O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

GULISTON DAVLAT UNIVERSITETI

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MATEMATIKADAN OLIMPIADA MASALALARI

Maktab, akademik litsey va Oliy o'quv yurtlarining III bosqich talabalari uchun qo'llanma

Toshkent

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Mazkur o’quv qo’llanma iqtidorli o’quvchilarni matematika olimpiadasiga tayyorlash uchun yaratilgan bo’lib, undan maktab o’qituvchilari hamda oliy o’quv yurti talabalari uchun ham mo’ljallangan. Kitobda matematika olimpiada masalalari uchun umumiy ma’lumotlar, teorema, lemma va xossalar, misol va masalalar yechib ko’rsatilgan. Jami 200 dan ortiq misol va masalalar yechilib, teoremalar va lemmalar isbotlab ko’rsatilgan. Mustaqil yechish uchun esa 600 ga yaqin misol va masalalar, 605 ta olimpiada testlari javoblar kaliti bilan va 43 ta variant 5 talik masaladan 215 ta masalalar berilgan.

MUNDARIJA

SO’Z BOSHI

Mamlakatimizda barkamol avlodni voyaga yetkazish borasida keng ko’lamli ishlar amalga oshirilmoqda. Bugungi kunda yurtimizdagi umumta’lim maktablarining qiyofasi tubdan o’zgarib, ularning moddiy-texnik bazasi mustaxkamlanib borayotganligi Milliy dastur ijrosining natijasidir.

Shu bilan birga, maktab ta’limini rivojlantirishga qaratilgan tadbirlar ichida Respublika fan olimpiadasining barcha bosqichlarini uyushgan holda o’tkazish va takomillashtirish, o’quvchilarning har tomonlama iste’dodlarini, fanlarga bo’lgan qiziqishlarini, qobiliyatlarini, zamonaviy texnika vositalaridan foydalana olish va mustaqil fikrlash jarayonlarini uyg’unlashtirish, o’quvchilarni nufuzli xalqaro olimpiada va musobaqalarda muvaffaqiyatli ishtirok etishini ta’minlash ishlari katta ahamiyatga ega. Mamlakatimiz o’quvchilarining xalqaro matematika olimpiadalaridagi ishtirok etish geografiyasi juda tor doirada bo’lib qolmoqda. Masalan, ko’pchilik ishtirokchilar Buxoro va Samarqand viloyatlari o’quvchilari bo’lib qolmoqda. Shu sababli, bu geografiyani kengaytirish maqsadida butun mamlakat bo’yicha iqtidorli o’quvchilarni jahon olimpiadalariga tayyorlash zarur deb hisoblaymiz.

Bunday ishlarning samarali amalga oshirishda yetuk kadrlar tayyorlash va o’zbek tilida yozilgan zamonaviy o’quv-uslubiy qo’llanmalarning o’rni kattadir. Shu maqsadda Guliston davlat universiteti matematika yo’nalishi o’quv rejasiga “Akademik litsey va kasb-hunar kollejlarida matematika” fanini tanlov fan sifatida olib, bunda olimpiada masalalarini o’rganish uchun 22 soat ajratilgan. Ushbu qo’llanmani 2014 yilda chop etilgan H. Norjigitov va J. A. Bahramovlarning “Matematik olimpiada masalalarini yechish uchun qo’llanma” nomli o’quv qo’llanmaning mantiqiy davomi deyish mumkin. Qo’llanmada tuman, viloyat, Respublika va jahon fan olimpiadalarida tushgan masalalar yechib ko’rsatilgan. Qo’llanma to’rt bobdan iborat bo’lib, 1-bob sonlar nazariyasi va tenglamalar, 2-bob tengsizliklar, 3-bob geometriya va 4-bob olimpiada masalalari va testlariga oiddir.

Mazkur o’quv qo’llanmadagi ma’lumotlardan matematika yo’nalishi bakalavriat hamda magistratura bosqichi talabalari foydalanishlari mumkin. Masalan, avtorlardan Nuraliyev A. o’zining magistrlik ishida Gyolder, Koshi-Bunyakovskiy-Shvars, Bernulli tengsizliklaridan muvaffaqiyatli foydalanilgan.

O’ylaymizki, o’z kasbiga fidoiy bo’lgan o’qituvchi-pedagoglar bu qo’llanmani o’quvchilar foydalanishga to’g’ri ko’rsatma bersalar, ularning aqliy salohiyati o'sishiga, rivojlanishiga qo'l keladi. Bu yo'lda Biz matematika fan olimpiadasi ishtirokchilariga muvaffaqiyatlar tilaymiz.

Qo'llanma to'g'risida bildirilgan fikr-mulohazalarni mualliflar mamnuniyat bilan qabul qiladilar.

I BOB

SONLAR NAZARIYASI VA TENGLAMALAR

1-§. Bo'linishga doir masalalar

Tayanch so'zlar:*Lemma, qoldiq, bo'luvchi, juft son, toq son, faktorial.*

Ta'rif. Agar noldan farqli a va b butun sonlar uchun $a = bq$ tenglikni qanoatlantiradigan q butun son mavjud bo'lsa, u holda a son b songa qoldiqsiz bo'linadi deyiladi hamda $a \vdots b$ kabi yoziladi.

$a = bq$ tenglikdagi a son bo'linuvchi yoki b soniga karrali son, b son a sonning bo'luvchisi, q son esa bo'linma deb yuritiladi.

x, y, z butun sonlar bo'lsa, u holda quyidagi sodda xossalar o'rinli:

(a) $x \vdots x$;
(b) Agar $x \vdots y$ va $y \vdots z$ bo'lsa, u holda $x \vdots z$;
(d) Agar $x \vdots y$ va $y \neq 0$ bo'lsa, u holda $|x| \geq |y|$;
(e) Agar $x \vdots z$ va $y \vdots z$ bo'lsa, u holda barcha butun a,b sonlar uchun $ax + by \vdots z$;
(f) Agar $x \vdots z$ va $x \pm y \vdots z$ bo'lsa, u holda $y \vdots z$;
(h) Agar $x \vdots y$ va $y \vdots x$ bo'lsa, u holda $|x| = |y|$;
(i) Agar $x \vdots y \Leftrightarrow |x| \vdots |y|$.

2 ga karrali butun sonlar (ya'ni $2k, k \in Z$ ko'rinishdagi sonlar) *juft*, 2 ga karrali bo'lmagan butun sonlar (ya'ni $2k+1, k \in Z$ ko'rinishdagi sonlar) esa *toq* sonlar deb yuritiladi.

Bunda quyidagilar o'rinli:

(a) Ikkita *toq sonlar* yig'indisi va ayirmasi *juft*, ko'paytmasi esa *toq son* bo'ladi.
(b) Ikkita *juft sonlar* yig'indisi, ayirmasi va ko'paytmasi *juft son* bo'ladi.

Lemma. Agar a,b,k,n lar natural sonlar bo'lsa, u holda $(ak+b)^n$ ni a ga bo'lgandagi qoldiq b^n ni a ga bo'lgandagi qoldiq bilan bir xil bo'ladi.

Isbot. Nyuton binom formulasidan foydalanamiz:

$$(ak+b)^n = \underbrace{(ak)^n + n(ak)^{n-1}b + \frac{n(n-1)}{2!}(ak)^{n-2}b^2 + \ldots + nakb^{n-1}}_{\vdots a} + b^n = aN + b^n .$$

Qoida. $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \ldots + ab^{k-2} + b^{k-1}) \Rightarrow a^k - b^k \vdots a - b$,

$$a^{2k+1}+b^{2k+1}=(a+b)\left(a^{2k}-a^{2k-1}b+...+a^2b^{2k-2}-ab^{2k-1}+b^{2k}\right)\Rightarrow a^{2k+1}+b^{2k+1}\vdots a+b.$$

1-masala. Berilgan yettita butun sondan ixtiyoriy oltitasining yig'indisi 5 ga bo'linadi. Bu sonlar har biri 5 ga bo'linishini isbotlang.

Isbot. Berilgan sonlarni a,b,c,d,e,f,g orqali, ularning yig'indisini esa m orqali belgilaymiz. Masalaning shartiga ko'ra

$$m-a,\ m-b,\ m-c,\ m-d,\ m-e,\ m-f,\ m-g$$

ayirmalar barchasi 5 ga bo'linadi. Ularni qo'shib

$$7m-(a+b+c+d+e+f+g)=6m$$

tenglikni hosil qilamiz. Bundan $6m$ soni 5 ga bo'linishi kelib chiqadi. $(5;6)=1$ ekanligidan m ning 5 ga bo'linishi kelib chiqadi.

Shunday qilib, m va $m-a$ sonlar 5 ga bo'linadi, demak $m-(m-a)$ son ham 5 ga bo'linadi.

Xuddi shunday, qolgan b,c,d,e,f va g 5 ga bo'linishi isbotlanadi.

2-masala. Agar $a+2$ va $35-b$ sonlar 11 ga bo'linsa, $a+b$ son ham 11 ga bo'linishi isbotlang.

Isbot. $a+2\vdots 11,\ 35-b\vdots 11\Rightarrow (a+2)-(35-b)\vdots 11\Rightarrow a+b-33\vdots 11\Rightarrow 33\vdots 11\Rightarrow a+b\vdots 11.$

3-masala. 3 ta ketma-ket natural sonlarning ko'paytmasi 6 ga bo'linishini isbotlang.

Isbot. Berilgan sonlardan kamida bittasi juft son bo'lgani uchun ko'paytma 2 ga bo'linadi. Xuddi shunday, berilgan sonlardan kamida bittasi 3 ga karrali bo'lgani uchun, ko'paytma 3 ga bo'linadi. Demak, ko'paytma $6=2\cdot 3$ ga bo'linadi.

Teorema. n ta ketma-ket kelgan natural sonlar ko'paytmasi $n!$ ga qoldiqsiz bo'linadi.

Bu teoremaning isboti yuqoridagi 3- masala isboti kabi isbotlanadi.

4-masala. Ma'lumki, a,b,c,d butun sonlar barchasi $ab-cd$ ga bo'linadi. $ab-cd$ ning qiymatini toping.

Yechish. a,b,c,d butun sonlar barchasi $ab-cd$ ga bo'lingani uchun $a=(ab-cd)m,\ b=(ab-cd)n,\ c=(ab-cd)k,\ d=(ab-cd)t$ tengliklar bajariladi, bu yerda m,n,k,t – butun sonlar. Bundan

$$ab-cd=(ab-cd)m(ab-cd)n-(ab-cd)k(ab-cd)t=(ab-cd)^2(mn-kt)$$

tengliklarni hosil qilamiz, ya'ni $ab-cd$ son $(ab-cd)^2$ ga bo'linadi. Bu esa $ab-cd=1$ va $ab-cd=-1$ bo'lganda o'rinli.

Javob: $ab-cd=1$ yoki $ab-cd=-1$.

5-masala. Ixtiyoriy natural k son uchun $7+7^2+...+7^{4k}$ yig'indi 400 ga bo'linishini isbotlang.

Isbot. Berilgan yig'indini quyidagicha yozamiz:

$$\left(7+7^2+7^3+7^4\right)+\left(7^5+7^6+7^7+7^8\right)+\ldots+\left(7^{4k-3}+7^{4k-2}+7^{4k-1}+7^{4k}\right)=$$
$$=\left(7+7^2+7^3+7^4\right)\cdot\left(1+7^4+7^8+\ldots+7^{4k-4}\right)=7\cdot 400\cdot\left(1+7^4+7^8+\ldots+7^{4k-4}\right).$$

6-masala. $n>1$natural son berilgan bo'lsin. 2^n son ikkita ketma-ket natural toq son yig'indisi ko'rinishida ifodalanishini isbotlang.

Isbot. Agar ikkita ketma-ket toq sonlar mos ravishda $2k-1,\ 2k+1$ko'rinishda bo'lsa $(k\in Z)$, u holda $2^n=(2k-1)+(2k+1)$ tengklikdan quyidagini topamiz:

$$2^n=(2k-1)+(2k+1)=4k\Rightarrow k=2^{n-2}.$$

Demak, $2^n=\left(2^{n-1}-1\right)+\left(2^{n-1}+1\right)$.

7-masala. Ixtiyoriy m va n butun sonlar uchun $mn(m+n)$ soni juft son bo'lishini isbotlang.

Isbot. Agar m va n sonlardan biri juft bo'lsa, u holda $mn(m+n)$ son juft son bo'lishi ravshan. Shuning uchun m va n ikkalasi ham toq bo'ladi deb faraz qilamiz. U holda $m+n$ yig'indi juft son bo'lganligi sababli $mn(m+n)$ soni juft bo'ladi.

8-masala. Ixtiyoriy n natural son uchun $3^{2^n}+1$ sonini 4 ga bo'linmasligini isbotlang.

Isbot. Ravshanki, 3^{2^n} soni toq, demak $3^{2^n}+1$ soni juft bo'ladi. Quyidagiga egamiz:

$$3^{2^n}=\left(3^2\right)^{2^{n-1}}=9^{2^{n-1}}=(8+1)^{2^{n-1}}=8A+1$$

Bu yerda A – natural son.

Demak, $3^{2^n}+1=8A+2=4B+2$. Oxirgi son esa 4 ga bo'linmaydi.

9-masala. Ma'lumki, n^2+1 va $(n+1)^2+1,\ (n\in N)$ sonlar bir vaqtda d natural songa bo'linadi. d sonini toping.

Yechish. $\left(n^2+1\right)\vdots d,\ \left((n+1)^2+1\right)\vdots d\Rightarrow\left(n^2+1\right)\vdots d,\ \left(n^2+2n+2\right)\vdots d\Rightarrow$

$$\Rightarrow\left(\left(n^2+2n+2\right)-\left(n^2+1\right)\right)\vdots d\Rightarrow(2n+1)\vdots d\Rightarrow\left(4n^2+4n+1\right)\vdots d\Rightarrow$$
$$\left(4\left(n^2+2n+2\right)-\left(4n^2+4n+1\right)\right)\vdots d\Rightarrow(4n+7)\vdots d\Rightarrow((4n+7)-2(2n+1))\vdots d\Rightarrow 5\vdots d.$$

Oxirgi munosabat $d=1$ yoki $d=5$ bo'lgandagina bajariladi. Bu ikkita hol ham $n=2$ da o'rinli.

10-masala. Barcha butun n sonlar uchun n^3+23n sonini 6 ga bo'linishini isbotlang.

Isbot. Isbotni uchta ketma-ket kelgan sonlar ko'paytmasi 3! ga ya'ni 6 ga bo'linishidan ko'rsatamiz:

$$n^3+23n=n^3-n+24n=\underbrace{(n-1)n(n+1)}_{\vdots 6}+\underbrace{24n}_{\vdots 6}\vdots 6.$$

Faollashtiruvchi savollar.

1. *Juft son ta'rifini ayting?*
2. *Toq son ta'rifini ayting?*
3. *Qanday sonlar 5 ga bo'linadi?*
4. *4 ga bo'linish qoidasini ayting?*
5. *Qoldiqli bo'lish formulasini ayting?*
6. *n! nimaga teng? 0! chi?*

Mustaqil yechish uchun masalalar

1. $1^{2017}+2^{2017}+...+16^{2017}$natural son 17 ga bo'linishini isbotlang.
2. $n>1$ natural son berilgan bo'lsin. 3^n son uchta ketma-ket natural toq son yig'indisi ko'rinishida ifodalanishini isbotlang.
3. 5 ta ketma-ket natural sonlarning ko'paytmasi 120 ga bo'linishini isbotlang.
4. $a+1$ son 3 ga bo'linsa, $7a+4$ son ham 3 ga bo'linishini isbotlang.
5. $3n+7$ va $8n-5$ natural sonlar bir vaqtda qandaydir p natural soniga qoldiqsiz bo'linadi $(p\neq 1)$. p sonini toping.
6. Barcha butun n sonlar uchun n^3+11n sonini 6 ga bo'linishini isbotlang.
7. Ixtiyoriy natural n uchun $27n+4$ va $18n+3$ sonlari o'zaro tub ekanligini isbotlang.
8. $\frac{11n+3}{13n+4}$ kasr qisqaradigan va $(2;30]$ kesmaga tegishli natural n sonlarni toping.
9. $p=n^5+n+1, n\in N$ p tub son bo'ladigan n larning yig'indisini toping.
10. $\frac{a}{b}$ kasr $(a,b\in N)$ qisqarmas kasr ekanligi ma'lum. $\frac{5a+3b}{2a+b}$ kasr ham qisqarmas ekanligini isbotlang.
11. $3x+8y$ ifoda 17 ga bo'linsa, u holda $35x+65y$ ni 17 ga bo'linishini isbotlang.
12. x,y – butun sonlar bo'lsin. $2x+3y$ soni 17 ga bo'linsa, u holda $9x+5y$ son ham 17 ga bo'linishini isbotlang.
13. Barcha natural $m,k>2$ toq sonlari uchun $1^k+2^k+...+(m-1)^k$ sonini m ga bo'linishini isbotlang.
14. Barcha natural n sonlar uchun n^5-5n^3+4n sonini 120 ga bo'linishini isbotlang.
15. $3n+2$ va $8n+3$ natural sonlar bir vaqtda p natural soniga qoldiqsiz bo'linadi $(p\neq 1)$. p sonini toping.
16. Barcha natural n sonlar uchun n^3+17n sonini 6 ga bo'linishini isbotlang.

17. Ixtiyoriy natural $n>1$ da $n^n - n^2 + n - 1 \vdots (n-1)^2$ isbotlang.

18. $\frac{2a+5}{3a+4}$ kasr a ning qanday qiymatlarida qisqaruvchan bo'ladi.

19. $n \in N,\ n<100$ son uchun $\frac{n^3+23}{24}$ kasr natural son bo'ladigan barcha n larni toping.

20. $\forall n \in N$ son uchun $\frac{n^3}{6}+\frac{n^2}{2}+\frac{n}{3}$ kasr natural son ekanligini isbotlang.

21. Barcha natural n sonlar uchun $n^5 - n$ sonini 5 ga bo'linishini isbotlang.

22. n^2+3n+5 ifoda hech qanday butun n larda 121 ga bo'linmasligini isbotlang.

23. Barcha natural n sonlar uchun $\frac{2n+13}{n+7}$ kasr qisqarmas ekanligini isbotlang.

24. $2^{19}+1$ va $2^{98}-1$ sonlarning eng katta umumiy bo'luvchisini toping.

25. $p = n^5 + n^4 + 1,\ n \in N$ p tub son bo'ladigan n larning yig'indisini toping.

26. Agar d son $(n+19)(n+98)(n+1998)$ ko'rinishdagi sonni natural n ning barcha qiymatlarida bo'luvchi bo'lsa, shunday d larning eng katta natural qiymati topilsin.

27. $p>3$ tub son 6 ga bo'linganda hosil bo'lganda qoldiq 1 yoki 5 ga teng bo'lishini isbotlang.

28. $n-1$ ning 15 ga va 1001 ning $n+1$ ga bo'linishi ma'lum bo'lsa, n ni toping.

29. n ning qanday natural qiymatlarida n^2+3 soni $n+3$ ga bo'linadi?

30. Ushbu $\frac{2n+1}{9n+4}$ $(n \in N)$ ifoda qisqarmas kasr ekanligini isbotlang.

31. 16, 50 va A sonlarining EKUKi 1200 ga teng. Nechta natural A son shu shartni qanoatlantiradi?

2-§. Bo'linishga doir lemma va uning tadbiqlari

Tayanch so'zlar: *Lemma, tub son, EKUB, EKUK, kasr, o'zaro tub sonlar, bo'luvchi, Evklid algoritmi, mod, qoldiq.*

Qoida 1. Har qanday n xonali sonni

$$\overline{a_1 a_2 a_3 ... a_n} = 10^{n-1} \cdot a_1 + 10^{n-2} \cdot a_2 + 10^{n-3} \cdot a_3 + ... + 10 \cdot a_{n-1} + a_n$$

ko'rinishda yozish mumkin. Bu yerda $a_i,\ i = 1,2,...,n$ lar raqamlar.

Masalan, 2 xonali sonni $\overline{xy} = 10x + y$, 3 xonali sonni $\overline{xyz} = 100x + 10y + z$ ko'rinishda yozish mumkin.

a) $\overline{xy}=10x+y,\ 1\le x\le 9,\ 0\le y\le 9$;

b) $\overline{xyz}=100x+10y+z,\ 1\le x\le 9,\ 0\le y\le 9, 0\le z\le 9$.

Qoida 2. Har qanday A natural sonni n natural songa bo'lganda qoldiqlar 0 dan $n-1$ gacha bo'lishi mumkin va u quyidagicha yoziladi:

$$A\equiv 0,1,2,...,n-1(\operatorname{mod} n)$$

Lemma 1. Agar $(a,b)=1$ bo'lib, $A\vdots a$ va $A\vdots b$ bo'lsa, u holda $A\vdots ab$ bo'ladi.

Isbot. $A\vdots a\Rightarrow A=an,\ n\in N$ va $A\vdots b\Rightarrow A=bm,\ m\in N$ shartdan $an=bm$ tenglikni hosil qilamiz. $(a,b)=1$ bo'lgani uchun oxirgi tenglikdan $m\vdots a$ yoki $n\vdots b$ kelib chiqadi. Bundan esa $n=bk,\ k\in N$. Bu tenglikni $A=an$ ga olib borib qo'yamiz: $A=an\Rightarrow A=abk\Rightarrow A\vdots ab$ kelib chiqadi. Lemma isbotlandi.

Lemma 2. Agar $(a,b)=1,\ a,b\in Z$ bo'lsa, u holda a^2+b^2 ko'rinishdagi sonning $4k+3$ ko'rinishdagi bo'luvchisi mavjud emas.

Isbot. Teskarisini faraz qilaylik, ya'ni yuqoridagi shartlarni qanoatlantirgan holda a^2+b^2 sonini $4k+3$ ko'rinishdagi bo'luvchisi mavjud bo'lsin. U holda, o'sha bo'luvchining kamida bitta $p=4m+3$ ko'rinishda tub bo'luvchisi bo'ladi. Aks holda, barchasi $p_i=4t_i+1$ ko'rinishda bo'lsa, sonning o'zi

$a^2+b^2=2^{\beta}(4t_1+1)^{\alpha_1}(4t_2+1)^{\alpha_2}...(4t_k+1)^{\alpha_k}\Rightarrow a^2+b^2=2^{\beta}(4T+1)$ ko'rinishda bo'lar edi. Demak, $\exists p,\ 4k+3\vdots p=4m+3$.

Endi Ferma teoremasiga ko'ra $(a,p)=1,\ (b,p)=1,$ $p-$ tub son. Bundan, $a^{p-1}-1\vdots p$ va $b^{p-1}-1\vdots p$ larni hadma-had qo'shib yuborsak $a^{p-1}+b^{p-1}-2\vdots p$ ni, $p=4m+3$ dan $a^{4m+2}+b^{4m+2}-2\vdots p$ va $(a^2)^{2m+1}+(b^2)^{2m+1}-2\vdots p$ ni hosil qilamiz. Bundan, $(a^2+b^2)A-2\vdots p$ va farazimiz $a^2+b^2\vdots p$ dan $2\vdots p$ kelib chiqadi. Ammo $p>2$ tub son bo'lgani uchun oxirgi natija o'rinsiz. Bundan esa farazimizning xatoligi kelib chiqadi. Lemma isbotlandi.

