проблемами при численном моделировании в бионауках.
- Полученные результаты дают понять биологам для улучшения свойств моделей и экспериментальных данных.

Эълон қилинган ишлар рўйхати
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Автореферат “Ўзбекистон математика журналы” таҳририятида
таҳрирдан ўтказилди (06.03.2019 йил)

Босишга рухсат этилди: 06.03.2019. Ҳажми 5 босма табақ,
Бичими 60x84 1/16, «Times New Roman»
гарнитурда рақамли босма усулида босилди.
Адади: 60. Буюртма: № 33

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