1-masala. Uch xonali sonning uni raqamlari teskari tartibda yozilgan uch xonali sondan ayirmasi 99 ga bo'linishini isbotlang.

Isbot. Uch xonali sonni quyidagicha yozamiz: $\overline{xyz}=100x+10y+z$

$\overline{xyz}-\overline{zyx}=100x+10y+z-100z-10y-x=99x-99z=99(x-z)$.

2-masala. Uch xonali sonning oxiriga yana shu uch xonali son yozilib olti xonali son hosil qilindi. Hosil bo'lgan olti xonali sonni 1001 ga bo'linishini isbotlang.

Isbot. Uch xonali sonni quyidagicha yozib olamiz: $\overline{xyz}=100x+10y+z$

$$\overline{xyzxyz}=100000x+10000y+1000z+100x+10y+z=100100x+10010y+1001z=1001\overline{xyz}.$$

3-masala. Natural a va b sonlar uchun $a^2+b^2 \vdots ab$ bo'lsa, u holda $a=b$ isbotlang.

Isbot. $a^2+b^2 \vdots ab \Rightarrow a^2+b^2=abk,\ k\in N \Rightarrow a^2=b(ak-b) \Rightarrow a^2 \vdots b$

$$a^2+b^2 \vdots ab \Rightarrow a^2+b^2=abk,\ k\in N \Rightarrow b^2=a(bk-a) \Rightarrow b^2 \vdots a$$

Endi yuqoridagi shartni hisobga olsak, $a^2 \vdots b \Rightarrow a \vdots b$ va $b^2 \vdots a \Rightarrow b \vdots a$ ekanligi kelib chiqadi. Bundan esa $a=b$.

4-masala. $p>3$ tub son bo'lsa, u holda p^2-1 ni 24 ga bo'linishini isbotlang.

Isbot. $p>3$ tub bo'lsa, albatta u toq son bo'ladi. Bundan esa $p-1$ va $p+1$ ikkita ketma-ket kelgan juft son va shu juft sonlardan kamida bittasi 4 ga karrali ekanligi kelib chiqadi. U holda ularning ko'paytmasi 8 ga bo'linadi, ya'ni

$$(p-1)(p+1)=p^2-1 \vdots 8.$$

Endi $p-1,\ p,\ p+1$ sonlarni olaylik. Bu sonlar uchta ketma-ket kelgan sonlar bo'lgani uchun ulardan kamida bittasi 3 ga bo'linadi. p tub son bo'lgani uchun 3 ga bo'linmaydi. Bundan esa $p-1$ yoki $p+1$ larning bittasi 3 ga bo'linishi va ularning ko'paytmasi albatta 3 ga bo'linishi kelib chiqadi, ya'ni

$$(p-1)(p+1)=p^2-1 \vdots 3.$$

Lemma 1 ga asosan $(3,8)=1,\ p^2-1 \vdots 3,\ \ p^2-1 \vdots 8$. Bundan esa $p^2-1 \vdots 24$ ekanligi kelib chiqadi.

5-masala. $\forall n\in N$ da $\frac{21n+4}{14n+3}$ kasr qisqarmas ekanligini isbotlang.

Isbot. Kasr qisqarmas bo'lishi uchun uning surati va maxraji o'zaro tub bo'lishi kerak. Demak, kasrning surati va maxrajini EKUB ini topaylik.

$14n+3 \vdots d, 21n+4 \vdots d \Rightarrow 3(14n+3) \vdots d,\ 2(21n+4) \vdots d \Rightarrow 42n+9 \vdots d, 42n+8 \vdots d \Rightarrow$

$\Rightarrow ((42n+9)-(42n+8)) \vdots d \Rightarrow 1 \vdots d$. Bundan esa $d=1$ ekanligi kelib chiqadi.

Yuqoridagi 5- masala odatda Evklid algoritmi deb ham yuritiladi.

Evklid algoritmi quyidagicha: ikkita sonning EKUB ini topishda shu sonlardan kichigi bilan ularning ayirmasining modulini, hosil bo’lgan sonlarning kichigi bilan shu sonlar ayirmasining moduli va hokazo. Oxirgi qadamda bir xil son chiqqunicha davom ettiriladi. Natijada hosil bo’lgan son shu sonlarning EKUBi bo’ladi.

Ikkita sonning EKUB va EKUK ini odatda mos ravishda $B(a,b)$ va $K(a,b)$ yoki (a,b) va $[a,b]$ kabi yoziladi. $D(a,b)$ orqali a va b sonlarning umumiy bo’luvchilari to’plamini belgilaymiz.

Masalan, $B(18,24) \to (18,6) \to (6,12) \to (6,6) = 6$.

6-masala. $n \in N$ uchun $(\sqrt{2}+1)^n = a_n + \sqrt{2} b_n$ tenglik o’rinli bo’lsa, $EKUB(a_n, b_n) = 1$ ekanligini isbotlang.

Isbot. Bizga ma’lumki, $\begin{cases} (\sqrt{2}+1)^n = a_n + \sqrt{2} b_n \\ (\sqrt{2}-1)^n = a_n - \sqrt{2} b_n \end{cases}$ tengliklar o’rinli. Bu tengliklarni hadma-had ko’paytirib

$$(\sqrt{2}+1)^n (\sqrt{2}-1)^n = (a_n + \sqrt{2} b_n)(a_n - \sqrt{2} b_n) \Rightarrow a_n^2 - 2b_n^2 = 1 \Rightarrow a_n^2 = 2b_n^2 + 1$$

tenglikni hosil qilamiz. $EKUB(a_n, b_n) = 1$ ekanligini isbotlash uchun $EKUB(a_n^2, b_n^2) = 1$ isbotlash yetarli. Evklid algoritmiga asosan

$$EKUB(a_n^2, b_n^2) \to EKUB(2b_n^2 + 1, b_n^2) \to EKUB(b_n^2, b_n^2 + 1) \to EKUB(b_n^2, 1) = 1$$

ekanligi kelib chiqadi.

7-masala. Tenglamani natural sonlarda yechimi yo’qligini isbotlang.

$$m^2 - n^2 = 222$$

Isbot. Tenglamaning har ikkala tomonini 4 soni bilan baholaymiz. O’ng tomoni juft bo’lganligidan chap tomoni ham juftligi kelib chiqadi. Bundan esa m va n larning har ikkalasi bir vaqtda juft yoki toq bo’lganida bajariladi. U holda $m-n$ va $m+n$ sonlar juft va ko’paytmasi 4 ga karrali ekanligi kelib chiqadi. Tenglikning chap tomoni 4 ga karrali o’ng tomoni 4 ga bo’lganda 2 qoldiq qoladi. Bundan esa tenglama natural sonlarda yechimi yo’qligi kelib chiqadi.

8-masala. Quyidagi tenglamani qanoatlantiruvchi barcha butun (x, y) sonlarni juftliklarini toping:

$$y^2 = x^3 + 7$$

Yechish. x juft son bo'lsin ya'ni, $x = 2k$ u holda $y^2 = 8k^3 + 7$ dan $y = 2m + 1$, $(2m+1)^2 = 8k^3 + 7$ dan $4m(m+1) - 8k^3 = 6$ ni hosil qilamiz. Ko'rinib turibdiki oxirgi tenglik o'rinli emas. Chunki, tenglikning chap tomoni 4 ga karrali, o'ng tomonini 4 ga bo'lganda 2 qoldiq qoladi. Bundan kelib chiqadiki, x toq son ekan.

$x = 4k + 1$ yoki $x = 4k + 3$ bo'lishi mumkin.

1-hol. $x = 4k + 1$ bo'lsin. $y^2 = x^3 + 7$ dan

$y^2 + 1 = x^3 + 8 = x^3 + 2^3 = (x+2)(x^2 - 2x + 4) = (4k+3)A$ kelib chiqadi. $(y,1) = 1$ dan lemma 2 ga ko'ra $y^2 + 1 \vdots 4k + 3$ formula o'rinsizdir. Bundan kelib chiqadiki, bu holda tenglamning yechimi yo'q.

2-hol. $x = 4k + 3$ bo'lsin. U holda $y^2 = x^3 + 7$ dan

$y^2 = (4k+3)^3 + 7 = 4B + 27 + 7 = 4C + 2$ kelib chiqadi. Biror sonning kvadratini 4 ga bo'lganda 0 yoki 1 qoldiq qoladi, ya'ni $y^2 \equiv 0{,}1 (\mathrm{mod}\, 4)$ bo'lganligi bois, $y^2 \neq 4C + 2$ bo'ladi. Bundan kelib chiqadiki, bu holda ham yechim yo'q.

Demak, barcha hollardan ko'rinib turibdiki berilgan tenglama butun sonlarda yechimga ega emas ekan.

9-masala. d_1 va d_2 sonlari n – natural sonning bo'luvchilari uchun $d_1 > d_2$ shart bajarilsa, u holda $d_1 > d_2 + \frac{d_2}{n}$ ekanligini isbotlang.

Isbot. Berilgan shartlardan foydalanamiz:

$$\begin{cases} n \vdots d_1 \\ n \vdots d_2 \end{cases} \Rightarrow \begin{cases} n = d_1 x \\ n = d_2 y \end{cases} \quad x, y \in N$$

$$d_1 > d_2 \Rightarrow y > x \quad x, y \in N \Rightarrow y - x \geq 1, y > x \Rightarrow y(y - x) > x$$

$$\begin{cases} d_1 = \dfrac{n}{x} \\ d_2 = \dfrac{n}{y} \end{cases} \Rightarrow \frac{n}{x} > \frac{n}{y} + \frac{n^2}{ny^2} \Rightarrow \frac{1}{x} > \frac{1}{y} + \frac{1}{y^2} \Rightarrow \frac{y - x}{xy} > \frac{1}{y^2} \Rightarrow y - x > \frac{x}{y} \Rightarrow$$

$$\Rightarrow y(y - x) > x.$$

10-masala. Quyidagi tengliklarni qanoatlantiruvchi barcha (a, b, c) natural sonlar uchliklarini toping.

$$a + b = \big(B(a, b)\big)^2, b + c = \big(B(b, c)\big)^2, c + a = \big(B(c, a)\big)^2$$

Yechish. $B(a, b, c) = d \Rightarrow B(b, c) = dx, B(c, a) = dy, B(a, b) = dz$ deylik.

$$d = B(a, b, c) = B\big(B(a, b), B(b, c), B(c, a)\big) = B(dx, dy, dz) = dB(x, y, z) \Rightarrow$$

$$\Rightarrow dB(x, y, z) = d \Rightarrow B(x, y, z) = 1$$

$x \geq y \geq z$ bo'lsin. Aytaylik, $B(x, y) > 1$ bo'lsin. $x \vdots l, y \vdots l$ son mavjud,

$$b \vdots dx \vdots dl, a \vdots dy \vdots dl \Rightarrow B(a, b) \vdots dl \Rightarrow dz \vdots dl \Rightarrow z \vdots l \Rightarrow B(x, y, z) \vdots l \Rightarrow l \vdots 1$$

Bu esa $B(x, y) > 1$ ekanligiga zid!

Demak, $B(x, y) = 1$. Xuddi shunday $B(y, z) = 1, B(z, x) = 1$

$a \vdots dy$ va $a \vdots dz \Rightarrow a \vdots K(dy, dz) \Rightarrow a \vdots dyz \Rightarrow a = dyzp$

Xuddi shunday $b \vdots dx$ va $b \vdots dz \Rightarrow b \vdots K(dx, dz) \Rightarrow b \vdots dxz \Rightarrow a = dxzq$

$$c = dxyr, \qquad B(p, q, r) = 1.$$

Shartdan:

$$\begin{cases} dyzp + dxzq = d^2z^2 \\ dxzq + dxyr = d^2x^2 \\ dyzp + dxyr = d^2y^2 \end{cases} \Rightarrow \begin{cases} yzp + xzq = dz^2 \\ xzq + xyr = dx^2 \\ yzp + xyr = dy^2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 2xyr = d(x^2 + y^2 - z^2) \\ 2yzp = d(y^2 + z^2 - x^2) \\ 2xzq = d(x^2 + z^2 - y^2) \end{cases} \Rightarrow \begin{cases} 2xyr \vdots d \\ 2yzp \vdots d \\ 2xzq \vdots d \end{cases} \Rightarrow$$

$$\Rightarrow B(2xyr, 2yzp, 2xzq) \vdots d \Rightarrow 2B(xyr, yzp, xzq) \vdots d$$

$$d = B(a, b, c) = B(dyzp, dxzq, dxyr) = dB(xyr, yzp, xzq) \Rightarrow$$

$$\Rightarrow dB(xyr, yzp, xzq) = d \Rightarrow B(xyr, yzp, xzq) = 1$$

$$2B(xyr, yzp, xzq) \vdots d \Rightarrow 2 \vdots d \Rightarrow d = \{1,2\}$$

Shartdan $p, q, r \geq 1$

$$yp + xq = dz, \qquad x \geq y \geq z$$

$$x + y \leq xq + yp = dz \leq 2z = z + z \leq x + y \Rightarrow q = p = 1,$$

$$x = y = z, d = 2$$

$$1 = B(x, y, z) = x = y = z \Rightarrow x = y = z = 1, q = p = 1$$

$$zq + yr = dz \Rightarrow 1 + r = 2 \Rightarrow r = 1 \Rightarrow a = b = c = 2$$

Faollashtiruvchi savollar:

1. *Tub son ta'rifini ayting?*
2. *O'zaro tub sonlar deb nimaga aytiladi?*
3. *Eng katta umumiy bo'luvchi deb nimaga aytiladi?*
4. *Eng kichik umumiy karrali deb nimaga aytiladi?*
5. *Qisqarmas kasr deb nimaga aytiladi?*
6. *Evklid algoritmi nima uchun foydalaniladi?*
7. *Toq sonlar har doim tub son bo'ladimi?*
8. *Tub sonlar har doim toq sonlar bo'ladimi?*
9. *Barcha raqamlar ko'paytmasi nechchiga teng?*
10. *Barcha raqamlar yig'indisi nechchiga teng?*
11. *Eng katta raqamni ayting?*
12. *Eng kichik raqamni ayting?*

Mustaqil yechish uchun masalalar

1. $\forall m \in N$ da $\frac{27m+7}{18m+5}$ kasr qisqarmas ekanligini isbotlang.
2. $p, q > 3$ tub sonlar bo'lsa, u holda $p^2 - q^2$ ni 24 ga bo'linishini isbotlang.
3. Uch xonali abc son 37 ga bo'linadi. bca va cab uch xonali sonlar yig'indisi ham 37 ga bo'linishini isbotlang.
4. a, b, c raqamlar va $\overline{abc7} + \overline{abc} = 2955$ bo'lsa, $a + b + c$ ning qiymatini toping.
5. To'rtta sonning yig'indisi 396. Agar birinchisiga 5 qo'shsak, ikkinchisidan 5 ni ayirsak, uchinchisini 5 ga ko'paytirsak va to'rtinchisini 5 ga bo'lsak, u holda bir xil son chiqadi. Bu sonlarni toping.
6. Quyidagi tenglamani qanoatlantiruvchi barcha butun (x, y) sonlarni juftliklarini toping: $x^5 + 31 = y^2$
7. $a^4 + 4b^4$ son tub son bo'ladigan barcha natural a va b sonlar toping.
8. To'rt xonali sonni 9 ga ko'paytirsak, raqamlari teskari tartibda yozilgan to'rt xonali son hosil bo'ladi. Shu xossaga ega bo'lgan barcha to'rt xonali sonlar topilsin.

9. Birliklar xonasidagi raqamning kubiga teng bo’lgan barcha uch xonali sonlarni toping.
10. Bir qatorga dastlabki 30 ta son yozilgan. Dastlab barcha toq sonlar o’chirildi. Qolgan juft sonlar yana bir qatorga yozilib, toq o’rinlarda turgan barcha sonlar o’chirildi. So’ng bu ish yana takrorlandi. Eng oxirida qanday son qoladi?
11. Birliklar xonasidan 7 marta katta bo’lgan sonni toping.
12. $7^{2018}-5^{2018}$ ni 24 ga bo’lgandagi qoldiqni toping.
13. Olti xonali $\overline{abcdef}$ va $\overline{fdebca}$ sonlarining ayirmasi 271 ga bo’linadi. $b=d$ va $c=e$ ekanligini isbotlang.
14. $\overline{abc}$ va $\overline{ab}$ mos ravishda uch va ikki xonali sonlar. $\overline{abc}+\overline{ab}=1073$ bo’lsa, $a+b+c$ ning qiymatini toping.
15. To’rt xonali $\overline{xxyy}$ son $\overline{xxyy}=\overline{xx}^2+\overline{yy}^2$ shartni qanoatlantiradi. Shu shartni qanoatlantiruvchi barcha *x* va *y* raqamlarni toping.
16. Ikki xonali son berilgan. Uning orqasiga, shu sonni raqamlari o’rinlarini almashtirib yozdik. Yangi to’rt xonali son 11 ga qoldiqsiz bo’linishini ko’rsating.
17. Uch xonali son berilgan. Uning orqasiga shu sonni yana yozib olti xonali son hosil qildik. Yangi son berilgan sondan necha marta katta?
18. Besh xonali sonning chap tomoniga 6 raqamini yozib hosil qilingan son, o’ng tomoniga 6 raqami yozib hosil qilingan sondan 4 baravar katta bo’ldi. Shu besh xonali sonni toping.
19. Ikkinchisi birinchisidan 2 marta, uchinchisi birinchisidan 3 marta katta, yig’indisi 222 bo’lgan uchta sonni toping.
20. $m^2+n^2=291$ tenglamani natural sonlarda yeching.
21. Quyidagi shartni qanoatlantiruvchi barcha $x\in Z$ larni toping: $x^2+(x+1)^2 \vdots 2019$
22. Agar n natural son 3 ga bo’linmasa, u holda n^2+2 ning 3 ga bo’linishini isbotlang.
23. Agar natural son 3 ga bo’linmasa, u holda shu sonning kvadratini 3 ga bo’lishdan hosil bo’ladigan qoldiq 1 ga teng bo’lishini isbotlang.
24. Ixtiyoriy n naturalda $3n+2$ son biror sonning kvadratiga teng bo’lmasligini isbotlang.
25. Ixtiyoriy n natural son uchun $7n^2+1$ ifodani 3 ga bo’linmasligini isbotlang.
26. Agar m va n butun sonlar 5 ga bo’linmasa, u holda m^4-n^4 ifodani 5 ga bo’linishini isbotlang.
27. Aytaylik n,m – natural sonlar va $m-1$ son 3^n ga bo’linsin. m^3-1 son 3^{n+1} ga bo’linishini isbotlang.

28. p va q lar tub son bo'lib, bunda $q = p - 2$ bo'lsa, $p^q + q^p$ ifodani $p + q$ ga bo'linishini isbotlang.
29. Ikkita natural sonning yig'indisi 1244 ga teng. Agar birinchi sonning oxiriga 3 raqamini yozib, ikkinchi sonning oxiridagi 2 raqamini o'chirsak ikkita bir xil son hosil bo'ladi. Shu sonlarni toping.
30. O'zining raqamlari yig'indisidan 59 marta katta bo'lgan natural sonlarni toping.
31. $2^n + 65$ soni biror natural sonning kvadrati bo'ladigan barcha n natural sonlarni toping.
32. Raqamlari ko'paytmasining uchlanganiga teng bo'lgan barcha ikki xonali sonlarni toping.
33. Ikki xonali sonning raqamlari orasiga yana o'sha ikki xonali son yozilsa berilgan sondan 66 marta katta son hosil bo'ldi. Berilgan sonni toping.
34. Yig'indisi ko'paytmasiga teng bo'lgan 5 ta natural sonni toping. Mumkin bo'lgan barcha hollarni ko'rsating.
35. Uchta tub sonning ko'paytmasi ular yig'indisidan 5 marta katta bo'lsa, bu sonlarni toping.
36. Yig'indisi ko'paytmasiga teng bo'lgan 3 ta natural sonni toping. Mumkin bo'lgan barcha hollarni ko'rsating.
37. Ikkita ketma-ket toq sonlarning yig'indisi 4 ga bo'linishini isbotlang.
38. Ikkita ketma-ket toq sonlarning kvadratlarining ayirmasining modulini 8 ga bo'linishini isbotlang.
39. 96 son natural sonlarning yig'indisi ko'rinishida yozilgan. Qo'shiluvchilarning har biri 1 dan katta va ixtiyoriy 2 tasi o'zaro tub. U holda qo'shiluvchilar soni eng ko'pi bilan nechta bo'lishi mumkin?
40. Ikki xonali $\overline{bc}$ son qandaydir sonning kvadrati bo'lsin. $a = b + c$ tenglikni qanoatlantiradigan barcha 11 ga karrali to'rt xonali $\overline{abcd}$ sonlarni toping.
41. Bir qatorga dastlabki 2018 ta son yozilgan. Dastlab barcha toq sonlar o'chirildi. Qolgan juft sonlar yana bir qatorga yozilib, toq o'rinlarda turgan barcha sonlar o'chirildi. So'ng bu ish yana takrorlandi. Eng oxirida qanday son qoladi?
42. Uchta berilgan ikki xonali sonlar shunday tanlanganki, bunda ixtiyoriy ikkitasining yig'indisi uchinchisining raqamlarini o'rni almashishidan hosil bo'lgan songa teng. Uchchala son yig'indisini toping.
43. Agar ikki xonali sonning raqamlarini o'rnini almashtirsak hosil bo'lgan son berilgan ikki xonali sondan kamida 3 marta katta bo'ldi. Shu shartni qanoatlantiruvchi nechta ikki xonali son mavjud?

44. Shunday eng kichik natural sonni topingki, bunda u ham 4 ga va ham 6 ga bo'linsin. Uning barcha raqamlari yoki 4 yoki 6 bo'lsin hamda unda kamida 1 ta 4 raqami va kamida 1 ta 6 raqami ishtirok etsin.
45. Istalgan ikkita a va b sonlari uchun $a*b$ ifoda $a+b-\frac{2019}{2}$ ni anglatadi. Quyidagi ifodaning qiymatini hisoblang: $1*2*3*...*2018*2019$.
46. 49 ta turli xil natural sonlarning yig'indisi 2016 ga teng. Ulardan eng kamida nechtasi toq son bo'lishi kerak?
47. 25 ta natural son yig'indisi 2016 ga teng. Ularning eng katta umumiy bo'luvchisining eng katta qiymati nechaga teng bo'lishi mumkin?
48. Raqamlari yig'indisidan 20 marta katta bo'lgan bitta uch xonali sonni toping.
49. Ikki son yig'indisi 1465 ga teng. Birinchi sonning o'ng tomoniga 5 yozilsa va ikkinchisining oxirgi raqami o'chirilsa, teng sonlar hosil bo'ladi. Berilgan sonlarni toping.
50. Ketma-ket kelgan to'rtta son ko'paytmasiga 1 qo'shilsa hosil bo'lgan son to'la kvadrat bo'lishini ko'rsating.
51. Agar olti xonali N soni uchun $2N, 3N, 4N, 5N, 6N$ sonlari ham olti xonali sonlar bo'lsa, bu son 0 dan farqli turli raqamlardan tashkil topgan bo'lsa, u holda N ni toping.
52. 5 raqami faqat bir marta qatnashgan nechta uch xonali sonlar mavjud?
53. Ikki xonali son 3 ga bo'linmaydi. Uning raqamlari kvadratlari yig'indisi 3 ga bo'linadimi?
54. Agar ko'p xonali sonni chapdan o'ngga va o'ngdan chapga qarab o'qiganda ham bir xil bo'lsa, u holda uni simmetrik son deymiz. Besh xonali simmetrik sonlardan nechtasi 37 ga bo'linadi?
55. Uch xonali M soni, uning ixtiyoriy bir raqami o'chirilib hosil qilingan ikki xonali songa bo'linadi. Shunday xossaga ega bo'lgan barcha uch xonali sonlarni toping.
56. Ikki xonali son raqamlari yig'indisini shu yig'indi kvadratiga qo'shsak berilgan son hosil bo'ladi. Berilgan ikki xonali son topilsin.
57. Qandaydir to'rt xonali sonni, shu sonni raqamlari teskari tartibda yozilganiga ko'paytirilganda oxirida 3 ta nol bo'lgan 8 xonali son hosil bo'ldi. Shu shartni qanoatlantiruvchi barcha 4 xonali sonlar topilsin.
58. Aytaylik, A va B ikkita har xil raqamlar bo'lsin. Agar $\overline{AB}\cdot\overline{AAA}=\overline{ABA}\cdot\overline{AA}+1$ munosabat o'rinli bo'lsa, bu raqamlarni toping.
59. k^5+3 ning k^2+1 ga bo'linadigan k ning barcha butun qiymatlarini toping.
60. $\overline{abcde}$ besh xonali son 41 ga qoldiqsiz bo'linsa, u holda sonning istalgan siklik tartibda yozilgan shakli ham 41 ga qoldiqsiz bo'linishini isbotlang.

61. Agar $m,n\in N$ sonlar $n^3+(n+1)^3+(n+2)^3=m^3$ tenglikni qanoatlantirsa, u holda $n+1\vdots 4$ ekanligini isbotlang.

62. $\frac{a^2+b}{b^2-a}$ va $\frac{b^2+a}{a^2-b}$ sonlarining ikkalasi ham natural son bo'ladigan a,b natural sonlarni toping.

3-§. Matematik induksiya metodi

Tayanch so'zlar: *Teorema, induksiya, deduksiya, kvadrat ildiz, tengsizlik, gipoteza, parallelogramm, uchburchak, diagonal, simmetriya.*

Xulosalar ikki turga bo'linadi: umumiy va xususiy.

Umumiy xulosalarga misollar keltiramiz:

Har qanday parallelogrammning diagonallari kesishish nuqtasida teng ikkiga bo'linadi.

Oxiri nol bilan tugovchi barcha sonlar beshga bo'linadi.

Istalgan teng yonli uchburchak simmetriya o'qiga ega.

Bu misollarga mos keluvchi xususiy xulosalar:

ABCD parallelogrammning diagonallari kesishish nuqtasida teng ikkiga bo'linadi.

150 soni 5 ga bo'linadi.

Berilgan uchburchak teng yonli bo'lsa, u holda bunday uchburchak simmetriya o'qiga ega.

Umumiy xulosalardan xususiy xulosalar chiqarish *deduksiya* deyiladi. Deduksiya so'zi o'zbek tilida „xulosa chiqarish" degan ma'noni bildiradi.

Fizika, kimyo, biologiya kabi fanlarda kuzatish va tajribalarga suyanib, induktiv mulohazalar yuritish keng qo'llaniladi. Induksiya so'zi o'zbek tilida „boshqarib borish" yoki „yetaklab borish" kabi ma'nolarni bildiradi.

Xususiy xulosalardan umumiy xulosalar chiqarish *induksiya* deyiladi.

Matematik induksiya usuli asosida matematik induksiya prinsipi yotadi. Bu prinsipning mazmunini izohlaylik. Biror n natural songa bog'liq fikrimizni (gipotezani) $A(n)$ orqali belgilaylik. Bu fikrimizning (mulohazamizning) ixtiyoriy

n natural son uchun to'g'riligini isbotlash kerak bo'lsin. Lekin $A(n)$ mulohazaning to'g'riligini barcha n uchun bevosita tekshirib ko'rishning iloji bo'lmasin. $A(n)$ mulohaza matematik induksiya prinsipiga ko'ra quyidagicha isbotlanadi: $A(n)$ mulohazaning to'g'riligi, avvalo, $n=1$ uchun tekshiriladi. So'ngra aytilgan mulohazani $n=k$ uchun to'g'ri deb faraz qilib, uning to'g'riligi $n=k+1$ uchun isbotlanadi. Shundan so'ng $A(n)$ mulohazamiz barcha n natural sonlar uchun isbotlangan hisoblanadi. Shunday qilib, matematik induksiya prinsipiga asoslangan isbot matematik induksiya usuli bilan isbotlash deyiladi. Bunday isbot ikkita qismdan iborat bo'lib, ikkita mustaqil teoremani isbotlashdan iborat.

1-teorema. $A(n)$ tasdiq $n=1$ uchun o'rinli.

2-teorema. Agar $A(n)$ tasdiqning $n=k$ uchun to'g'riligidan uning $n=k+1$ uchun to'g'riligi kelib chiqsa, u holda $A(n)$ tasdig'i ixtiyoriy n uchun o'rinli bo'ladi.

Agar bu ikki teorema isbotlangan bo'lsa, u holda matematik induksiya prinsipiga ko'ra $A(n)$ tasdig'imiz ixtiyoriy n uchun o'rinli bo'ladi.

Matematik induksiya usuli bilan yaqinroq tanishish uchun bir nechta misollar ko'ramiz.

1-masala. Tenglikni isbotlang: $1+2+3+...+n=\frac{n(n+1)}{2}$

1-qadam. $n=1$ da $1=\frac{1\cdot 2}{2}\Rightarrow 1=1$. 1-qadam isbotlandi.

2-qadam. $n=k$ da $1+2+3+...+k=\frac{k(k+1)}{2}$ tenglikning bajarilishi berilgan.

$n=k+1$ da $1+2+3+...+k+(k+1)=\frac{(k+1)(k+2)}{2}$ tenglikning bajarilishini isbotlash kerak.

Isbot. $\underbrace{1+2+3+...+k}_{\frac{k(k+1)}{2}}+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{(k+1)(k+2)}{2}$

2-masala. Tenglikni isbotlang. $\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+...+\frac{1}{(5n-3)(5n+2)}=\frac{n}{2(5n+2)}$

1-qadam. $n=1$ da $\frac{1}{2\cdot 7}=\frac{1}{2\cdot(5+2)}\Rightarrow\frac{1}{2\cdot 7}=\frac{1}{2\cdot 7}$. 1-qadam isbotlandi.

2-qadam. $n=k$ da $\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+...+\frac{1}{(5k-3)(5k+2)}=\frac{k}{2(5k+2)}$ tenglikning bajarilishi berilgan.

$n=k+1$ da $\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+...+\frac{1}{(5k-3)(5k+2)}+\frac{1}{(5k+2)(5k+7)}=\frac{k+1}{2(5k+7)}$ tenglikning bajarilishini isbotlash lozim.

Isbot. $\underbrace{\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+...+\frac{1}{(5k-3)(5k+2)}}_{\frac{k}{2(5k+2)}}+\frac{1}{(5k+2)(5k+7)}=\frac{k}{2(5k+2)}+$

$+\frac{1}{(5k+2)(5k+7)}=\frac{5k^2+7k+2}{2(5k+2)(5k+7)}=\frac{(k+1)(5k+2)}{2(5k+2)(5k+7)}=\frac{k+1}{2(5k+7)}$. 2-qadam isbotlandi.

3-masala. Barcha natural n larda $4^n+15n-1$ ifodani 9 ga bo'linishini isbotlang.

1-qadam. $n=1$ da $4+15-1=18 \vdots 9$. 1-qadam isbotlandi.

2-qadam. $n=k$ da $4^k+15k-1 \vdots 9$ bajarilishi berilgan.

$n=k+1$ da $4^{k+1}+15(k+1)-1 \vdots 9$ bajarilishini isbotlash kerak.

Isbot. $4^{k+1}+15(k+1)-1=4\cdot 4^k+15k+15-1=\underbrace{4^k+15k-1}_{\vdots 9}+3\cdot(4^k+5)$.

Endi 4^k+5 ifodani 3 ga bo'linishini isbotlash yetarli.

$$4^k+5=(3+1)^k+5=3A+1+5=3A+6=3B.$$

4-masala. Ixtiyoriy nomanfiy butun n sonlarda $3^{3n+2}+2^{4n+1}$ ifodani 11 ga bo'linishini isbotlang.

1-qadam. $n=0$ da $9+2=11 \vdots 11$. 1-qadam isbotlandi.

2-qadam. $n=k$ da $3^{3k+2}+2^{4k+1} \vdots 11$ bajarilishi berilgan.

$n=k+1$ da $3^{3(k+1)+2}+2^{4(k+1)+1} \vdots 11$ bajarilishini isbotlash lozim.

Isbot.

$$3^{3(k+1)+2}+2^{4(k+1)+1}=3^{3k+2+3}+2^{4k+1+4}=27\cdot 3^{3k+2}+16\cdot 2^{4k+1}=16\cdot(\underbrace{3^{3k+2}+2^{4k+1}}_{\vdots 11})+\underbrace{11\cdot 3^{3k+2}}_{\vdots 11} \vdots 11.$$

2-qadam isbotlandi.

5-masala. $\forall n\in N$ da n^7-n ifodani 7 ga bo'linishini isbotlang.

1-qadam. $n=1$ da $1^7-1=0\vdots 7$. 1-qadam isbotlandi.

2-qadam. $n=k$ da $k^7-k\vdots 7$ bajarilishi berilgan.

$n=k+1$ da $(k+1)^7-(k+1)\vdots 7$ bajarilishini isbotlash lozim.

Isbot. $(k+1)^7-(k+1)=k^7+7k^6+...+7k+1-k-1=\underbrace{k^7-k}_{\vdots 7}+\underbrace{7A}_{\vdots 7}=7B\vdots 7$.

2-qadam isbotlandi.

6-masala. $\forall n\in N$ da $1\cdot 2+2\cdot 3+3\cdot 4+...+n\cdot(n+1)=\frac{n(n+1)(n+2)}{3}$ ayniyatni isbotlang.

1-qadam. $n=1$ da$1\cdot 2=\frac{1\cdot 2\cdot 3}{3}\Rightarrow 2=2$. 1-qadam isbotlandi.

2-qadam. $n=k$ da $1\cdot 2+2\cdot 3+3\cdot 4+...+k\cdot(k+1)=\frac{k(k+1)(k+2)}{3}$ tenglikning bajarilishi berilgan.

$n=k+1$ da $1\cdot 2+2\cdot 3+3\cdot 4+...+k\cdot(k+1)+(k+1)\cdot(k+2)=\frac{(k+1)(k+2)(k+3)}{3}$ tenglikning bajarilishini isbotlash lozim.

Isbot. $\underbrace{1\cdot 2+2\cdot 3+3\cdot 4+...+k\cdot(k+1)}_{\frac{k(k+1)(k+2)}{3}}+(k+1)\cdot(k+2)=\frac{k(k+1)(k+2)}{3}+$

$$+(k+1)(k+2)=(k+1)(k+2)\left(\frac{k}{3}+1\right)=\frac{(k+1)(k+2)(k+3)}{3}.$$

2-qadam isbotlandi.

Matematik induksiya metodining tengsizliklarni isbotlashda qo’llanilishi.

7-masala. $\forall n\in N,\ n\geq 5$ da $2^n>n^2$ tengsizlikni isbotlang.

1-qadam. $n=5$ da$2^5>5^2\Rightarrow 32>25$. 1-qadam isbotlandi.

2-qadam. $n=k$ da $2^k>k^2$ tengsizlikning bajarilishi berilgan.

$n=k+1$ da $2^{k+1}>(k+1)^2$ tengsizlikning bajarilishini isbotlash lozim.

Isbot. $2^{k+1}=2\cdot\underbrace{2^k}_{>k^2}>k^2+k^2\geq\left[k\geq 5\Rightarrow k^2\geq 15+2k\right]\geq k^2+2k+15>(k+1)^2$. 2-qadam isbotlandi.

8-masala. $\forall n \in N,\ n \geq 3$ da $n^{n+1} > (n+1)^n$ tengsizlikni isbotlang.

1-qadam. $n = 3$ da $3^4 > 4^3 \Rightarrow 81 > 64$. 1-qadam isbotlandi.

2-qadam. $n = k$ da $k^{k+1} > (k+1)^k$ tensizlikning bajarilishi berilgan.

$n = k+1$ da $(k+1)^{k+2} > (k+2)^{k+1}$ tengsizlikning bajarilishini isbotlash lozim.

Isbot. Agar $k^{k+1} > (k+1)^k$ bo'lsa, u holda $1 > \frac{(k+1)^k}{k^{k+1}}$ tengsizlik bajariladi. Oxirgi tengsizlikni musbat $(k+1)^{k+2}$ songa ko'paytirib, quyidagi tenglik va tengsizlik zanjirini hosil qilamiz:

$$(k+1)^{k+2} > \frac{(k+1)^k (k+1)^{k+2}}{k^{k+1}} = \frac{((k+1)^2)^{k+1}}{k^{k+1}} = \left(\frac{k^2+2k+1}{k}\right)^{k+1} = \left(k+2+\underbrace{\frac{1}{k}}_{>0}\right)^{k+1} > (k+2)^{k+1}.$$

2-qadam isbotlandi.

9-masala. $\forall n \in N,\ n \geq 2$ da $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > \sqrt{n}$ tengsizlikni isbotlang.

1-qadam. $n = 2$ da $1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{3}} > \frac{1+1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$ ega bo'lamiz. 1-qadam isbotlandi.

2-qadam. $n = k$ da $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} > \sqrt{k}$ tengsizlik berilgan.

$n = k+1$ da $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ tengsizlikni isbotlash lozim.

Isbot.

$$\underbrace{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}}}_{>\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k^2+k}+1}{\sqrt{k+1}} > \frac{\sqrt{k^2}+1}{\sqrt{k+1}} = \sqrt{k+1}.$$

2-qadam isbotlandi.

Matematik induksiya metodining trigonometrik ayniyatlarni isbotlashda qo'llanilishi.

10-masala. Ixtiyoriy n natural son uchun $|\sin nx| \leq n|\sin x|$ tengsizlikni isbotlang.

1-qadam. $n = 1$ da $|\sin 1 \cdot x| \leq 1 \cdot |\sin x|$. 1-qadam isbotlandi.

2-qadam. $n=k$ da $|\sin kx|\le k|\sin x|$ tengsizlikning bajarilishi berilgan.

$n=k+1$ da $|\sin(k+1)x|\le (k+1)|\sin x|$ tengsizlikning bajarilishini isbotlash lozim.

Isbot. $|\sin(k+1)x| = |\sin kx\cos x+\sin x\cos kx| \le \underbrace{|\sin kx|}_{\le k|\sin x|}\underbrace{|\cos x|}_{\le 1} + |\sin x|\underbrace{|\cos kx|}_{\le 1} \le (k+1)|\sin x|$

Faollashtiruvchi savollar.

1. *Deduksiya deb nimaga aytiladi?*
2. *Induksiya deb nimaga aytiladi?*
3. *3 ga karrali son 9 ga karralimi?*
4. *9 ga karrali son 3 ga karralimi?*
5. *Induksiya metodini har qanday tasdiqni to'g'riligini isbotlashda qo'llasa bo'ladimi?*
6. *Teskarisini faraz qilib isbotlash mazmunini ayting?*
7. *Induksiya yo'lida isbotlashning mazmunini ayting?*

Mustaqil yechish uchun masalalar

1. Tenglikni isbotlang. $1^2+2^2+3^2+...+n^2=\dfrac{n(n+1)(2n+1)}{6}, n\in N$
2. Tenglikni isbotlang. $\dfrac{1}{1\cdot 3}+\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+...+\dfrac{1}{(2n-1)(2n+1)}=\dfrac{1}{2}\left(1-\dfrac{1}{2n+1}\right), n\in N$
3. Tenglikni isbotlang. $1+x+x^2+...+x^n=\dfrac{x^{n+1}-1}{x-1}, n\in N$
4. Isbotlang: $6^{2n}-1 \vdots 35,\ n\in N$
5. Isbotlang: $6^{2n}+3^{n+2}+3^n \vdots 11,\ n\in N$
6. Isbotlang: $6^{2n-1}+1 \vdots 7,\ n\in N$
7. Isbotlang: $1+\dfrac{1}{2}+\dfrac{1}{3}+...+\dfrac{1}{2^n-1}>\dfrac{n}{2},\ n\in N$
8. Faraz qilaylik, to'g'ri burchakli uchburchakda a,b – katetlar uzunlikari, c – gipotenuza uzunligi. U holda barcha $n\ge 2$ natural sonlar uchun $a^n+b^n\le c^n$ tengsizlikni isbotlang.
9. Isbotlang: $1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+...+\dfrac{1}{\sqrt{n}}<2\sqrt{n},\ n\in N$
10. Isbotlang: $arctg\dfrac{1}{2}+arctg\dfrac{1}{8}+arctg\dfrac{1}{18}+...+arctg\dfrac{1}{2n^2}=arctg\dfrac{n}{n+1},\ n\in N$
11. Isbotlang: $\cos x\cos 2x\cos 4x...\cos 2^n x=\dfrac{\sin 2^{n+1}x}{2^{n+1}\sin x},\ n\in N$
12. Isbotlang: $1+2+2^2+...+2^{n-1}=2^n-1,\ n\in N$
13. Isbotlang: $9^n-8n-1 \vdots 64,\ n\in N$

14. Isbotlang: $\left|\sin\left(\sum_{k=1}^{n} x_k\right)\right| \le \sum_{k=1}^{n}\sin x_k,\ 0 \le x_k \le \pi; k = 1,2,...,n$

15. Isbotlang: $7^{n+2} + 8^{2n+1} \vdots 57, n \in N$

16. Isbotlang: $n^4 + 6n^3 + 11n^2 + 6n \vdots 24, n \in N$

17. Isbotlang: $7^{2n} - 1 \vdots 48, n \in N$

18. Isbotlang: $2^{5n+3} + 5^n \cdot 3^{n+2} \vdots 17, n \in N$

19. Isbotlang: $3^{2n+1} + 2^{n+2} \vdots 7, n \in N$

20. Isbotlang: $15^n + 6 \vdots 7, n \in N$

21. Isbotlang: $5 \cdot 2^{3n-2} + 3^{3n-1} \vdots 19, n \in N$

22. Isbotlang: $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)...\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+2}{2n+2}, n \in N$

23. Isbotlang: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)...\left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}, n \in N$

24. Isbotlang: $3^{2n+1} + 40n - 67 \vdots 64, n \in N$

25. Isbotlang: $n^3 + (n+1)^3 + (n+2)^3 \vdots 9, n \in N$

26. Isbotlang: $1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2, n \in N$

27. Isbotlang: $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + ... + n \cdot n! = (n+1)! - 1,\ n \in N$

28. Isbotlang: $\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \frac{3^2}{5\cdot 7} + ... + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}, n \in N$

29. Isbotlang: $\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot...\cdot\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}},\ n \in N\cdot$

30. Isbotlang: $n! > 2^{n-1},\ n \ge 3,\ n \in N$

31. Isbotlang: $2^n > 2n+1,\ n \ge 3,\ n \in N$

32. Isbotlang: $2^n > n^3,\ n \ge 11,\ n \in N$

33. Isbotlang: $\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{3n+1} > 1,\ n \in N\cdot$

34. Isbotlang: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}, n \in N$

35. Isbotlang: $\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + ... + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}, n \in N$

36. Isbotlang: $1 + 3 + 5 + ... + (2n-1) = n^2, n \in N$

37. Isbotlang: $\cos x + \cos 2x + \cos 3x + ... + \cos nx = \frac{\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}}{\sin\frac{x}{2}}, n \in N$

38. Isbotlang: $\sin x+\sin 2x+\sin 3x+...+\sin nx=\dfrac{\sin\frac{nx}{2}\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}, n\in N$

39. Isbotlang: $\dfrac{7}{1\cdot 8}+\dfrac{7}{8\cdot 15}+\dfrac{7}{15\cdot 22}+...+\dfrac{7}{(7n-6)(7n+1)}=1-\dfrac{1}{7n+1}, n\in N$

40. Isbotlang: $4^n>n^2,\ n\in N$

41. Isbotlang: $2^n\geq n+1,\ n\in N$

42. Isbotlang: $n^3+17n \vdots 6,\ n\in N$

43. Isbotlang: $n>1$ da $n^n-n^2+n-1 \vdots (n-1)^2,\ n\in N$

4-§. Sonlarda qoldiqli bo'lish

Tayanch so'zlar: *Teorema, Ferma teoremasi, Eyler teoremasi, Nyuton binomi, Eyler funksiyasi, tub son, induksiya.*

Sonlarda qoldiqli bo'lish bilan asosan P.Ferma va L.Eylerlar shug'ullanishgan. Ferma P'er (1601-1655 y.y) – fransiyalik advokat va matematik. Analitik geometriyaning asoschisi. Leonard Eyler (1707-1783 y.y) – shvetsariyalik matematik olim, Iogan Bernullining shogirdi. Ferma va Eyler teoremalari bilan tanishaylik.

Teorema. (Ferma) p tub son uchun $a^p\equiv a(\mathrm{mod}\, p)$ taqqoslama o'rinli bo'ladi.

Isbot. a bo'yicha induksiyani qo'llaymiz. $a=1$ da natija ravshan. Faraz qilamiz, $a^p-a \vdots p$. U holda Nyuton binom formulasiga ko'ra

$$(a+1)^p-(a+1)=a^p-a+\sum_{k=1}^{p-1}C_p^k a^k$$

$C_p^k=\dfrac{p!}{(p-k)!k!}=p\cdot\dfrac{(p-1)!}{(p-k)!k!}=pt \vdots p,\ k=1,2,...,p-1$ munosabatdan va induksiya farazimizga ko'ra $(a+1)^p-(a+1) \vdots p$. Demak,

$$(a+1)^p\equiv(a+1)(\mathrm{mod}\, p)$$

Teorema isbot bo'ldi.

Izoh. Agar $(a,p)=1$ bo'lsa, u holda Ferma teoremasidan quyidagi munosabat kelib chiqadi:

$$a^{p-1} \equiv 1 \pmod p$$

Teorema. (Eyler) Agar $(a,m)=1$ bo'lsa, u holda $a^{\varphi(m)} \equiv 1 \pmod m$. Bu yerda $\varphi(m)$ – 1 dan m gacha bo'lgan sonlar orasida shu m soni bilan o'zaro tub bo'lgan sonlar soni. Odatda uni Eyler funksiyasi ham deyiladi. Bu yerda $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$, p_i – lar tub sonlar, α_i – lar esa natural sonlar $i = 1,2,\ldots,k$ bo'lsa,

$$\varphi(m) = m\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_k}\right)$$

orqali topiladi.

Isbot. $n = \varphi(m)$ deb belgilaymiz.

$x_1, x_2, \ldots, x_n$ sonlar $\{1,2,\ldots,m\}$ to'plam ichida joylashgan va m soni bilan o'zaro tub bo'lgan o'zaro teng bo'lmagan sonlarni ajratamiz. Ravshanki ular bir-biri bilan m modul bo'yicha taqqoslanmaydi. Quyidagi sonlarni kiritamiz:

$$ax_1, ax_2, \ldots, ax_n$$

Bu ketma-ketlikda ham ikkita turli hadlari m modul bo'yicha taqqoslanmaydi. Haqiqatdan ham, $x_i \neq x_j$ va $x_i a \equiv x_j a \pmod m$ bo'lsin. U holda $(a,m)=1$ bo'lgani uchun $x_i \equiv x_j \pmod m$ bo'ladi. Bu esa $x_1, x_2, \ldots, x_n$ sonlarning bir-biri bilan m modul bo'yicha taqqoslanmasligiga zid.

$(a,m)=1$, $(x_i,m)=1$ bo'lgani uchun $(ax_i,m)=1$ bo'ladi, ya'ni

$$ax_i \equiv x_j \pmod m$$

Bu taqqoslamalarni $i = 1,2,\ldots,n$ bo'yicha ko'paytirib chiqsak

$$ax_1 \cdot ax_2 \cdot \ldots \cdot ax_n = a^n \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n \equiv x_1 \cdot x_2 \cdot \ldots \cdot x_n \pmod m.$$

ni hosil qilamiz. $(x_1 \cdot x_2 \cdot \ldots \cdot x_n, m)=1$ bo'lgani uchun $a^n \equiv 1 \pmod m$ bo'ladi.

Teorema isbot bo'ldi.

Quyidagi ikkita sonli funksiyani ham keltirib o'tamiz.

I. n natural sonning natural bo'luvchilar soni τ (tau)

$$\tau(n) = (\alpha_1+1)(\alpha_2+1)\ldots(\alpha_k+1)$$

formula orqali topiladi.

II. n natural sonning natural bo'luvchilar yig'indisi σ (sigma)

$$\sigma(n)=\frac{p_1^{\alpha_1+1}-1}{p_1-1}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2-1}\cdot\ldots\cdot\frac{p_k^{\alpha_k+1}-1}{p_k-1}$$

formula orqali topiladi.

Bu yerda $n=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots\cdot p_k^{\alpha_k}$, p_i – lar tub sonlar, α_i – lar esa natural sonlar $i=1,2,\ldots,k$.

Qoida 1. Har qanday sonning yuqori darajasining oxirgi ikkita raqamini topish uchun shu sonni 100 ga bo'lgandagi qoldiqni topish yetarli.

Qoida 2. Agar sonning darajasini 100 ga bo'lganda Eyler teoremasining o'zaro tublik sharti bajarilmasa, u holda 100 soni bilan darajali sonning asosining EKUBi ga taqqoslamaning har ikkala tomoniga bo'lib yuborib, quyidagi masalaga olib kelinadi: $(a,m)=d,\ \frac{a}{d}\equiv\frac{r}{d}\left(\operatorname{mod}\frac{m}{d}\right)\Rightarrow A\equiv R(\operatorname{mod}M)$. Bu oxirgi munosabat Eyler teoremasiga tushadi: $(A,M)=1,\ A^{\varphi(M)}\equiv 1(\operatorname{mod}M)$.

Endi Ferma va Eyler teoremalarining qo'llanilishiga doir bir nechta misollarni ko'rib chiqaylik.

1-masala. 7^{112} sonni 11 ga bo'lgandagi qoldiqni toping.

Yechish. $(7,11)=1$ va 11 – tub son, u holda Ferma teoremasiga ko'ra

$$7^{11-1}\equiv 1(\operatorname{mod}11)\Rightarrow 7^{10}\equiv 1(\operatorname{mod}11)\Rightarrow\left(7^{10}\right)^{11}\equiv(1)^{11}(\operatorname{mod}11)\Rightarrow 7^{110}\equiv 1(\operatorname{mod}11)\Rightarrow$$

$$\Rightarrow 7^{112}\equiv 7^2(\operatorname{mod}11)\Rightarrow 7^{112}\equiv 49(\operatorname{mod}11)\Rightarrow 7^{112}\equiv 5(\operatorname{mod}11).$$

Javob: $r=5$.

2-masala. 14^{79} sonni 9 ga bo'lgandagi qoldiqni toping.

Yechish. $(14,9)=1$va 9 – tub son emas, u holda Eyler teoremasiga asosan $14^{\varphi(9)}\equiv 1(\operatorname{mod}9)$ munosabat o'rinli. Endi $\varphi(9)$ ni hisoblaymiz: $\varphi(9)=9\left(1-\frac{1}{3}\right)=6$.

$$14^{\varphi(9)}\equiv 1(\operatorname{mod}9)\Rightarrow 14^6\equiv 1(\operatorname{mod}9)\Rightarrow 14^{78}\equiv 1(\operatorname{mod}9)\Rightarrow 14^{79}\equiv 14(\operatorname{mod}9)\Rightarrow 14^{79}\equiv 5(\operatorname{mod}9).$$

Javob: $r=5$.

3-masala. 2^{2006} sonning oxirgi ikki raqamini toping.

Yechish. 2^{2006} sonning oxirgi ikki raqamini topish uchun uni 100 ga bo'lgandagi qoldiqni topamiz. Eyler teoremasiga keltirib olamiz:

$$2^{2006} \equiv r(\mathrm{mod}\,100) \Rightarrow 4\cdot 2^{2004} \equiv r(\mathrm{mod}\,100) \Rightarrow 2^{2004} \equiv \frac{r}{4}(\mathrm{mod}\,25).$$

Endi 2^{2004} sonini 25 ga bo'lgandagi qoldiqni topamiz. $(2,25)=1$ va 25– tub son emas, u holda Eyler teoremasiga asosan $2^{\varphi(25)} \equiv 1(\mathrm{mod}\,25)$ munosabat o'rinli. Endi $\varphi(25)$ ni hisoblaymiz: $\varphi(25) = 25\left(1-\frac{1}{5}\right) = 20$.

$$2^{\varphi(25)} \equiv 1(\mathrm{mod}\,25) \Rightarrow 2^{20} \equiv 1(\mathrm{mod}\,25) \Rightarrow 2^{2000} \equiv 1(\mathrm{mod}\,25) \Rightarrow 2^{2004} \equiv 2^4(\mathrm{mod}\,25) \Rightarrow$$

$$\Rightarrow 2^{2004} \equiv 16(\mathrm{mod}\,25) \Rightarrow \frac{r}{4} = 16 \Rightarrow r = 64.$$

Demak, 2^{2006} sonning oxirgi ikki raqami 64 bilan tugar ekan.

Javob: 64

4-masala. $2^{60} + 7^{30}$ sonni 13 ga bo'linishini isbotlang.

Isbot. 2^{60} va 7^{30} sonlarini 13 ga bo'lgandagi qoldiqlarni topamiz.

$(2,13)=1$ va 13 – tub son, u holda Ferma teoremasiga ko'ra

$$2^{12} \equiv 1(\mathrm{mod}\,13) \Rightarrow 2^{60} \equiv 1(\mathrm{mod}\,13) \Rightarrow 2^{60} = 13n + 1 \qquad (1)$$

Xuddi shunday $(7,13)=1$ va 13 – tub son, u holda Ferma teoremasiga ko'ra

$$7^{12} \equiv 1(\mathrm{mod}\,13) \Rightarrow 7^{24} \equiv 1(\mathrm{mod}\,13) \Rightarrow 7^{30} \equiv 7^6(\mathrm{mod}\,13).$$

Endi 7^6 ni 13 ga bo'lgandagi qoldiqni topamiz.

$$7^6 = 343^2 = (13\cdot 26 + 5)^2 = 13t + 5^2 = 13t + 13 + 12 = 13m + 12 \Rightarrow 7^6 = 13m + 12 \qquad (2)$$

(1) va (2) tengliklarni hadlab qo'shamiz: $2^{60} + 7^{30} = 13(n+m) + 13 = 13k$.

5-masala. p tub son bo'lsa, u holda 2^{p^2} ni 13 ga bo'lgandagi qoldiqni toping.

Yechish. $p = 2$ bo'lsin. $2^4 = 16 = 13 + 3 \Rightarrow r = 3$.

$p = 3$ bo'lsin. $2^9 = 512 = 13\cdot 39 + 5 \Rightarrow r = 5$.

$p > 3$ tub son bo'lsin. U holda p albatta toq son bo'ladi. $p-1$va $p+1$ juft son ketma-ketligi bo'ladi. Bundan $(p-1)(p+1) \vdots 4 \Rightarrow p^2 - 1 \vdots 4$. Endi $p-1,\ p,\ p+1$ sonlar 3 ta ketma-ket kelgan natural son bo'lgani uchun ulardan albatta bittasi 3 ga bo'linadi. p – tub son bo'lgani uchun 3 ga bo'linmaydi. Bundan esa $p-1 \vdots 3$ yoki $p+1 \vdots 3$ ekanligi kelib chiqadi. Demak, $p^2-1 \vdots 3$ ekan.

$$\begin{cases} p^2-1 \vdots 4 \\ p^2-1 \vdots 3 \end{cases}, (3,4)=1 \Rightarrow p^2-1 \vdots 12 \Rightarrow p^2=12k+1,\ k \in N.$$

Endi Fermaning kichik teoremasiga asosan $(2,13)=1$ va 13 – tub. Bundan

$$2^{12} \equiv 1(\mathrm{mod}13) \Rightarrow 2^{12k} \equiv 1(\mathrm{mod}13) \Rightarrow 2^{12k+1} \equiv 2(\mathrm{mod}13) \Rightarrow 2^{p^2} \equiv 2(\mathrm{mod}13) \Rightarrow r=2.$$

Javob: $r=\{2,3,5\}$.

Faollashtiruvchi savollar.

1. *Natural sonning natural bo'luvchilar soni qanday topiladi?*
2. *Natural sonning natural bo'luvchilari yig'indisi qanday topiladi?*
3. *Natural sonning yuqori darajasini oxirgi 2 ta raqami qanday topiladi?*
4. *Ferma teoremasini ayting?*
5. *Eyler teoremasini ayting?*
6. *Eyler va Ferma teoremalarining farqini ayting?*

Mustaqil yechish uchun masalalar

1. 13^7 sonni 7 ga bo'lgandagi qoldiqni toping.
2. 9^{258} sonni 17 ga bo'lgandagi qoldiqni toping.
3. 3^{101} sonni 101 ga bo'lgandagi qoldiqni toping.
4. 8^{122} sonni 11 ga bo'lgandagi qoldiqni toping.
5. 12^{61} sonni 25 ga bo'lgandagi qoldiqni toping.
6. 3^{2007} sonning oxirgi ikki raqamini toping.
7. 4^{2005} sonning oxirgi ikki raqamini toping.
8. 2222^{5555} sonni 7 ga bo'lgandagi qoldiqni toping.
9. 3333^{6666} sonni 5 ga bo'lgandagi qoldiqni toping.
10. $7^{7^{7}}$ sonning oxirgi ikki raqamini toping.
11. 17^{502} sonni 12 ga bo'lgandagi qoldiqni toping.
12. 12^{1201} sonni 13 ga bo'lgandagi qoldiqni toping.
13. 3^{20} sonni 7 ga bo'lgandagi qoldiqni toping.
14. 4^{12} sonni 9 ga bo'lgandagi qoldiqni toping.
15. 2002^{2002} sonni 5 ga bo'lgandagi qoldiqni toping.
16. 4^8 sonni 9 ga bo'lgandagi qoldiqni toping.
17. 11^{322} sonni 17 ga bo'lgandagi qoldiqni toping.
18. 3^{37} sonni 7 ga bo'lgandagi qoldiqni toping.

19. 17^{609} sonni 16 ga bo'lgandagi qoldiqni toping.
20. 25^{111} sonni 26 ga bo'lgandagi qoldiqni toping.
21. 2^{500} sonni 7 ga bo'lgandagi qoldiqni toping.
22. 171^{2147} sonni 52 ga bo'lgandagi qoldiqni toping.
23. 5^{2023} sonning oxirgi ikki raqamini toping.
24. 6^{2003} sonning oxirgi ikki raqamini toping.
25. 7^{2021} sonning oxirgi ikki raqamini toping.
26. 8^{2005} sonning oxirgi ikki raqamini toping.
27. 9^{2002} sonning oxirgi ikki raqamini toping.
28. $8^{8^{87}}$ sonning oxirgi raqamini toping.
29. 1001^{9n} sonning oxirgi raqamini toping.
30. 65^{6n+1} sonini 9 ga bo'lgandagi qoldiqni toping.
31. $3^{32}+2^{41}-5$ sonini 11 ga bo'lgandagi qoldiqni toping.
32. $10^{70}-361$ sonini 27 ga bo'linishini isbotlang.
33. $11^{10}-1$ sonini 100 ga bo'linishini isbotlang.
34. 19^{8^7} sonining oxirgi 3 ta raqamini toping.
35. $1^4+2^4+3^4+...+2019^4$ sonni 16 ga bo'lgandagi qoldiqni toping.
36. $3^{2002^{2001}}$ sonining oxirgi 3 ta raqamini toping.
37. 227^{227} sonining oxirgi 4 ta raqamini toping.

5-§. Evklid algoritmi

Tayanch so'zlar: *Teorema, lemma, Fermaning kichik teoremasi, Evklid algoritmi, EKUB, o'zaro tub sonlar, qoldiq, bo'luvchi, tub son, juft son, toq son.*

Sonlar nazariyasiga doir masalalar barcha olimpiadalarda, xususan Xalqaro Matematika Olimpiadasi (XMO) da ham doim "urf" bo'lib kelgan va hozir ham shunday. Biz, ushbu paragrafda sonlar nazariyasiga oid chet el olimpiadalarida va XMO da uchraydigan ayrim masalalarni ko'ramiz. Ularni yechishda yuqorida keltirilgan Fermaning kichik teoremasidan va quyidagi keltiriladigan Evklid algoritmidan unumli foydalanamiz.

1-teorema. Agar p tub son, $a \in Z$ bo'lsa, u holda $a^p - a \vdots p$ o'rinli.

Bu teorema quyidagi ko'rinishda ham ifodalash mumkin:

Fermaning kichik teoremasi. Agar p tub son, $a \in Z$ va $(a, p) = 1$ bo'lsa, u holda $a^{p-1} \equiv 1 \pmod p$ munosabat o'rinli bo'ladi.

Endi Evklid algoritmini ta'riflashga o'tamiz. Biz, $EKUB(a,b)=(a,b)$ deb olamiz va uni quyida faqat shunday tushunamiz.

1-ta'rif (Evklid algoritmi). Berilgan ikkita a,b natural sonlarning (a,b) topish uchun, a va b larni, $|a-b|$ va $\min\{a,b\}$ lar bilan almashtiraveramiz, qachonki, ikkita teng son hosil bo'lguncha. Bu algoritm Evklid algoritmi deyiladi va uni ikkitadan ko'p sonlar uchun ham umumlashtirish ham mumkin. Bu yerda va bundan keyin $EKUB(a,b)=(a,b)$.

Evklid algoritmidan quyidagi natijalar kelib chiqadi:

Natija-1. Ixtiyoriy $a,b\in N$ sonlar uchun $(a,b)=ax+by$ tenglikni qanoatlantiruvchi $x,y\in Z$ sonlar mavjud. Agar $x,y\in Z$ sonlar topilmasa, u holda a,b sonlar o'zaro tub bo'ladi.

Isbot. $(a,b)=d$ bo'lsin. Evklid algoritmini qo'llaymiz:

$$a=bq_1+r_1,\ b=r_1q_2+r_2,\ r_1=r_2q_3+r_3,...,r_{n-2}=r_{n-1}q_{n-1}+r_n,\ r_{n-1}=r_nq_n$$

Bunda r_k qoldiqlar uchun $r_k=\alpha_k a+\beta_k b$ tengliklar bajarilishini ko'rsatamiz, bu yerda α_k,β_k – butun sonlar.

r_1 uchun ushbu mulohaza o'rinliligi $r_1=a-bq_1$ dan kelib chiqadi. Faraz qilamiz, barcha $r_1,r_2,...,r_{n-1}$ qoldiqlar $r_k=\alpha_k a+\beta_k b$ tenglikni qanoatlantirsin. U holda

$$r_n=\alpha_{n-2}a+\beta_{n-2}b-(\alpha_{n-1}a+\beta_{n-1}b)q_{n-1}=(\alpha_{n-2}-\alpha_{n-1})a+(\beta_{n-2}-\beta_{n-1}q_{n-1})b$$

Shuning uchun $(a,b)=ax+by$, bu yerda x va y sonlari mos ravishda

$(\alpha_{n-2}-\alpha_{n-1})$ va $(\beta_{n-2}-\beta_{n-1}q_{n-1})$ larga teng.

Natijadan quyidagi xossa kelib chiqadi:

Xossa-1. a,b sonlar o'zaro tub bo'lishi uchun $ax+by=1$ tenglikning bajarilishi zarur va yetarli, bu yerda $x,y\in Z$.

Natija-2. $a,m,n\in N$ va $a>1$ sonlar uchun quyidagi tenglik o'rinlidir:

$$(a^m-1,a^n-1)=a^{(m,n)}-1$$

Isbot. Umumiylikka zarar yetkazmasdan $m\geq n$ deb olish mumkin. Ravshanki, $(a^n,a^n-1)=1$. Demak,

$$\left(a^m-1,a^n-1\right)=\left(a^m-a^n,a^n-1\right)=\left(a^n\left(a^{m-n}-1\right),a^n-1\right)=\left(a^{n-m}-1,a^n-1\right)$$

Shuning uchun a^m-1,a^n-1 sonlar uchun Evklid algoritmi m va n darajalar uchun Evklid algoritmiga o'tadi hamda (m,n) da tugallanadi.

Natija-2 ni yanada umumiylashtirib,

$a,b,m,n\in N$ va $a>1,b\geq 1$ sonlar uchun

$$\left(a^m-b^m,a^n-b^n\right)=a^{(m,n)}-b^{(m,n)}$$

tenglik o'rinli ekanligiga ishonch hosil qilish mumkin.

Endi yuqoridagilarni qo'llab ba'zi masalalarni ko'rib chiqmiz.

1-masala.

$$\frac{a^p+1}{a+1} \qquad (1)$$

sonining p dan kichik tub bo'luvchisi mavjud emasligini isbotlang.

Isbot. Teskarisini faraz qilaylik, ya'ni $\frac{a^p+1}{a+1}\vdots q,\ (p>q)$ bo'lsin. $\frac{a^p+1}{a+1}\vdots q$, bundan $a^p+1\vdots q$, undan esa

$$a^{2p}-1\ \vdots\ q \qquad (2)$$

ga kelamiz. $(a,q)=1$ dan Fermaning kichik teoremasiga ko'ra:

$$a^{q-1}-1\ \vdots\ q \qquad (3)$$

(2) va (3) lardan Evklid algoritmiga ko'ra $\left(a^{2p}-1,a^{q-1}-1\right)=a^{(2p;q-1)}-1$ ni hosil qilamiz. $(2p,q-1)=d$ deylik. $p>q$ va p,q – o'zaro tubligidan $d=1$ yoki $d=2$ bo'lishi mumkin.

1-hol. $d=1$ bo'lsin. U holda $a-1\ \vdots\ q$ dan $a=qk+1,\ k\in N$ va nihoyat, uni (1) ga olib borib qo'ysak

$$\frac{a^p+1}{a+1}=a^{p-1}-a^{p-2}+...+a^2-a+1=(qk+1)^{p-1}-(qk+1)^{p-2}+...+(qk+1)^2-(qk+1)+1=$$

$$=qA+1-1+1-1+...+1-1+1=qA+1$$

Oxirgi ifoda q ga bo'linmaydi, bundan esa $\frac{a^p+1}{a+1} \vdots q$ ifodaning xatoligi kelib chiqadi.

2-hol. $d=2$ bo'lsin. U holda $a^2-1 \vdots q$ va undan $(a-1)(a+1) \vdots q$ ni q – tub sonligidan $a-1 \vdots q$ va $a+1 \vdots q$ lardan biri bo'lishi mumkin. Biz $a-1 \vdots q$ holni avvalroq ko'rganligimiz sababli, $a+1 \vdots q$ holni tekshiramiz. $a=qm-1,\ m\in N$ deb olaylik. U holda

$$\frac{a^p+1}{a+1}=a^{p-1}-a^{p-2}+...+a^2-a+1=(qm-1)^{p-1}-(qm-1)^{p-2}+...+(qm-1)^2-(qm-1)+1=$$

$$=qB+1+1+1+1+...+1+1+1=qB+p$$

ni va $\frac{a^p+1}{a+1} \vdots q$ dan $qB+p \vdots q$ va $p \vdots q$ kelib chiqadi. Farazga ko'ra $p>q$ va q – tub son deb olgan edik. Demak, 2 ta holdan ko'rinadiki, farazimiz xato, ya'ni $\frac{a^p+1}{a+1}$ sonining p dan kichik tub bo'luvchisi mavjud emas. Shu bilan masala to'liq isbotlandi.

2-masala. $n\in N$ va $n>1$ toq son bo'lsin. U holda 3^n+1 soni n ga bo'linmasligini isbotlang.

Isbot. Teskarisini faraz qilaylik, ya'ni, 3^n+1 ga bo'linadigan $n>1$ toq natural son mavjud bo'lsin. n sonining eng kichik tub bo'luvchisi p bo'lsin. U holda $3^n+1 \vdots n$ va $n \vdots p$ lardan $3^n+1 \vdots p$ kelib chiqadi. Bundan,

$$3^{2n}-1 \vdots p \qquad (4)$$

ni hosil qilamiz. $(3,p)=1$ dan Fermaning kichik teoremasiga ko'ra:

$$3^{p-1}-1 \vdots p \qquad (5)$$

ga egamiz. (4) va (5) ga 2-natijani qo'llasak, $(3^{2n}-1, 3^{p-1}-1)=3^{(2n,p-1)}-1$ ni olamiz. $3^{2n}-1 \vdots p$ va $3^{p-1}-1 \vdots p$ lardan $3^{(2n,p-1)}-1 \vdots p$ ni, n ning toqligi va $(n,p-1)=1$ dan $(2n,p-1)=2$ ni hosil qilamiz. Bundan esa $3^2-1 \vdots p$, ya'ni $8 \vdots p$ ga egamiz. Lekin oxirgi natija o'rinli emas. Bundan farazimizning noto'g'riligi kelib chiqadi. Demak, 3^n+1 soni ixtiyoriy birdan katta toq natural n ga bo'linmas ekan.

3-masala (Xitoy Matematika Olimpidasi). Quyidagi shartni qanoatlantiruvchi barcha p,q tub sonlarni toping.

$$5^p + 5^q \vdots pq$$

Yechish. Fermaning kichik teoremasiga ko'ra

$$5^p - 5 \vdots p \qquad (6)$$

$$5^q - 5 \vdots q \qquad (7)$$

lar o'rinlidir.

p yoki q lardan biri 5 ga teng bo'lgan holni qaraylik. p, q – tub sonlarning simmetrikligidan biz faqat $p = 5$ holni qarab, boshqa yechimlarni topilgan juftliklarning joyini almashtirish orqali hosil qilamiz. $p = 5$, u holda

$$5^q + 5^5 \vdots 5q \vdots q \qquad (8)$$

ni topamiz. (8) dan (7) ni ayirib, $5^5 + 5 \vdots q$ ga egamiz. $5^5 + 5 = 2 \cdot 5 \cdot 313$ tenglikdan $q = \{2,5,313\}$, ya'ni $(5,2),(5,5),(5,313)$ juftliklarni aniqlaymiz. p, q – tub sonlarning simmetrikligidan esa $(2,5),(5,5),(313,5)$ yechimlarni hosil qilamiz.

$p, q \neq 5$ bo'lsin. $5^p + 5^q$ ifoda ustida shakl almashtirishlar bilan quyidagi ko'rinishga keltiramiz:

$$5^p + 5^q = 5^p - 5 + 5^q + 5 = \left(5^p - 5\right) + 5\left(5^{q-1} + 1\right)$$

$$5^p + 5^q = 5^q - 5 + 5^p + 5 = \left(5^q - 5\right) + 5\left(5^{p-1} + 1\right).$$

Oxirgi ikki tenglikdan (6) va (7) larni tatbiq qilib qilib

$$\begin{cases} 5^{q-1} + 1 \vdots p \\ 5^{p-1} + 1 \vdots q \end{cases}$$

Bulardan quyidagilarni aniqlaymiz:

$$\begin{cases} 5^{2(q-1)} - 1 \vdots p \\ 5^{2(p-1)} - 1 \vdots q \end{cases} \left| \begin{array}{l} 5^{p-1} - 1 \vdots p \\ 5^{q-1} - 1 \vdots q \end{array} \right. \Rightarrow$$

Evklid algoritmidan foydalniib:

$$\begin{cases} \left(5^{2(q-1)} - 1, 5^{p-1} - 1\right) = 5^{\left(2(q-1), p-1\right)} - 1 \vdots p \\ \left(5^{2(p-1)} - 1, 5^{q-1} - 1\right) = 5^{\left(2(p-1), q-1\right)} - 1 \vdots q \end{cases}$$

natijalarni olamiz. $\begin{cases} p - 1 = a \\ q - 1 = b \end{cases}$ deb belgilaylik, u holda:

$$\begin{cases} 5^{(2b,a)} - 1 \vdots p \\ 5^{(2a,b)} - 1 \vdots q \end{cases} \left| \begin{array}{l} (2b,a) = d_1 \Rightarrow 2b = d_1 k,\ a = d_1 l,\ (k,l) = 1 \\ (2a,b) = d_2 \Rightarrow 2a = d_2 k,\ b = d_2 n,\ (m,n) = 1 \end{array} \right.$$

Bulardan quyidagilarni aniqlaymiz:

$$\begin{cases} 2d_2 n = d_1 k \\ 2d_1 l = d_2 m \end{cases} \Rightarrow \begin{cases} \dfrac{d_2}{d_1} = \dfrac{k}{2n} \\ \dfrac{d_2}{d_1} = \dfrac{2l}{m} \end{cases} \Rightarrow \begin{cases} \dfrac{k}{2n} = \dfrac{2l}{m},\ (k,l) = 1 \\ km = 4nl,\ (m,n) = 1 \end{cases}$$

shartlardan quyidagi 3 ta holdan biri bo'lishi mumkin:

1-hol. $k = 2n$ va $m = 2l$ bo'lsin. U holda $d_1 = d_2$ va undan

$(2b,a) = (2a,b)$ kelib chiqadi. $5^{(2(q-1),p-1)} - 1 \vdots p$ va $5^{(2(p-1),q-1)} - 1 \vdots q$ larda $(2(p-1),p-1) = (2(p-1),q-1)$ ni e'tiborga olsak, $5^{(2(q-1),p-1)} - 1 \vdots q$ ni, $\left(5^{2(q-1)} - 1, 5^{p-1} - 1\right) = 5^{(2(q-1),p-1)} - 1$ dan esa $5^{p-1} - 1 \vdots q$ ni hosil qilamiz. Yuqorida topgan $5^{p-1} + 1 \vdots q$ natijamizdan va $5^{p-1} - 1 \vdots q$ dan $2 \vdots q$ ya'ni, $q = 2$ ni topamiz. Buni berilgan shartga olib borib qo'yib, $p = 2,3,5$ lardan biri bo'lishini aniqlaymiz. Demak, bu holda $(2,2),(3,2),(5,2)$ juftliklarga egamiz. p,q larning simmetrikligidan $(2,3),(2,5)$ javoblarni ham osongina aniqlaymiz.

2 – va 3 – hollarda ham 1 – holdagi kabi yo'l tutib, osongina yana yuqoridagi javoblarni topamiz. Demak, $5^p + 5^q \vdots pq$ shartni qanoatlantiruvchi barcha p,q – tub sonlarning juftliklari quyidagilardan iborat:

Javob: $(2,2),(2,3),(2,5),(3,2),(5,2),(5,5),(5,313),(313,5)$.

4-masala. $\dfrac{\left(7^p - 2^p\right)\left(7^q - 2^q\right)}{pq}$ soni butun bo'ladigan barcha p,q tub sonlarni toping.

Yechish. Ko'rinib turibdiki, p,q toq tub sonlar. p,q tub sonlar bo'lganligi uchun quyidagi 4 ta hol bo'lishi mumkin:

1-hol. $\left(7^p - 2^p\right) \vdots p,q$,

2-hol. $\left(7^q - 2^q\right) \vdots p,q$,

3-hol. $\left(7^p - 2^p\right) \vdots p$ va $\left(7^q - 2^q\right) \vdots q$

4-hol. $\left(7^p - 2^p\right) \vdots q$ va $\left(7^q - 2^q\right) \vdots p$

Endi har bir holni ko'rib chiqamiz:

1-hol. $(7^p-2^p)\vdots p,q$ bo'lsin. U holda Fermaning kichik teoremasiga ko'ra $7^p-7\vdots p$ va $2^p-2\vdots p$ larni hadma-had ayirib, $7^p-2^p-5\vdots p$ ni, $7^p-2^p\vdots p$ ni e'tiborga olib, $5\vdots p$ ni hosil qilamiz. p ning tubligidan $p=5$ kelib chiqadi. $7^5-2^5=5^2\cdot 11\cdot 61\vdots q$ dan $(5,5),(5,11),(5,61)$ yechimlar kelib chiqadi.

2-hol. $(7^q-2^q)\vdots p,q$ bo'lsin. Bu holda ham yuqoridagi kabi yo'l tutib, $q=5$ ni aniqlaymiz. $7^5-2^5=5^2\cdot 11\cdot 61\vdots p$ dan $(5,5),(11,5),(61,5)$ yechimlar kelib chiqadi.

3-hol. $(7^p-2^p)\vdots p$ va $(7^q-2^q)\vdots q$ bo'lsin. $7^p-2^p-5\vdots p$ va $7^q-2^q-5\vdots q$ lardan $p=q=5$ yechimlarni aniqlaymiz.

4-hol. $(7^p-2^p)\vdots q$ va $(7^q-2^q)\vdots p$ bo'lsin. Biz $p=q$ dan yana yuqoridagi kabi $p=q=5$ yechimni aniqlaymiz. Endi $p\neq q$ bo'lsin. U holda $p>q$ deb olaylik va undan $p=qk+r$ bu yerda $0<r<q$.

$$7^p-2^p=7^{qk+r}-2^{qk+r}=7^{qk+r}-7^r\cdot 2^{qk}+7^r\cdot 2^{qk}-2^{qk+r}=7^r\left(7^{qk}-2^{qk}\right)+2^{qk}\left(7^r-2^r\right)$$

$7^{qk}-2^{qk}\vdots 7^q-2^q\vdots p$ va $p=qk+r$, $0<r<q$ lardan $7^r-2^r\vdots p$ ni, ya'ni $(7^p-2^p)\vdots p$ kelib chiqadi. Buni biz yuqorida ko'rib chiqqan edik. $p<q$ holda ham xuddi shunday yo'l tutiladi. Ko'rib chiqilgan barcha hollardan quyidagi yechimlarni aniqlaymiz:

Javob: $(5,5),(5,11),(5,61),(11,5),(61,5)$.

5-masala ("American Mathematical Monthly" jurnali). Quyidagi shartni qanoatlantiruvchi barcha natural n larni toping:

$$2^n-1\vdots n$$

Yechish. $n=1$ da o'rinli. $n>1$ bo'lsin. U holda n sonining toqligi ma'lum. n ning eng kichik tub bo'luvchisi p bo'lsin. U holda $2^n-1\vdots n$ va $n\vdots p$ lardan $2^n-1\vdots p$ kelib chiqadi. n ning toqligidan p ning toqligi va undan $(2,p)=1$ kelib chiqadi. Fermaning kichkina teoremasiga ko'ra $2^{p-1}-1\vdots p$ ni hosil qilamiz. $2^n-1\vdots p$ va $2^{p-1}-1\vdots p$ lar Evklid algoritmiga ko'ra: $\left(2^n-1,2^{p-1}-1\right)=2^{(n,p-1)}-1\vdots p$ va $(n,p-1)=1$ dan $2^{(n,p-1)}-1=2^1-1\vdots p$ ga kelamiz. Xato narsaga kelganligimiz sababli, $n>1$ da 2^n-1 soni n ga bo'linmaydi. Bundan yuqoridagi shart faqat $n=1$ da bajarilishi aniqlanadi.

2-teorema (Eyler teoremasi). $n \in N$ va a butun sonlar $(a,n)=1$ ni qanoatlantirsa, $a^{\varphi(n)} \equiv 1 (\bmod n)$ bo'ladi.

Bu yerda $\varphi(n)$ – Eyler funksiyasi, ya'ni, 1 dan n gacha bo'lgan natural sonlar ichida n bilan o'zaro tub bo'lgan sonlar soni.

Quyida bir yangi belgilashni kiritamiz va shunga doir teorema keltiramiz.

2-ta'rif. p – tub va α – manfiy bo'lmagan butun sonlar bo'lsin. $a \| p^{\alpha}$ belgi bilan a sonini p – tub sonining eng katta α darajasiga bo'linishini aytamiz, ya'ni, a soni p^{α} ga bo'linadi, ammo $p^{\alpha+1}$ ga bo'linmaydi.

3-teorema. p – toq tub son bo'lsin. $\alpha \in N$ son $a-1 \| p^{\alpha}$ va ixtiyoriy nomanfiy butun β lar uchun $a^n - 1 \| p^{\alpha+\beta}$ formula, $n \| p^{\beta}$ bo'lganda bajarilishi zarur va yetarli.

$p=2$ hol uchun: $\alpha \in N$ da $a^2 - 1 \| 2^{\alpha}$ va ixtiyoriy nomanfiy butun β lar uchun $a^n - 1 \| 2^{\alpha+\beta}$ formula, $n \| 2^{\beta}$ bo'lganda bajarilishi zarur va yetarli.

6-masala (Xalqaro Matematika Olimpiada-1990). $2^n + 1 \vdots n^2$ shartni bajaradigan $n \in N$ larni toping.

Yechish. $n=1$ da o'rinli. $n>1$ bo'lsin. U holda n sonining toqligi ma'lum. n^2 ning eng kichik tub bo'luvchisi p bo'lsin. Undan $n \vdots p$ lardan kelib chiqadi. $2^n + 1 \vdots n^2$ dan $2^{2n} - 1 \vdots 2^n + 1$ va undan $2^{2n} - 1 \vdots n^2,\ n^2 \vdots p$ lardan

$$2^{2n} - 1 \vdots p \qquad (9)$$

kelib chiqadi. n ning toqligidan p ning toqligi va undan $(2,p)=1$ kelib chiqadi. Fermaning kichik teoremasiga ko'ra

$$2^{p-1} - 1 \vdots p \qquad (10)$$

ni hosil qilamiz.

(9) va (10) ga Evklid algoritmini qo'llasak, $\left(2^{2n} - 1, 2^{p-1} - 1\right) = 2^{(2n,p-1)} - 1 \vdots p$ ga kelamiz. p ning toqligidan $(2n, p-1) = 2$ va $2^{(2n,p-1)} - 1 = 2^2 - 1 = 3 \vdots p$, undan esa $p=3$ ni topamiz. Endi, $n = 3^{\alpha} k,\ k \not\vdots 3$ deb olaylik. U holda $2^{2n} - 1 \vdots n^2 = 3^{2\alpha} k^2$, ya'ni $2^{2n} - 1 \vdots 3^{2\alpha}$ ga egamiz. $2^2 - 1 \| 3$ va $n \| 3^{\alpha}$ lardan 3-teoremaga ko'ra $2^{2n} - 1 \| 3^{2\alpha+1}$ ni

hosil qilamiz. $2^{2n}-1\vdots 3^{2\alpha}$ dan $2^{2n}-1\| 3^{2\alpha+1}\vdots 3^{2\alpha}$, ya'ni $\alpha+1\geq 2\alpha$ bundan $\alpha=1$ ni topamiz. $n=3k,\ k\vdots 3$ ni aniqladik. Endi, k ning eng kichik tub bo'luvchisi q bo'lsin. U holda $2^{2n}-1\vdots n^2,\ n=3k$ dan $2^{6k}-1\vdots 9k^2$ va $k\vdots q$ dan $8^{2k}-1\vdots q$ kelib chiqadi. Fermaning kichkina teoremasiga ko'ra, $8^{q-1}-1\vdots q$ ni $8^{2k}-1\vdots q$ va $8^{q-1}-1\vdots q$ larga Evklid algoritmini qo'llasak

$$\left(8^{2k}-1, 8^{q-1}-1\right)=8^{(2k,q-1)}-1\vdots q$$

ga egamiz. q – soni k ning eng kichik tub bo'luvchisiligidan va toqligidan

$(2k,q-1)=2$ bo'lishini osongina aniqlaymiz. $8^{(2k,q-1)}-1=8^2-1=63=7\cdot 9\vdots q$ dan $q=7$ kelib chiqadi. Demak, $k=7m$ va undan $n=21m$ ekan. Ammo, 2^n+1 soni 7 ga bo'linmasligi quyidagidan ma'lum: $2^n+1=2^{3k}+1=8^k+1=(7+1)^k+1=7A+2$ Oxirgidan $k=1$, undan esa $n=3$ kelib chiqadi. Berilgan shart faqat $n=1$ va $n=3$ da o'rinli ekan.

7-masala. $8^n-1\vdots 7^n$ shartni qanoatlantiruvchi barcha $n\in N$ larni toping.

Yechish. $8^1-1\vdots 7^1$ va $n\| 7^{\alpha}$ larga 3-teoremani tatbiq qilsak, $8^n-1\| 7^{\alpha+1}$ ni hosil qilamiz. $8^n-1\vdots 7^n$ va $8^n-1\| 7^{\alpha+1}$ lardan $7^{\alpha+1}\vdots 7^n$ ga egamiz. Oxirgidan $\alpha+1\geq n$, bundan esa $\alpha\geq n-1$ ni hosil qilamiz. $n\| 7^{\alpha}$ dan $n|\vdots 7^{\alpha}$ ni, $\alpha\geq n-1$ dan $n\geq 7^{n-1}$ ga kelamiz. Ma'lumki, oxirgi tengsizlik $n\geq 2$ da o'rinli emas. Demak, $8^n-1\vdots 7^n$ shart faqat $n=1$ da bajarilar ekan.

8-masala (Xalqaro Matematika Olimpiadasi -1990). p – tub son, $n\in N$ va $n\leq 2p$ bo'lsa, quyidagi shartni qanoatlantiruvchi barcha (n,p) juftliklarni toping:

$$(p-1)^n+1\vdots n^{p-1}$$

Yechish. Agar $n=1$ bo'lsa, u holda p – tub son ixtiyoriy, ya'ni, $(1,p)$ yechimni aniqlaymiz. $n>1$ bo'lsin. n ning eng kichik tub bo'luvchisi q bo'lsin. $n\vdots q$ dan $n^{p-1}\vdots q$ va $(p-1)^n+1\vdots n^{p-1}$ dan $(p-1)^n+1\vdots q$ ni aniqlaymiz. $(p-1)^{2n}-1\vdots (p-1)^n+1\vdots q$ dan esa

$$(p-1)^{2n}-1\vdots q \qquad (11)$$

ni hosil qilamiz. $n\vdots q$ dan $(p-1,q)=1$ va bundan 1-teoremaga ko'ra

$$(p-1)^{q-1}-1\vdots q \qquad (12)$$

ni, (11) va (12) larga Evklid algoritmini qo'llab,

$$\left((p-1)^{2n}-1,(p-1)^{p-1}-1\right)=(p-1)^{(2n,q-1)}-1\vdots q$$

ga kelamiz. $(2n,q-1)=d$ deb olsak, 2 ta hol bo'lishi mumkin:

1-hol. $d=1$ bo'lsin. U holda $q=2$, $(p-1)^{q-1}+1\vdots 2$ va p – tub sonligidan $p=2$ va undan $n=2$ natijani olamiz. Bu holda biz $(2,2)$ yechimga egamiz.

2-hol. $d=2$ bo'lsin. U holda $(p-1)^2-1=p(p-2)\vdots q$ ni hosil qilamiz. Yana 2 ta hol bo'lishi mumkin:

2.1.-hol. $p-2\vdots q$ bo'lsin. Bundan $p=qk+2$ ga va buni masalaning berilgan shartiga olib borib qo'ysak, $(qk+1)^n+1\vdots n^{qk+1}\vdots q$ ni, undan esa $2\vdots q$ ni hosil qilamiz. Ammo bu mumkin emas. Demak, bu holda yechim yo'q.

2.2.-hol. $p\vdots q$ bo'lsin. p,q lar tub sonligidan $p=q$ kelib chiqadi, undan $n\vdots p$ va berilgan $n\le 2p$ shartdan $n=p$ yoki $n=2p$ bo'lishi mumkin. $n=p$ holda, $(p-1)^p+1\vdots p^{p-1}$ va undan

$$(p-1)^p+1=p^p-p\cdot p^{p-1}+...+p\cdot p-1+1=p^2(pA+1)\vdots p^{p-1}$$

ni, $2\ge p-1$ dan $n=p=3$ yoki $(3,3)$ yechimni hosil qilamiz.

$n=2p$ hol bo'lishi mumkin emas. Chunki, n ning eng kichik tub bo'luvchisini $p=q$ va $d=2$ dan p ning toqligi ma'lum. $n=2p$ da esa p dan kichik 2 tub bo'luvchisi mavjud bo'lib qoldi. Shu sababli, bu holda yechim mavjud emas.

Demak, berilgan shartni qanoatlantiruvchi juftliklar quyidagilardan iborat:

Javob: $(1,p),(2,2),(3,3)$.

Faollashtiruvchi savollar.

1. *O'zaro tub sonlar ta'rifini ayting?*
2. *Tub son deb nimaga aytiladi?*
3. *Murakkab son deb nimaga aytiladi?*
4. *Har qanday natural sonni n ga bo'lganda qanday qoldiqlar qoladi?*
5. *Evklid algoritmi qanday qo'llaniladi?*

Mustaqil yechish uchun masalalar

1. $2^n - 1 \vdots 3^k$ shartni qanoatlantiruvchi barcha $n, k \in N$ sonlarni toping.
2. $n, a \in N$ va p – toq tub sonlar uchun $a^p \equiv 1 \pmod{p^n}$ o'rinli bo'lsa, u holda $a \equiv 1 \pmod{p^{n-1}}$ shart ham bajarilishini isbotlang.
3. $n \in N$ da $4(a^n + 1)$ soni aniq kub bo'ladigan barcha natural a larni toping.
4. $k > 1, k \in N$ son uchun $1^n + 2^n + 3^n + 4^n + ... + k^n \vdots n$ shartni qanoatlantiradigan $n \in N$ lar cheksiz ko'pligini isbotlang.
5. $n, a \in N$ va p – tub sonlar uchun $2^p + 3^p = a^n$ tenglik o'rinli bo'lsa, $n = 1$ bo'lishini ko'rsating.
6. $\prod_{k=0}^{n-1}(2^n - 2^k) \vdots n!$ ni isbotlang. Bunda, $n \in N$ va $k \in Z$
7. Quyidagi tengliklarni qanoatlantiruvchi barcha p – tub sonlarni toping:

   a) $\dfrac{7^{p-1} - 1}{p} = k^2,\ k \in N$

   b) $\dfrac{11^{p-1} - 1}{p} = l^2,\ l \in N$
8. $m, n \in N$ larni toping, bunda $(n-1)! + 1 = n^m$.
9. $x, y, k, n \in N$ va $n > 1$, $(x, y) = 1$ sonlar $x^k + y^k = 3^n$ ni qanoatlantiradigan barcha n larni aniqlang.
10. Quyidagi shart o'rinli bo'ladigan k ning maksimumini toping:

$$1990^{1991^{1992}} + 1992^{1991^{1990}} \vdots 1991^k$$

6-§. Yig'indilarni hisoblash

Tayanch so'zlar: *Natural son, butun son, kvadrat, kub, arifmetik progressiya, geometrik progressiya, kvadrat ildiz, kasr, yig'indi.*

Qoida. Har qanday $\underbrace{aaa...aa}_{n\ ta}$ ko'rinishdagi sonni quyidagicha yozish mumkin:

$$\underbrace{aaa...aa}_{n\ ta} = \frac{a}{9}(10^n - 1)$$

Quyidagicha belgilashlar kiritamiz: $\sum$ summa belgisi (yig'indi).

$$1 + 2 + 3 + ... + n = \sum_{k=1}^{n} k$$

$$1^2+2^2+3^2+...+n^2=\sum_{k=1}^{n}k^2$$

$$1^3+2^3+3^3+...+n^3=\sum_{k=1}^{n}k^3$$

va hokazo.

Sodda kasrlarga yoyish:

$$ax^2+bx+c=a(x-x_1)(x-x_2),\ \frac{k}{x^2+px+q}=\frac{n}{x-x_1}+\frac{m}{x-x_2}.$$

Masalan, $\frac{1}{x^2+x-2}=\frac{1}{3}\left(\frac{1}{x+2}-\frac{1}{x-1}\right)$. Chunki $x+2$ va $x-1$ larning farqi 3 ga teng.

Endi quyidagi masalalarni ko'rib chiqaylik.

1-masala. $\underbrace{111..11}_{100\,ta}\underbrace{222..22}_{100\,ta}$ soni ikkita ketma-ket natural sonning ko'paytmasidan iborat ekanligini isbotlang.

Isbot. $\underbrace{111..11}_{100\,ta}\underbrace{222..22}_{100\,ta}=\underbrace{111..11}_{100\,ta}\cdot10^{100}+\underbrace{222...22}_{100\,ta}=\frac{1}{9}\left(10^{100}-1\right)\cdot10^{100}+\frac{2}{9}\left(10^{100}-1\right)=$

$$=\frac{1}{9}\left(10^{200}-10^{100}+2\cdot10^{100}-2\right)=\frac{1}{9}\left(10^{200}+10^{100}-2\right)=\frac{1}{9}\left(10^{100}-1\right)\left(10^{100}+2\right)=$$

$$=\frac{10^{100}-1}{3}\cdot\frac{10^{100}+2}{3}=\frac{3}{9}\left(10^{100}-1\right)\cdot\frac{10^{100}-1+3}{3}=\frac{3}{9}\left(10^{100}-1\right)\cdot\left(\frac{10^{100}-1}{3}+1\right)=$$

$$=\frac{3}{9}\left(10^{100}-1\right)\cdot\left(\frac{3}{9}\left(10^{100}-1\right)+1\right)=\underbrace{333...33}_{100\,ta}\cdot\left(\underbrace{333...33}_{100\,ta}+1\right)=\underbrace{333...33}_{100\,ta}\cdot\underbrace{333...3}_{99ta}4.$$

2-masala. $\forall n\in N$ da $\frac{10^n+8}{9}$ ifoda natural son ekanligini isbotlang.

Isbot. $\frac{10^n+8}{9}=\frac{10^n-1+9}{9}=\frac{10^n-1}{9}+1=\frac{1}{9}\left(10^n-1\right)+1=\underbrace{111...11}_{n-1\,ta}2.$

3-masala. Yig'indini hisoblang:

$$\sum_{k=1}^{n}k=1+2+3+...+n$$

Yechish. Quyidagi ayniyatdan foydalanamiz: $(k+1)^2 = k^2 + 2k + 1$. Bu tenglikni har ikkala tomonini yig'indiga o'tamiz.

$$\sum_{k=1}^{n}(k+1)^2 = \sum_{k=1}^{n}k^2 + 2\sum_{k=1}^{n}k + \sum_{k=1}^{n}1$$

Bu tenglikning $2\sum_{k=1}^{n}k$ hadini bir tomonda qoldirib, qolgan hadlarini bir tomonga o'tkazamiz.

$$2\sum_{k=1}^{n}k = \sum_{k=1}^{n}(k+1)^2 - \sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}1 = 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 - 1^2 - 2^2 - \ldots - n^2 - n =$$

$$(n+1)^2 - 1 - n = n^2 + n = n(n+1) \Rightarrow \sum_{k=1}^{n}k = \frac{n(n+1)}{2}$$

Javob: $\frac{n(n+1)}{2}$.

4-masala. Yig'indini hisoblang:

$$\sum_{k=1}^{n}k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

Yechish. Quyidagi ayniyatdan foydalanamiz: $(k+1)^3 = k^3 + 3k^2 + 3k + 1$. Bu tenglikni har ikkala tomonini yig'indiga o'tamiz.

$$\sum_{k=1}^{n}(k+1)^3 = \sum_{k=1}^{n}k^3 + 3\sum_{k=1}^{n}k^2 + 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}1$$

Bu tenglikning $3\sum_{k=1}^{n}k^2$ hadini bir tomonda qoldirib, qolgan hadlarini bir tomonga o'tkazamiz.

$$3\sum_{k=1}^{n}k^2 = \sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n}k^3 - 3\sum_{k=1}^{n}k - \sum_{k=1}^{n}1 = (n+1)^3 - 1 - \frac{3n(n+1)}{2} - n =$$

$$= (n+1)\left((n+1)^2 - \frac{n}{2} - 1\right) = (n+1)\left(n^2 + 2n - \frac{n}{2}\right) = \frac{n(n+1)(2n+1)}{2}$$

Bundan esa

$$\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$$

ekanligi kelib chiqadi.

Javob: $\frac{n(n+1)(2n+1)}{6}$.

5-masala. Yig'indini hisoblang:

$$\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+\frac{1}{12\cdot 17}+...+\frac{1}{(5n-3)\cdot(5n+2)}$$

Yechish. $5n-3$ va $5n+2$ sonlar farqi 5 ga teng bo'lgani uchun yig'indining umumiy hadini quyidagicha sodda kasrlarga yoyamiz:

$$\frac{1}{(5n-3)\cdot(5n+2)}=\frac{1}{5}\left(\frac{1}{5n-3}-\frac{1}{5n+2}\right)\Rightarrow\sum_{k=1}^{n}\frac{1}{(5k-3)\cdot(5k+2)}=\frac{1}{5}\left(\sum_{k=1}^{n}\frac{1}{5k-3}-\sum_{k=1}^{n}\frac{1}{5k+2}\right).$$

Hosil bo'lgan yig'indini hisoblaymiz:

$$\frac{1}{5}\left(\sum_{k=1}^{n}\frac{1}{5k-3}-\sum_{k=1}^{n}\frac{1}{5k+2}\right)=\frac{1}{5}\left(\frac{1}{2}-\frac{1}{7}+\frac{1}{7}-\frac{1}{12}+\frac{1}{12}-\frac{1}{17}+...+\frac{1}{5n-3}-\frac{1}{5n+2}\right)=\frac{n}{2(5n+2)}$$

Bundan esa

$$\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+\frac{1}{12\cdot 17}+...+\frac{1}{(5n-3)\cdot(5n+2)}=\frac{n}{2(5n+2)}$$

ekanligi kelib chiqadi.

Javob: $\frac{n}{2(5n+2)}$.

6-masala. Hisoblang: $\sqrt{\underbrace{111...11}_{2n\ ta}-\underbrace{222...22}_{n\ ta}}$

Yechish. Dastlab ildiz ostidagi ifodaning qiymatini hisoblaymiz:

$$\underbrace{111...11}_{2n\ ta}-\underbrace{222...22}_{n\ ta}=\frac{1}{9}\left(10^{2n}-1\right)-\frac{2}{9}\left(10^{n}-1\right)=\frac{1}{9}\left(10^{2n}-1-2\cdot 10^{n}+2\right)=\frac{1}{9}\left(10^{2n}-2\cdot 10^{n}+1\right)=$$

$$=\frac{1}{9}\left(10^{n}-1\right)^{2}=\left(\frac{10^{n}-1}{3}\right)^{2}=\left(\frac{3}{9}\cdot\left(10^{n}-1\right)\right)^{2}=\left(\underbrace{333...33}_{n\ ta}\right)^{2}$$

Bundan esa

$$\sqrt{\underbrace{111...11}_{2n\ ta}-\underbrace{222...22}_{n\ ta}}=\underbrace{333...33}_{n\ ta}$$

ekanligi kelib chiqadi.

Javob: $\underbrace{333...33}_{n\ ta}$.

7-masala. Yig’indini hisoblang:

$$\left(a+\frac{1}{a}\right)^2+\left(a^2+\frac{1}{a^2}\right)^2+...+\left(a^n+\frac{1}{a^n}\right)^2$$

Yechish. Arifmetik va geometrik progressiyalarning dastlabki n ta hadi yig’indisini hisoblash formulasidan foydalanamiz:

$$\left(a+\frac{1}{a}\right)^2+\left(a^2+\frac{1}{a^2}\right)^2+...+\left(a^n+\frac{1}{a^n}\right)^2=a^2+a^4+...+a^{2n}+\frac{1}{a^2}+\frac{1}{a^4}+...+\frac{1}{a^{2n}}+2n=$$

$$=\frac{a^2\left(a^{2n}-1\right)}{a^2-1}+\frac{\frac{1}{a^2}\left(\frac{1}{a^{2n}}-1\right)}{\frac{1}{a^2}-1}+2n=\frac{a^2\left(a^{2n}-1\right)}{a^2-1}+\frac{\left(a^{2n}-1\right)}{a^{2n}\left(a^2-1\right)}+2n=\frac{\left(a^{2n}-1\right)\left(a^{2n+2}+1\right)}{a^{2n}\left(a^2-1\right)}+2n$$

Bundan esa

$$\left(a+\frac{1}{a}\right)^2+\left(a^2+\frac{1}{a^2}\right)^2+...+\left(a^n+\frac{1}{a^n}\right)^2=\frac{\left(a^{2n}-1\right)\left(a^{2n+2}+1\right)}{a^{2n}\left(a^2-1\right)}+2n$$

tengligi kelib chiqadi.

Javob: $\frac{\left(a^{2n}-1\right)\left(a^{2n+2}+1\right)}{a^{2n}\left(a^2-1\right)}+2n$.

8-masala. Yig’indini hisoblang: $1+2x+3x^2+4x^3+...+(n+1)x^n,\ x\neq 1$

Yechish. Yig’indini S orqali belgilab, tenglikni har ikkala qismiga x ni ko’paytiramiz.

$$xS=x+2x^2+3x^3+4x^4+...+(n+1)x^{n+1}$$

Yuqoridagi berilgan tenglikdan oxirgi hosil bo’lgan tenglikni ayiramiz.

$$S-xS=1+x+x^2+x^3+x^4+...+x^n-(n+1)x^{n+1}\Rightarrow(1-x)S=\frac{x^{n+1}-1}{x-1}-(n+1)x^{n+1}$$

Oxirgi tenglikdan S yig’indining

$$S=\frac{(n+1)x^{n+2}-(n+2)x^{n+1}+1}{(x-1)^2}$$

tengligi kelib chiqadi.

Javob: $\dfrac{(n+1)x^{n+2}-(n+2)x^{n+1}+1}{(x-1)^2}$.

9-masala. Yig'indini hisoblang: $1\cdot 2+2\cdot 3+3\cdot 4+...+n\cdot(n+1)$

Yechish. Yuqoridagi 3 va 4 - masalalardagi ayniyatlardan foydalanamiz.

$$1\cdot 2+2\cdot 3+3\cdot 4+...+n\cdot(n+1)=1\cdot(1+1)+2\cdot(2+1)+3\cdot(3+1)+...+n\cdot(n+1)=$$

$$1^2+2^2+3^2+...+n^2+1+2+3+...+n=\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{3}.$$

Javob: $\dfrac{n(n+1)(n+2)}{3}$.

10-masala. Agar $a_i,\ i=1,2,...,n$ arifmetik progressiyaning hadlari bo'lsa, u holda quyidagini hisoblang:

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+...+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

Yechish. Bu yig'indidagi har bir kasrning maxrajini irratsionallikdan qutqarib hisoblaymiz:

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+...+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{\sqrt{a_2}-\sqrt{a_1}}{d}+\frac{\sqrt{a_3}-\sqrt{a_2}}{d}+...+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{d}=$$

$$=\frac{1}{d}\left(\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+...+\sqrt{a_n}-\sqrt{a_{n-1}}\right)=\frac{1}{d}\left(\sqrt{a_n}-\sqrt{a_1}\right).$$

Bundan esa

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+...+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{1}{d}\left(\sqrt{a_n}-\sqrt{a_1}\right)$$

tengligi kelib chiqadi.

Javob: $\dfrac{1}{d}\left(\sqrt{a_n}-\sqrt{a_1}\right)$.

11-masala. Agar $b_i,\ i=1,2,...,n$ cheksiz kamayuvchi geometrik progressiyaning hadlari bo'lsa, u holda quyidagini hisoblang:

$$b_1^3+b_2^3+b_3^3+...$$

Yechish. Bu yig'indidagi har bir qo'shiluvchilarini boshqa bir, hadlari $c_i,\ i=1,2,...,n$ bo'lgan cheksiz kamayuvchi geometrik progressiyaga almashtirib yig'indini hisoblaymiz. Bizga ma'lumki cheksiz kamayuvchi geometrik progressiyaning hadlari yig'indisi $S=\frac{c_1}{1-p}$ orqali topiladi. Bizni yig'indimizda $c_1=b_1^3,\ p=q^3$ ga teng.

$$S=\frac{c_1}{1-p}=\frac{b_1^3}{1-q^3}.$$

Javob: $\frac{b_1^3}{1-q^3}$.

12-masala. Yig'indini hisoblang: $1+4\cdot 2+7\cdot 2^2+10\cdot 2^3+...+64\cdot 2^{21}+67\cdot 2^{22}$

Yechish. Yig'indini S orqali belgilab, tenglikni har ikkala qismiga 2 ni ko'paytiramiz.

$$S=1+4\cdot 2+7\cdot 2^2+10\cdot 2^3+...+64\cdot 2^{21}+67\cdot 2^{22}$$

$$2S=2+4\cdot 2^2+7\cdot 2^3+10\cdot 2^4+...+64\cdot 2^{22}+67\cdot 2^{23}.$$

Hosil bo'lgan tenglikdan yuqoridagi berilgan tenglikni hadma had ayiramiz:

$$S=1-4\cdot 2-3\cdot 2^2-3\cdot 2^3-...-3\cdot 2^{21}+67\cdot 2^{22}\Rightarrow S=67\cdot 2^{22}-7-3\cdot 2^2\left(1+2+...+2^{20}\right)\Rightarrow$$

$$\Rightarrow S=67\cdot 2^{22}-7-12\cdot\left(2^{21}-1\right)\Rightarrow S=2^{29}+5$$

Javob: $2^{29}+5$.

13-masala. Yig'indini hisoblang.

$$1\cdot 4+2\cdot 7+3\cdot 10+...+n(3n+1)$$

Yechish. Quyidagi $1+2+3+...+n=\frac{n(n+1)}{2},\ 1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$ tengliklardan foydalanamiz.

$$1\cdot 4+2\cdot 7+3\cdot 10+...+n(3n+1)=1\cdot(3\cdot 1+1)+2\cdot(3\cdot 2+1)+3\cdot(3\cdot 3+1)+...+n\cdot(3n+1)=$$

$$=3\left(1^2+2^2+3^2+...+n^2\right)+\left(1+2+3+...+n\right)=3\cdot\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}=n(n+1)^2$$

ekanligi kelib chiqadi.

Javob: $n(n+1)^2$.

14-masala. $\sqrt{\frac{6^8+8^8+10^8}{2}}$ ni hisoblang.

Yechish: 6,8,10 lar qanday sonlar? Ha, u *Pifagor* sonlari, ya'ni $10^2 = 8^2 + 6^2$ bo'ladi. Shundan g'oya olgan holda $8^2 = x,\ 6^2 = y$ desak, $10^2 = x + y$ bo'ladi.

$$\sqrt{\frac{6^8+8^8+10^8}{2}} = \sqrt{\frac{x^4+y^4+(x+y)^4}{2}} = \sqrt{\frac{x^4+y^4+x^4+4x^3y+6x^2y^2+4xy^3+y^4}{2}} =$$

$$= \sqrt{y^4 + x^4 + 2x^3y + 3x^2y^2 + 2xy^3} = x^2 + y^2 + xy =$$

$$= (x+y)^2 - xy = 10^4 - 48^2 = 10000 - 2304 = 7696$$

Javob: 7696

15-masala. Quyidagi ayniyatni isbotlang:

$$\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{n}{2^{n+1}} + \cdots = 1$$

Isbot: (1-usul) Belgilash kiritamiz: $S = \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{n}{2^{n+1}} + \cdots$

Tenglikning ikkala qismini 2 ga ko'paytirib, hosil bo'lgan tenglikdan dastlabki tenglikni hadlab ayiramiz:

$$2S - S = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots \Rightarrow S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

(2-usul) Quyidagi $f(x) = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots + \frac{1}{x^n} + \cdots$ funksiyaning hosilasini qaraylik.

$$f'(x) = -\left(\frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \cdots + \frac{n}{x^{n+1}} + \cdots\right)$$

$$f'(2) = -\left(\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{n}{2^{n+1}} + \cdots\right)$$

$$\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{n}{2^{n+1}} + \cdots = -f'(2)$$

$$f(x) = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots + \frac{1}{x^n} + \cdots = \frac{\frac{1}{x}}{1 - \frac{1}{x}} = \frac{1}{x-1} \Rightarrow f(x) = \frac{1}{x-1}$$

$$f'(x) = -\left(\frac{1}{x-1}\right)^2 \Rightarrow f'(2) = -\left(\frac{1}{2-1}\right)^2 = -1 \Rightarrow f'(2) = -1$$

$$\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{n}{2^{n+1}} + \cdots = -f'(2) = -(-1) = 1.$$

Faollashtiruvchi savollar.

1. *1 dan n gacha bo'lgan sonlar yig'indisi formulasini ayting?*
2. *1 dan n gacha bo'lgan sonlar kvadratlari yig'indisi formulasini ayting?*
3. *1 dan n gacha bo'lgan sonlar kublari yig'indisi formulasini ayting?*
4. *Kasr ifoda qachon natural son bo'ladi?*
5. *Arifmetik progressiyaning n – hadi formulasi qanday topiladi?*
6. *Arifmetik progressiyaning ayirmasi qanday topiladi?*
7. *Arifmetik progressiyaning dastlabki n ta hadi yig'indisini hisoblash formulasini ayting?*
8. *Arifmetik progressiya qanday holda o'suvchi bo'ladi?*
9. *Arifmetik progressiya qanday holda kamayuvchi bo'ladi?*
10. *Geometrik progressiyaning n – hadi formulasi qanday topiladi?*
11. *Geometrik progressiyaning maxraji qanday topiladi?*
12. *Geometrik progressiya qanday holda o'suvchi bo'ladi?*
13. *Geometrik progressiya qanday holda kamayuvchi bo'ladi?*
14. *Geometrik progressiyaning dastlabki n ta hadi yig'indisini hisoblash formulasini ayting?*
15. *Cheksiz kamayuvchi geometrik progressiyaning hadilari yig'indisini hisoblash formulasini ayting?*

Mustaqil yechish uchun masalalar

1. Agar $A = \underbrace{444...44}_{2n\ ta}$ va $B = \underbrace{888...88}_{n\ ta}$, $n \in N$ bo'lsa, $A + 2B + 4$ ifoda biror sonning kvadrati ekanligini isbotlang.
2. $\forall n \in N$ da $\dfrac{10^n + 2}{3} + \dfrac{10^{3n} + 8}{9}$ ifoda natural son ekanligini isbotlang.
3. Yig'indini hisoblang: $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + ... + \dfrac{1}{(2n-1)\cdot(2n+1)}$
4. Yig'indini hisoblang: $\dfrac{1}{5\cdot 11} + \dfrac{1}{11\cdot 17} + ... + \dfrac{1}{(6n-1)\cdot(6n+5)}$
5. Yig'indini hisoblang: $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{2\cdot 3\cdot 4} + \dfrac{1}{3\cdot 4\cdot 5} + ... + \dfrac{1}{n(n+1)(n+2)}$

6. Yig'indini hisoblang: $\frac{1}{2\cdot5\cdot8}+\frac{1}{5\cdot8\cdot11}+\frac{1}{8\cdot11\cdot14}+...+\frac{1}{(3n-1)(3n+2)(3n+5)}$

7. $A=\underbrace{444...44}_{2m\ ta}$, $B=\underbrace{222...22}_{m+1\ ta}$ va $C=\underbrace{888...88}_{m\ ta}$ bo'lsa, $A+B+C+7$ ifoda biror sonning kvadrati ekanligini isbotlang.

8. Hisoblang: $\sqrt{\underbrace{444...44}_{2012ta}-11\cdot\underbrace{444...44}_{1006ta}+9}$

9. Yig'indini hisoblang: $(a+b)+(a^2+ab+b^2)+...+(a^n+a^{n-1}b+...+ab^{n-1}+b^n)$

10. Yig'indini hisoblang: $1+6+27+108+...+(n+1)3^n$

11. Yig'indini hisoblang: $1+2\cdot2+3\cdot2^2+4\cdot2^3+...+n\cdot2^{n-1}$

12. Hisoblang: $(1+2+2^2)(1+2^3+2^6)(1+2^9+2^{18})(1+2^{27}+2^{54})$

13. Yig'indini hisoblang: $1\cdot1!+2\cdot2!+3\cdot3!+...+n\cdot n!$

14. Hisoblang: $\sqrt{\underbrace{444...44}_{2n\ ta}+\underbrace{111...11}_{n+1\ ta}-\underbrace{666...66}_{n\ ta}}$

15. Hisoblang: $\sqrt[3]{\frac{1}{3}\left(\underbrace{111...11}_{3n\ ta}-\underbrace{333...33}_{n\ ta}\underbrace{000...00}_{n\ ta}\right)}$

16. Hisoblang: $\left(\underbrace{666...66}_{n\ ta}\right)^2+\underbrace{888...88}_{n\ ta}$

17. Hisoblang: $\underbrace{111...11}_{n\ ta}\underbrace{555...55}_{n-1ta}6$

18. $\left(\underbrace{111...11}_{n\ ta}\right)\cdot\left(\underbrace{100...005}_{n+1\ ta}\right)+1$ son natural sonning kvadrati bo'lishini isbotlang.

19. $\underbrace{999...99}_{n-1\ ta}7\underbrace{00...00}_{n+1\ ta}2\underbrace{999...99}_{n\ ta}$ son natural sonning kubi bo'lishini isbotlang.

20. $55\underbrace{999...99}_{1992ta}89\underbrace{333...33}_{1993ta}9$ son 3 ta butun son kvadratlarining yig'indisi ekanligini isbotlang.

21. $\underbrace{444...44}_{n\ ta}\underbrace{888...88}_{n-1\ ta}9$ son natural son kvadrati ekanini isbotlang.

22. $\left(\underbrace{111...11}_{3n\ ta}\right)\cdot\frac{1}{3}-\underbrace{111...11}_{n\ ta}\cdot10^n$ son natural sonning kubi bo'lishini isbotlang.

23. Hisoblang: $\sqrt{\underbrace{444...44}_{2n\ ta}-\underbrace{888...88}_{n\ ta}}$

24. Hisoblang: $\sqrt{\underbrace{111...11}_{1997ta}\underbrace{222...22}_{1998ta}5}$

25. Yig'indini hisoblang: $7+77+777+...+\underbrace{777...77}_{n\ ta}$

26. Yig'indini hisoblang: $5+55+555+...+\underbrace{555...55}_{n\ ta}$

27. Yig'indini hisoblang: $1^3+2^3+3^3+...+n^3$

28. Yig'indini hisoblang: $1^4+2^4+3^4+...+n^4$

29. Yig'indini hisoblang: $1^5+2^5+3^5+...+n^5$

30. Hisoblang: $\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+...+\frac{n-1}{n!}$

31. Hisoblang: $1+2\cdot 2+3\cdot 2^2+4\cdot 2^3+...+2019\cdot 2^{2018}$

32. Hisoblang: $(1+q)(1+q^2)(1+q^4)...(1+q^{2^n})$

33. Hisoblang: $\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)...\left(1+\frac{1}{3^{16}}\right)\left(1+\frac{1}{3^{32}}\right)$

34. Hisoblang: $\left(3^{2^0}+1\right)\left(3^{2^1}+1\right)\left(3^{2^2}+1\right)...\left(3^{2^n}+1\right)$

35. Hisoblang: $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)...\left(1-\frac{1}{n^2}\right)$

36. Agar $|x|<1$ bo'lsa, u holda quyidagi yig'indini hisoblang:

$$1+2x+3x^2+4x^3+...$$

37. Agar $a_i,\ i=1,2,...,n$ arifmetik progressiyaning hadlari bo'lsa, u holda quyidagini hisoblang: $\frac{1}{a_1\cdot a_2}+\frac{1}{a_2\cdot a_3}+\frac{1}{a_3\cdot a_4}+...+\frac{1}{a_{n-1}\cdot a_n}$

38. Agar $b_i,\ i=1,2,...,n$ cheksiz kamayuvchi geometrik progressiyaning hadlari bo'lsa, u holda quyidagilarni hisoblang: $b_1+b_2+b_3+...$

39. $b_1^2+b_2^2+b_3^2+...$

40. $(b_1+b_2)^2+(b_3+b_4)^2+(b_5+b_6)^2+...$

41. $b_1+\frac{1}{2}b_2+\frac{1}{4}b_3+...$

42. $\left(b_1+\frac{1}{2}\right)+\left(b_2-\frac{1}{4}\right)+\left(b_3+\frac{1}{8}\right)+\left(b_4-\frac{1}{16}\right)+...$

43. $\frac{b_2}{b_1}+\frac{b_4}{b_2}+\frac{b_6}{b_3}+\frac{b_8}{b_4}+...$

44. Agar $a,b\in Q$ va $n\in N$ sonlari uchun $a^{2n+1}+b^{2n+1}=2a^nb^n$ bo'lsa, u holda $1-ab$ son biror ratsional sonning kvadrati ekanligini isbotlang.

7-§. Sonlarning butun va kasr qismlari

Tayanch so'zlar: *Sonning butun qismi, sonning kasr qismi, butun son, kvadrat, kub, kvadrat ildiz, kasr, yig'indi.*

Ta'rif. Biror a sonini olib, $x \le a$ tengsizlikni ko'raylik. Tengsizlikning yechimi $(-\infty; a]$ bo'lib, bu oraliqdagi eng katta butun sonni $[a]$ ko'rinishda belgilaymiz va uni a sonning *butun qismi* deb olamiz.

Masalan, $[7]=7;\ [1,2]=1;\ [0]=0,\ [0,3]=0,\ [-1]=-1,\ [-2,3]=-3$ ekanini tekshirish qiyin emas. Demak, aniqlanishiga ko'ra $[a]$ albatta, butun son. Berilgan a sonning *kasr qismi* esa, $a-[a]$ ifodaga teng bo'lib, $\{a\}$ ko'rinishda belgilanadi.

Masalan, $\{5,7\}=0,7;\ \{2\}=0;\ \{0\}=0;\ \{-0,2\}=0,8;\ \left\{-\frac{17}{5}\right\}=\frac{3}{5}$.

Ta'rifga ko'ra, $0 \le \{a\} < 1$ tengsizlik o'rinli, $[a]$ esa butun son. Quyidagi xossalar o'rinli:

a) $0 \le \{x\} < 1$.
b) $x=[x]+\{x\}$.
c) $[x]=n$ bo'lsa, $n \le x < n+1$ tengsizlik bajariladi.
d) $[x+1]=[x]+1$, umuman, ixtiyoriy n butun son uchun $[x+n]=[x]+n$.
e) $\{x+1\}=\{x\}$, ixtiyoriy butun uchun $\{x+n\}=\{x\}$.
f) $[\{x\}]=0;\ \{[x]\}=0;\ [[x]]=[x];\ \{\{x\}\}=\{x\}$.
g) $\{x\}=\{y\}$ bo'lsa, $x-y$ ayirma butun son bo'ladi.
h) Ixtiyoriy n butun son uchun $[x]>n \Rightarrow x \ge n+1,\ [x] \ge n \Rightarrow x \ge n$ munosabatlar o'rinli.
i) $\forall x \in R$ da $x \ge [x]$.

Kiritilgan $[x]$ va $\{x\}$ tushunchalar bilan yaqinroq tanishish uchun bir nechta misollar ko'ramiz.

1-masala. Tenglamani yeching: $\left[\frac{4-x}{5}\right]=6$

Yechish. c) xossaga asosan:

$$6 \le \frac{4-x}{5} < 7 \Rightarrow 30 \le 4-x < 35 \Rightarrow 26 \le -x < 31 \Rightarrow -31 < x \le -26.$$

Javob: $x \in (-31; -26]$.

2-masala. Tenglamani yeching: $\left[\frac{x-1}{2}\right]=x$

Yechish. Tenglikning chap tomoni butun ekanligidan x ning ham butunligi kelib chiqadi. Endi a) va b) xossalaridan foydalanamiz, ya'ni

$$\left[\frac{x-1}{2}\right]=\frac{x-1}{2}-\left\{\frac{x-1}{2}\right\} \text{ va } 0\le\left\{\frac{x-1}{2}\right\}<1.$$

$\frac{x-1}{2}-\left\{\frac{x-1}{2}\right\}=x\Rightarrow\left\{\frac{x-1}{2}\right\}=-\frac{x+1}{2}\Rightarrow 0\le-\frac{x+1}{2}<1$. Bu tengsizlikni yechib $-3<x\le-1$ ni hosil qilamiz. x butunligidan $x=-2$ va $x=-1$ yechimlarni olamiz.

Javob: $x=-2$ va $x=-1$.

3-masala. Tenglamani yeching: $\left\{3x-\frac{9}{4}\right\}=\frac{2}{3}$

Yechish. Yuqoridagi *e*) xossadan foydalanamiz:

$$3x-\frac{9}{4}=\frac{2}{3}+n\Rightarrow x=\frac{35}{36}+\frac{n}{3},\ n\in Z$$

Javob: $x=\frac{35}{36}+\frac{n}{3},\ n\in Z$.

4-masala. Ushbu tenglamalar sistemasini yeching:

$$\begin{cases}[x]=1,\\ [x^2]=2,\\ [x^3]=3,\\ [x^4]=4.\end{cases}$$

Yechish. Ta'rifdan foydalanib, sistemani ushbu ko'rinishda yozamiz:

$$\begin{cases}1\le x<2,\\ 2\le x^2<3,\\ 3\le x^3<4,\\ 4\le x^4<5.\end{cases}$$

1- tengsizlikka ko'ra 2- va 4- tengsizliklarning manfiy yechimlarini e'tiborga olmasak ham bo'ladi. Demak, $1\le x<2,\ \sqrt{2}\le x<\sqrt{3},\ \sqrt[3]{3}\le x<\sqrt[3]{4},\ \sqrt[4]{4}\le x<\sqrt[4]{5}$ bo'lib, tengsizliklar sistemasining yechimi, bu qo'sh tengsizliklarning chap tomonlarida joylashgan $1,\sqrt{2},\sqrt[3]{3},\sqrt[4]{4}$ sonlarning eng kattasi bo'lgan $\sqrt[3]{3}$ dan katta yoki teng va o'ng tomonida joylashgan $2, \sqrt{3}, \sqrt[3]{4}, \sqrt[4]{5}$ sonlarning eng kichigi bo'lgan $\sqrt[4]{5}$ dan kichik bo'lishi shart. Haqiqatan ham, $\sqrt[3]{3} < \sqrt[4]{5}$ bo'lib, tenglamalar sistemasining yechimi $\left[\sqrt[3]{3}, \sqrt[4]{5}\right)$ oraliqdan iborat.

Javob: $\left[\sqrt[3]{3}, \sqrt[4]{5}\right)$

5-masala. Ushbu $[x+[x]] = 2001$ tenglamani yeching.

Yechish. d) xossasidan foydalanamiz: $[x+[x]] = [x]+[x] = 2\cdot[x]$. Demak, tenglikning chap tomoni juft son bo'lib, 2001 ga teng bo'lishi mumkin emas. Tenglama yechimi yo'q.

Javob: Tenglama yechimga ega emas.

6-masala. $[x]+2\{x\} = 22$ tenglamani yeching.

Yechish. Tenglikdan x ning kasr qismini topib, $[0;1)$ oraliqqa qo'yamiz:

$$[x]+2\{x\} = 22 \Rightarrow [x]+\{x\}+\{x\} = 22 \Rightarrow x+\{x\} = 22 \Rightarrow 0 \le 22-x < 1 \Rightarrow 21 < x \le 22$$

tengsizlikni hosil qilamiz. Endi $2\{x\}$ butun ekanligidan $\{x\} = \frac{n}{2}, n \in Z$ kelib chiqadi. Bundan $x = 21\frac{1}{2}$ va $x = 22$ yechimlarni olamiz.

Javob: $x = 21\frac{1}{2}$ va $x = 22$.

7-masala. $x^2 - 10[x] + 9 = 0$ tenglamani yeching.

Yechish. $[x] \ge 0$ ekanligi tushunarli. Chunki x^2+9. Bundan esa x ning musbatligi kelib chiqadi. Musbat son har doim o'zining butun qismidan kichikmas, ya'ni $x \ge [x]$. Natijada $x^2 - 10x + 9 \le 0$ tengsizlikni hosil qilamiz.

Bundan $1 \le x \le 9$ kelib chiqadi, bundan $1 \le [x] \le 9$. x^2+9 son 10 ga bo'linuvchi butun sondir.

1) $1 \le x < 2 \Rightarrow x^2 + 9 = 10 \Rightarrow x^2 = 1 \Rightarrow \boxed{x = 1}$

2) $2 \le x < 3 \Rightarrow x^2 + 9 = 20 \Rightarrow x^2 = 11 \Rightarrow x = \sqrt{11}$

3) $3 \le x < 4 \Rightarrow x^2 + 9 = 30 \Rightarrow x^2 = 21 \Rightarrow x = \sqrt{21}$

4) $4 \le x < 5 \Rightarrow x^2 + 9 = 40 \Rightarrow x^2 = 31 \Rightarrow x = \sqrt{31}$

5) $5 \leq x < 6 \Rightarrow x^2 + 9 = 50 \Rightarrow x^2 = 41 \Rightarrow x = \sqrt{41}$

6) $6 \leq x < 7 \Rightarrow x^2 + 9 = 60 \Rightarrow x^2 = 51 \Rightarrow x = \sqrt{51}$

7) $7 \leq x < 8 \Rightarrow x^2 + 9 = 70 \Rightarrow x^2 = 61 \Rightarrow \boxed{x = \sqrt{61}}$

8) $8 \leq x < 9 \Rightarrow x^2 + 9 = 80 \Rightarrow x^2 = 71 \Rightarrow \boxed{x = \sqrt{71}}$

9) $x = 9 \Rightarrow x^2 + 9 = 90 \Rightarrow x^2 = 81 \Rightarrow \boxed{x = 9}$

Javob: $1; \sqrt{61}; \sqrt{71}; 9$.

8-masala. Agar $[x]\cdot\{x\} = 100$ bo'lsa, u holda $[x^2] - [x]^2$ ifoda qiymatini toping.

Yechish. d) va f) xossalaridan foydalanamiz:

$$[x^2] - [x]^2 = [([x] + \{x\})^2] - [x]^2 = [[x]^2 + 2[x]\{x\} + \{x\}^2] - [x]^2 == [x]^2 + [2[x]\{x\} + \{x\}^2] - [x]^2 =$$

$$= [200 + \{x\}^2] = 200 + [\{x\}^2] = 200$$

Javob: 200

9-masala. Tenglamani yeching: $[x+2] + [x+3] - [x+4] = 3$

Yechish. Yuqoridagi d) xossadan foydalanamiz.

$$[x] + 2 + [x] + 3 - [x] - 4 = 3 \Rightarrow [x] = 2 \Rightarrow 2 \leq x < 3$$

Javob: $2 \leq x < 3$.

10-masala.Tenglamani yeching: $\left\{2x - \frac{1}{2}\right\} = \frac{3}{4}$

Yechish. Yuqoridagi e) xossadan foydalanamiz.

$$\left\{2x - \frac{1}{2}\right\} = \frac{3}{4} \Rightarrow 2x - \frac{1}{2} = \frac{3}{4} + n \Rightarrow x = \frac{5}{8} + \frac{1}{2}n, n \in Z$$

Javob: $x = \frac{5}{8} + \frac{1}{2}n, n \in Z$.

11-masala. Quyidagi tenglamani haqiqiy sonlarda yeching:

$$\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right] = x.$$

Yechish: $x = 30k + r, 0 \le r \le 29$ ko'rinishda izlaymiz.

$$15k + \left[\frac{r}{2}\right] + 10k + \left[\frac{r}{3}\right] + 6k + \left[\frac{r}{5}\right] = 30k + r \Rightarrow k = r - \left[\frac{r}{2}\right] - \left[\frac{r}{3}\right] - \left[\frac{r}{5}\right]$$

$$\begin{cases} r = 0 \Rightarrow k = 0 \Rightarrow x = 0 \\ r = 1 \Rightarrow k = 1 \Rightarrow x = 31 \\ r = 2 \Rightarrow k = 1 \Rightarrow x = 32 \\ r = 3 \Rightarrow k = 1 \Rightarrow x = 33 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ r = 29 \Rightarrow k = 1 \Rightarrow x = 59 \end{cases}$$

Javob: $x = \{0, 31, 32, 33,34,35,6,37,38,39,10,41,12,43,44,15,46,47,18\}$

$x = \{49,20,51,52,53,24,55,56,57,58,59\}$.

Faollashtiruvchi savollar.

1. *Sonning butun qismi deb nimaga aytiladi?*
2. *Sonning kasr qismi deb nimaga aytiladi?*
3. *e ning butun qismi nechchiga teng?*
4. *π ning butun qismi nechchiga teng?*
5. *Sonning kasr qismi qanday qiymatlar qabul qiladi?*
6. *Natural sonlarning kasr qismi nechchiga teng?*

Mustaqil yechish uchun masalalar

1. Tenglamani yeching: $\left[\frac{3x-1}{4}\right] = 5$.
2. Tenglamani yeching: $\left[\frac{2x-1}{3}\right] = 2x$.
3. Tengsizlikni yeching. $x^2 - 4[x] + 3 \ge 0$.
4. Tenglamani yeching: $[x] \cdot \{x\} = 3$.
5. Tenglamani yeching: $\left[\frac{x^2 - 3x}{2}\right] = 1$.
6. Tenglamani yeching: $[x]^2 = [x^2]$.
7. Tenglamani yeching: $[x + [x]] = 2017$.
8. Tenglamani yeching: $x \cdot \{x\} = 1$.
9. Tenglamani yeching: $[x^2 - 5x + 6] = 1$.
10. Tenglamani yeching: $x^2 - 5[x] - 3 = 0$.
11. Tenglamani yeching: $[x+1] + [x-2] + [x+3] = 2$.

12. Tenglamani yeching: $4^{[x]}-6\cdot 2^{[x]}+8=0$.

13. Tenglamani yeching: $\{x\}^2-\frac{3}{4}\{x\}+\frac{1}{8}=0$.

14. Tenglamani yeching: $[x]^2-2\cdot[x]-8=0$.

15. Tenglamani yeching: $\frac{1}{[x]-1}-\frac{1}{[x]+1}=\frac{1}{12}$.

16. Tenglamani yeching: $\left\{x+\frac{1}{3}\right\}=\frac{1}{2}$.

17. Tenglamani yeching: $\left\{2x-\frac{2}{5}\right\}=\frac{3}{4}$.

18. Tenglamani yeching: $36^{\{x\}}=6$.

19. Tenglamani yeching: $18\cdot[x]+87\cdot\{x\}=1887$.

20. Tenglamani yeching: $x+[x]+\{x\}=0$.

21. Tenglamani yeching: $[2x+4]=-5$.

22. Tenglamani yeching: $\left[\frac{3}{4}x-1\right]=15$.

23. Tenglamani yeching: $[3x-1]=-4$.

24. Tenglamani yeching: $\left[\frac{3x-1}{3}\right]=5$.

25. Tenglamani yeching: $\left[\frac{3x+1}{2}\right]=-x$.

26. Tenglamani yeching: $[3x+1]=\frac{x}{4}$.

27. Tenglamani yeching: $\{3x\}=\frac{1}{2}$.

28. Tenglamani yeching: $\{6x\}+\{x\}=1$.

29. Tenglamani yeching: $2\cdot[x]=[3x]$.

30. Tenglamani yeching: $[x]+2001\cdot\{3x\}=0$.

31. Tenglamani yeching: $6\cdot[x]=47\cdot\{x\}$.

32. Tenglamani yeching: $3\cdot[x-7]=5\cdot\{3-x\}$.

33. Tenglamani yeching: $[x^2]=9$.

34. Tenglamani yeching: $[x^2]=15$.

35. Tenglamani yeching: $[x]+3\cdot\{x\}=8$.

36. Tenglamani yeching: $\left[\frac{2x+1}{3}\right]=[x]$.

37. Tenglamani yeching: $[2x]+[3x]=3$.

38. Tenglamani yeching: $x=[x]\cdot\{x\}$.

39. Tenglamani yeching: $\left[\frac{x^2-3x+3}{2}\right]+1=x$.

40. Tenglamani yeching: $2^{[x]}=1+2x$.

41. Tenglamani yeching: $[\sin x+\cos x]=1$.

42. Tenglamani yeching: $[x-3]+[x-5]=8$.

43. Tenglamani yeching: $2+3\cdot[x]=4x$.

44. Tenglamani yeching: $[2x+1]=x+2$.

45. Tenglamani yeching: $4-2\cdot[x]=3x-1$.

46. Tenglamani yeching: $\left[\frac{x-3}{2}\right]=\left[\frac{x-2}{3}\right]$.

47. Tenglamani yeching: $[\sin x]=[\cos x]$.

48. Tenglamani yeching: $[2x]+[5x]=15$.

49. Tenglamani yeching: $x^3-[x]=3$.

50. Tengsizlikni yeching: $[x]\le 3\cdot\{x\}<\frac{1}{3}$.

51. Tengsizlikni yeching: $[x]\le\{x\}$.

52. Tengsizlikni yeching: $\frac{4[x]+2}{2[x]-3}>2$.

53. Tengsizlikni yeching: $\left[\frac{2x+3}{1-x}\right]\le\pi$.

54. Tengsizlikni yeching: $[x]^2-7[x]+6\ge 0$.

55. Tengsizlikni yeching: $[3x-2]<4$.

56. Tengsizlikni yeching: $\{x\}^2-\frac{5}{6}\{x\}+\frac{1}{6}<0$.

57. Tenglamani yeching: $\frac{x-1}{2}=\left[\frac{x+1}{3}\right]$.

58. Tenglamani natural yechimlari nechta? $\left[\frac{x}{2017}\right]=\left[\frac{x}{2018}\right]+1$.

59. Tenglamani yeching: $[x]+[x^2]=[x^3]$.

60. Tenglamani yeching: $19[x]-96\{x\}=0$.

61. Musbat x,y,z sonlarning yig’indisi 11 ga teng bo’lsa, u holda $[x]^4+[y]^4+[z]^4\ge 243$ tengsizlikni isbotlang.

8-§. Butun sonlarda yechiladigan tenglamalar

Tayanch so'zlar: *Diofant tenglamalari, butun son, natural son, tub son, kvadrat tenglama, mod, qoldiq, o'zaro tub sonlar, EKUB, juft son, toq son, diskriminant.*

Qadim yunon faylasufi va matematigi Diofant aniqmas tenglamalar bilan shug'ullangan. Keyinchalik bu tenglamalar Diofant tenglamalari deb atala boshlandi. Odatda Diofant tenglamalarining yechimi bir nechta, cheksiz ko'p yoki umuman bo'lmasligi mumkin. Quyida bir nechta Diofant tenglamalarini yechish usullarini tahlil etamiz.

Ta'rif. Butun sonlarda yechiladigan tenglamalar Diofant tenglamalar deyiladi.

Ko'p noma'lumli chiziqli Diofant tenglamalar

1-masala. Tenglamani butun sonlarda yeching. $5x+28y=59$

Yechish. $5x=59-28y \Rightarrow x=\frac{60-1-25y-3y}{5}=\frac{60-25y}{5}-\frac{3y+1}{5}$

$$x=12-5y-\frac{3y+1}{5} \Rightarrow 3y+1=5k, k\in Z \Rightarrow 3y=5k-1 \Rightarrow y=k+\frac{2k-1}{3} \Rightarrow$$

$$\Rightarrow 2k-1=3n, n\in Z \Rightarrow 2k=3n+1 \Rightarrow k=n+\frac{n+1}{2} \Rightarrow n+1=2m, m\in Z \Rightarrow$$

$$\Rightarrow n=2m-1 \Rightarrow k=3m-1 \Rightarrow y=5m-2 \Rightarrow x=23-28m.$$

Javob: $(x,y)\in(23-28m;5m-2),\ m\in Z$.

2-masala. Tenglamani butun sonlarda yeching. $36x-22y=12$

Yechish. $36x-22y=12 \Rightarrow y=\frac{18x-6}{11}=x+\frac{7x-6}{11} \Rightarrow 7x-6=11k, k\in Z \Rightarrow$

$$x=k+\frac{4k+6}{7} \Rightarrow 4k+6=7n, n\in Z \Rightarrow k=n+\frac{3n-6}{4} \Rightarrow 3n-6=4m, m\in Z \Rightarrow$$

$$\Rightarrow n=m+2+\frac{m}{3} \Rightarrow m=3l, l\in Z \Rightarrow n=4l+2 \Rightarrow k=7l+2 \Rightarrow x=11l+4,\ \ y=18l+6.$$

Javob: $(x,y)\in(11l+4;18l+6),\ l\in Z$

Ko'paytuvchilarga ajratish usuli

3-masala. Tenglamani butun sonlarda yeching. $y^2-xy-2x^2-13=0$

Yechish. $y^2 - xy - 2x^2 = 13 \Rightarrow (x+y)(y-2x) = 13$. Bu tenglikdan $x+y$ va $y-2x$ ifodalar 13 ning butun bo'luvchilariga tengligi kelib chiqadi. Bundan quyidagi tenglamalar sistemasi kelib chiqadi:

$$\begin{cases} x+y=1 \\ y-2x=13 \end{cases} \quad \begin{cases} x+y=13 \\ y-2x=1 \end{cases} \quad \begin{cases} x+y=-1 \\ y-2x=-13 \end{cases} \quad \begin{cases} x+y=-13 \\ y-2x=-1 \end{cases}.$$

Bu tenglamalar sistemasini yechib, quyidagi yechimlarni hosil qilamiz:

$$(x;y) \in (-4;5),(4;9),(4;-5),(-4;-9).$$

Javob: $(x;y) \in (-4;5),(4;9),(4;-5),(-4;-9)$

4-masala. Tenglamani butun sonlarda yeching. $y^2 - x^2 = 3$

Yechish. $y^2 - x^2 = 3 \Rightarrow (y-x)(y+x) = 3$. Bu tenglikdan $y-x$ va $y+x$ ifodalar 3 ning butun bo'luvchilariga tengligi kelib chiqadi. Bundan quyidagi tenglamalar sistemasi kelib chiqadi:

$$\begin{cases} y-x=1 \\ y+x=3 \end{cases} \quad \begin{cases} y-x=3 \\ y+x=1 \end{cases} \quad \begin{cases} y-x=-1 \\ y+x=-3 \end{cases} \quad \begin{cases} y-x=-3 \\ y+x=-1 \end{cases}.$$

Bu tenglamalar sistemasini yechib, quyidagi yechimlarni hosil qilamiz:

$$(x;y) \in (1;2),(-1;2),(-1;-2),(1;-2).$$

Javob: $(x;y) \in (1;2),(-1;2),(-1;-2),(1;-2)$

Noma'lumlardan biri chiziqli bo'lgan Diofant tenglamalar

5-masala. Tenglamani butun sonlarda yeching.

$$x^3 - x^2 - xy - 17x - 3y + 8 = 0$$

Yechish. $xy + 3y = x^3 - x^2 - 17x + 8 \Rightarrow (x+3)y = x^3 - x^2 - 17x + 8 \Rightarrow$

$$\Rightarrow y = \frac{x^3 - x^2 - 17x + 8}{x+3} = \frac{(x+3)(x^2 - 4x - 5) + 23}{x+3} \Rightarrow y = x^2 - 4x - 5 + \frac{23}{x+3} \Rightarrow$$

$$\Rightarrow 23 \vdots x+3 \Rightarrow x+3 = \pm 1;\ \pm 23$$

$$x+3=1 \Rightarrow x=-2, \quad y=30$$

$$x+3=-1 \Rightarrow x=-4, \quad y=4$$

$$x+3=23 \Rightarrow x=20, \quad y=316$$

$$x+3=-23 \Rightarrow x=-26, \quad y=774.$$

Javob: $(x, y) \in (-2;30),(-4;4),(20;316),(-26;774)$

6-masala. Tenglamani butun sonlarda yeching. $x^2-xy-y-4=0$

Yechish. $xy+y=x^2-4 \Rightarrow (x+1)y=(x-1)(x+1)-3 \Rightarrow y=x-1-\dfrac{3}{x+1} \Rightarrow$

$$\Rightarrow 3 \vdots x+1 \Rightarrow x+1=\pm 1; \pm 3$$

$$x+1=1 \Rightarrow x=0, \ y=-4$$

$$x+1=-1 \Rightarrow x=-2, \quad y=0$$

$$x+1=3 \Rightarrow x=2, \quad y=0$$

$$x+1=-3 \Rightarrow x=-4, \quad y=-4.$$

Javob: $(x, y) \in (0;-4),(-2;0),(2;0),(-4;-4)$

Bo'linish belgilaridan foydalanib yechiladigan Diofant tenglamalar

7-masala. Tenglamani butun sonlarda yechimi yo'qligini isbotlang.

$$x^2-y^2=1982$$

Isbot. Tenglamaning har ikkala tomonini 4 soni bilan baholaymiz. O'ng tomoni juft bo'lganligidan chap tomoni ham juftligi kelib chiqadi. Bundan esa *x* va *y* larning har ikkalasi bir vaqtda juft yoki toq bo'lganida bajariladi. U holda $x-y$ va $x+y$ sonlar juft va ko'paytmasi 4 ga karrali ekanligi kelib chiqadi. Tenglikning chap tomoni 4 ga karrali o'ng tomoni 4 ga bo'lganda 2 qoldiq qoladi. Bundan esa tenglama natural sonlarda yechimi yo'qligi kelib chiqadi.

8-masala. Tenglamani qanoatlantiruvchi *x* va *y* butun sonlar mavjud emasligi isbotlang.

$$15x^2=9+7y^2$$

Isbot. $3(5x^2-3)=7y^2 \Rightarrow y^2 \vdots 3 \Rightarrow y \vdots 3 \Rightarrow y=3k, k \in Z$

$$5x^2-3=21k^2 \Rightarrow 3(7k^2+1)=5x^2 \Rightarrow x^2 \vdots 3 \Rightarrow x \vdots 3 \Rightarrow x=3n, n \in Z$$

$$7k^2+1=15n^2 \Rightarrow 7k^2+1 \vdots 3 \Rightarrow 7k^2+1 \equiv 0 (\text{mod}\, 3).$$

$$k^2 \equiv 0,1(\mathrm{mod}\,3) \Rightarrow 7k^2 \equiv 0,1(\mathrm{mod}\,3) \Rightarrow 7k^2 + 1 \equiv 1,2(\mathrm{mod}\,3).$$

Oxirgi munosabat tenglamani butun sonlarda yechimi yo'qligini ko'rsatadi.

9-masala. Tenglamaning butun sonlarda yechimi yo'qligini isbotlang.

$$(x^2 + x + 1)^2 + (y^2 - y + 1)^2 = 2018^2$$

Isbot. $x^2 + x = x(x+1) \vdots 2 \Rightarrow x(x+1) = 2k \Rightarrow x^2 + x + 1 = 2k + 1,\ k \in Z$. Xuddi shunday

$$y^2 - y = y(y-1) \vdots 2 \Rightarrow y(y-1) = 2n \Rightarrow y^2 - y + 1 = 2n + 1,\ n \in Z$$

$$(x^2 + x + 1)^2 = (2k+1)^2 = 4A + 1, \quad (y^2 - y + 1)^2 = (2n+1)^2 = 4B + 1.$$

Bu tengliklarni hadma-had qo'shib, quyidagi tenglikni hosil qilamiz:

$$(x^2 + x + 1)^2 + (y^2 - y + 1)^2 = 4A + 4B + 2 = 4C + 2.$$

Tenglamaning chap qismi 4 ga bo'lganda 2 qoldiq, o'ng qismi esa 4 ga karrali. Ziddiyat! Bundan esa tenglamaning butun sonlarda yechimi yo'qligi kelib chiqadi.

Tub sonlarda yechiladigan tenglamalar

10-masala. Tenglamani tub sonlarda yeching. $p + q = (p - q)^3$

Yechish. $p + q > 0 \Rightarrow p - q > 0 \Rightarrow p > q \Rightarrow p = q + k,\ k \in N$

$$2q + k = k^3 \Rightarrow k^3 - k = 2q \Rightarrow (k-1)k(k+1) = 2q$$

$$(k-1)k(k+1) \vdots 6 \Rightarrow 2q \vdots 6 \Rightarrow q \vdots 3 \Rightarrow q = 3 \Rightarrow k = 2 \Rightarrow p = 5.$$

Javob: $p = 5;\ q = 3$.

11-masala. Tenglamani tub sonlarda yeching. $q^3 = p^2 - p + 1$

Yechish. $q^3 - 1 = p^2 - p \Rightarrow (q-1)(q^2 + q + 1) = p(p-1)$

$$q < p \Rightarrow q - 1 < p \Rightarrow q - 1 \not\vdots\, p \Rightarrow q^2 + q + 1 \vdots p \Rightarrow q^2 + q + 1 = pk,\ k \in N$$

$$p - 1 = (q-1)k \Rightarrow p = k(q-1) + 1 \Rightarrow q^2 + q + 1 = k^2(q-1) + k \Rightarrow$$

$$\Rightarrow q^2 - (k^2 - 1)q + k^2 - k + 1 = 0 \Rightarrow D = (k^2 - 1)^2 - 4(k^2 - k + 1) = A^2$$

1-hol.

$$k>3\Rightarrow (k^2-3)^2<(k^2-1)^2-4(k^2-k+1)=A^2<(k^2-2)^2\Rightarrow k^2-3<A<k^2-2\Rightarrow\varnothing$$

2-hol. $k=1\Rightarrow p=q\Rightarrow\varnothing$

3-hol. $k=2\Rightarrow q^2+q+1\,\vdots\,2$ bundan esa q^2+q toqligi kelib chiqadi. Bu esa $q^2+q=q(q+1)\,\vdots\,2$ ekanligiga zid!

4-hol. $k=3\Rightarrow p=3q-2\Rightarrow (q-1)(q^2+q+1)=(3q-2)(3q-3)\Rightarrow$

$$\Rightarrow q^2+q+1=9q-6\Rightarrow q^2-8q+7=0\Rightarrow q=7,\ p=19.$$

Javob: $p=19,\ q=7$

Endi har xil turdagi butun sonlarda yechiladigan tenglamalarni yechish usullarini ko'rib chiqamiz.

12-masala. Tenglamani qanoatlantiruvchi barcha x va y butun sonlarni toping.

$$\frac{x+y}{x^2-xy+y^2}=\frac{3}{7}$$

Yechish. Quyidagicha belgilash qilamiz: $a=x+y, b=xy$. x va y butunligidan a va b larning ham butunligi kelib chiqadi.

$$x^2-xy+y^2\geq 0\Rightarrow x+y\geq 0\Rightarrow a\geq 0.$$

$a=0$ bo'lmaydi.

$$\frac{a}{a^2-3b}=\frac{3}{7}\Rightarrow 7a=3a^2-9b\Rightarrow 9b=3a^2-7a\Rightarrow 3a^2-7a\,\vdots\,9\Rightarrow 3a^2-7a\,\vdots\,3\Rightarrow$$

$$\Rightarrow 7a\,\vdots\,3\Rightarrow a\,\vdots\,3\Rightarrow a=3k,\ k\in Z$$

$$3\cdot 9k^2-21k\,\vdots\,9\Rightarrow 27k^2-21k\,\vdots\,9\Rightarrow 9k^2-7k\,\vdots\,3\Rightarrow 7k\,\vdots\,3\Rightarrow$$

$$\Rightarrow k\,\vdots\,3\Rightarrow k=3l, l\in Z\Rightarrow a=9l\ (a>0\Rightarrow l>0)$$

$(t-x)(t-y)=0$ tenlamaning yechimi bor. Bundan $t^2-at+b=0$ tenglamaning yechimi borligi kelib chiqadi. Demak, $D=a^2-4b>0\Rightarrow a^2>4b$.

$$7a=3a^2-9b\Rightarrow 7\cdot 9l=3\cdot 81l^2-9b\Rightarrow b=27l^2-7l\Rightarrow 4b=108l^2-28l$$

$$a^2>4b\Rightarrow 81l^2>108l^2-28l\Rightarrow 28l>27l^2\Rightarrow l=1$$

Bundan esa $a=9,\ b=20$ ekanligi kelib chiqadi. Belgilashlardan $(x,y)=(4;5),(5;4)$.

Javob: $(x,y)=(4;5),(5;4)$

13-masala. Tenglamani butun sonlarda yeching. $x^3+xy+y^3=13$

Yechish. $x,y\in Z^+$ da yechim yo'q. Demak, x va y turli xil ishorali butun sonlar. Berilgan tenglama simmetrik bo'lgani uchun y ni $-y$ bilan almashtirib, tenglamani natural sonlarda yechamiz.

$$x^3-xy-y^3=13\Rightarrow (x-y)(x^2+xy+y^2)-xy=13\Rightarrow (x-y)((x-y)^2+3xy)-xy=13,$$

$$\begin{cases} x-y=a,\ a>0 \\ xy=b,\ b>0 \end{cases}$$

belgilash kiritamiz. U holda tenglama $a(a^2+3b)-b=13$ ko'rinishga keladi.

$$a(a^2+3b)-b=13\Rightarrow a^3+3ab-b=13\Rightarrow (3a-1)b=13-a^3$$

$(3a-1)b>0\Rightarrow 13-a^3>0\Rightarrow a<\sqrt[3]{13}\Rightarrow a=1,\ a=2$

1-hol. $a=1\Rightarrow b=6\Rightarrow \begin{cases} x-y=1 \\ xy=6 \end{cases}\Rightarrow \begin{cases} x=3 \\ y=2 \end{cases}, \begin{cases} x=-2 \\ y=-3 \end{cases}$

Topilgan yechimni y ni $-y$ bilan almashtirib quyidagi yechimni olamiz:

$$\begin{cases} x=3 \\ y=-2 \end{cases} \text{ yoki } \begin{cases} x=-2 \\ y=3 \end{cases}$$

2-hol. $a=2\Rightarrow b=1\Rightarrow \begin{cases} x-y=2 \\ xy=1 \end{cases}\Rightarrow x,y\notin Z$

Javob: $(x,y)\in\{(3;-2);(-2;3)\}$

14-masala. Tenglamani butun sonlarda yeching. $x+y=xy$

Yechish. Tenglamani ko'rinishini quyidagicha yozamiz:

$$x+y=xy\Rightarrow xy-x-y+1=1\Rightarrow (x-1)(y-1)=1$$

$\begin{cases} x-1=1 \\ y-1=1 \end{cases}$ yoki $\begin{cases} x-1=-1 \\ y-1=-1 \end{cases}$ bu tenglamalar sistemani yechib $(0;0),(2;2)$ yechimlarni olamiz.

Javob: $(x, y) \in \{(0;0);(2;2)\}$

15-masala. Tenglamani natural sonlarda yeching.

$$y^2 - x(x+1)(x+2)(x+3) = 1$$

Yechish. Tenglamaning o'ng qismini quyidagicha yozamiz:

$$x(x+1)(x+2)(x+3)+1 = (x(x+3))((x+1)(x+2))+1 =$$

$$= ((x^2+3x+1)-1)((x^2+3x+1)+1)+1 = (x^2+3x+1)^2$$

Bundan esa tenglamamiz quyidagi ko'rinishga keladi:

$$y^2 = (x^2+3x+1)^2$$

yoki

$$y = x^2+3x+1$$

Bundan esa tenglamaning barcha yechimlari $\{(x, x^2+3x+1) \mid x \in N\}$ ekanligi kelib chiqadi.

Javob: $\{(x, x^2+3x+1) \mid x \in N\}$

16-masala.Tenglamani butun sonlarda yeching. $x^2 + y^2 + z^2 = 2xyz$

Yechish. $x = y = z = 0$ yechimi ekanligi ko'rinib turibdi. Tenglamaning boshqa butun yechimi yo'qligini ko'rsatamiz. Teskarisini faraz qilaylik, ya'ni tenglamaning noldan farqli boshqa yechimi bo'lsin. U holda $x^2 + y^2 + z^2$ – juft son bo'lib, tenglamamiz simmetrikligidan ularda albatta bittasi juft ekanligi kelib chiqadi. Faraz qilaylik x juft ya'ni $x = 2x_1$ bo'lsin. U holda $y^2 + z^2 \vdots 4$ ekanligi kelib chiqadi. Faqat quyidagi holl y va z juft bo'lishi mumkin. Agar ulardan bittasi toq ikkinchisi juft bo'lsa, $y^2 + z^2$ son 4 ga karrali emasligi, agar ikkalasi ham toq son bo'lsa, ya'ni $y = 2u+1,\ z = 2v+1$. U holda

$$y^2 + z^2 = (2u+1)^2 + (2v+1)^2 = 4(u^2+v^2+u+v) + 2 \equiv 2(\mathrm{mod}\, 4).$$

Demak, $x = 2x_1,\ y = 2y_1,\ z = 2z_1$ ekan. Bu tengliklarni tenglamaga olib borib qo'yamiz:

$$x_1^2 + y_1^2 + z_1^2 = 2x_1 y_1 z_1$$

Xuddi shunday yuqoridagi mulohazani davom ettirasak, $x \vdots 2^2,\ y \vdots 2^2,\ z \vdots 2^2$ ekanligi kelib chiqadi va hokazo. Ko'rinib turibdiki, ixtiyoriy natural n uchun $x \vdots 2^n,\ y \vdots 2^n,\ z \vdots 2^n$ ni yozish mumkin. Bundan esa x, y va z sonlari 2 ning ixtiyoriy darajasiga bo'linadigan butun sonlar ekan. Bunday son esa yo'q. Ziddiyat! Demak, tenglama yagona $(0;0;0)$ yechimga ega ekan.

Javob: $(x;y;z)=(0;0;0)$

17-masala. Tenglamaning natural sonlarda yechimga ega emasligini isbotlang.

$$x(x+1)=4y(y+1)$$

Isbot. Tenglamani quyidagi ko'rinishda yozib olamiz:

$$x^2+x+1=(2y+1)^2$$

$\forall x \in N,\ x^2 < x^2+x+1 < x^2+2x+1 \Rightarrow x^2 < (2y+1)^2 < (x+1)^2$. Bundan $x < 2y+1 < x+1$ ekanligi kelib chiqadi. Hech qachon ketma-ket kelgan natural son orasida boshqa bir natural son yotmaydi. Ziddiyat! Demak, berilgan tenglama natural sonlarda yechimga ega emas.

18-masala. Tenglamaning natural sonlarda yechimga ega emasligini isbotlang.

$$\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}=1$$

Isbot. Faraz qilaylik $(x,y)=d,\ x_1=\frac{x}{d},\ y_1=\frac{y}{d},\ (x_1,y_1)=1$ bo'lsin.

$$x^2+xy+y^2=x^2y^2$$

Bundan,

$$x_1^2+x_1y_1+y_1^2=d^2x_1^2y_1^2$$

tenglikni hosil qilamiz. Oxirgi tenglikdan $y_1 \vdots x_1,\ x_1 \vdots y_1$ ekanligi kelib chiqadi.

$(x_1;y_1)=1$ bo'lgani uchun $x_1=y_1=1$. Bundan esa tenglama $d^2=3$ ko'rinishga keladi. d natural son bo'lgani uchun oxirgi tenglama yechimga ega emasligi kelib chiqadi.

19- masala. Tenglamani butun sonlarda yeching. $(xy-7)^2=x^2+y^2$

Yechish. $(xy-7)^2=x^2+y^2 \Rightarrow (xy-7)^2+2xy=(x+y)^2$. Belgilash kiritamiz: $x+y=a,\ xy=b$

$$(b-7)^2+2b=a^2 \Rightarrow b^2-14b+47+2b=a^2 \Rightarrow a^2-(b-6)^2=13 \Rightarrow (a-b+6)(a+b-6)=13$$

13 tub son bo'lgani uchun quyidagi hollar bo'lishi mumkin:

1-hol. $\begin{cases} a-b+6=13 \\ a+b-6=1 \end{cases} \Rightarrow \begin{cases} a=7 \\ b=0 \end{cases} \Rightarrow \begin{cases} x+y=7 \\ xy=0 \end{cases} \Rightarrow (x,y)\in(0;7),(7;0)$

2-hol. $\begin{cases} a-b+6=-13 \\ a+b-6=-1 \end{cases} \Rightarrow \begin{cases} a=-7 \\ b=12 \end{cases} \Rightarrow \begin{cases} x+y=-7 \\ xy=12 \end{cases} \Rightarrow (x,y)\in(-3;-4),(-4;-3)$

3-hol. $\begin{cases} a-b+6=1 \\ a+b-6=13 \end{cases} \Rightarrow \begin{cases} a=7 \\ b=12 \end{cases} \Rightarrow \begin{cases} x+y=7 \\ xy=12 \end{cases} \Rightarrow (x,y)\in(4;3),(3;4)$

4-hol. $\begin{cases} a-b+6=-1 \\ a+b-6=-13 \end{cases} \Rightarrow \begin{cases} a=-7 \\ b=0 \end{cases} \Rightarrow \begin{cases} x+y=-7 \\ xy=0 \end{cases} \Rightarrow (x,y)\in(-7;0),(0;-7)$

Javob: $(x,y)\in(7;0),(0;7),(-3;-4),(-4;-3),(4;3),(3;4),(-7;0),(0;-7)$

20- masala. Tenglamani yeching. $(x^2-2018^2)^2=8072x+1$

Yechish. Belgilash kiritamiz: $y=2018$. U holda tenglamamiz

$(x^2-y^2)^2=4xy+1$ ko'rinishga keladi. Bu tenglikni soddalashtiramiz.

$$x^4-2x^2y^2+y^4=4xy+1 \Rightarrow x^4+2x^2y^2+y^4=4x^2y^2+4xy+1 \Rightarrow$$

$$\Rightarrow (x^2+y^2)^2-(2xy+1)^2=0 \Rightarrow ((x+y)^2+1)((x-y)^2-1)=0$$

Oxirgi tenglikdan $x=y\pm1$ ekanligi kelib chiqadi. Bundan esa $x=2017,\ x=2019$ yechimlarni olamiz.

Javob: $x=2017,\ x=2019$

21-masala. Natural sonlarda yeching: $n^4+11n^2+4=m^2$

Yechish: tenglikning chap qismini baholaymiz:

$$n^4+11n^2+4=(n^2+2)^2+7n^2>(n^2+2)^2 \Rightarrow\ m>\ n^2+2\ \Rightarrow m \geq\ n^2+3$$

$$m^2 = n^4 + 11n^2 + 4 < n^4 + 12n^2 + 36 = (n^2+6)^2 \Rightarrow m < n^2 + 6 \Rightarrow m \leq n^2 + 5$$

Demak, $m = n^2 + 3$ yoki $m = n^2 + 4$ yoki $m = n^2 + 5$

1-hol: $m = n^2 + 3 \Rightarrow n^4 + 11n^2 + 4 = (n^2+3)^2 \Rightarrow n = 1, m = 4$

2-hol: $m = n^2 + 4 \Rightarrow n^4 + 11n^2 + 4 = (n^2+4)^2 \Rightarrow n = 2, m = 8$

3-hol: $m = n^2 + 5 \Rightarrow n^4 + 11n^2 + 4 = (n^2+5)^2 \Rightarrow n^2 = 21 \Rightarrow \emptyset$

Javob: $n = 1, m = 4$ va $n = 2, m = 8$

22-masala. Quyidagi tenglamalarni butun sonlarda yeching.

$$x^y + y^x = 2019$$

Yechish. 1-hol: $x < 0\,, y < 0$ bo'lsin.

$$0 < |x^y| \leq 1\,, 0 < |y^x| \leq 1 \Rightarrow x^y + y^x < 2 \Rightarrow x^y + y^x \neq 2019$$

2-hol: $x > 0\,, y < 0$ bo'lsin.

$$0 < |x^y| \leq 1\,, |y^x| > 1 \in Z$$

$x^y + y^x \in Z$ bo'lishi uchun $|x^y| = 1$ bo'lishi kerak.

2.1-hol: $x^y \Rightarrow x = 1 \Rightarrow 1^y + y^1 = 2019 \Rightarrow y = 2018 > 0\ \emptyset$

2.2-hol: $x^y = -1 \Rightarrow x = -1,\ y$ – toq, $(-1)^y + y^{-1} = 2019 \Rightarrow \frac{1}{y} = 2020 \Rightarrow$

$$\Rightarrow y = \frac{1}{2020} \notin Z\ \ \emptyset$$

3-hol: $x < 0\,, y > 0\ bo'lsin.$

x,*y* simmetrik bo'lgani uchun 2-holdagi kabi bu holda ham yechim yo'q.

4-hol: $x = 0$ bo'lsin.

$$0^y + y^0 = 1 \Rightarrow x \neq 0\,, y \neq 0$$

5-hol: $x = 1$

$$1^y + y^1 = 2019 \Rightarrow y = 2018$$

x va *y* simmetrik bo'lgani uchun $x = 2018, y = 1$ yechimni ham olamiz.

6-hol: $x > 1\,, y > 1$

x va y bir vaqtda juft yoki toq bo'lsa , $x^y + y^x$ juft bo'ladi.

$$x = 2n\,, y = 2k+1 \;\; k, n \in Z\,, \qquad k, n > 0$$

$$(2n)^{2k+1} + (2k+1)^{2n} = 4m + (4k^2+4k+1)^n = 4A+1$$

$x^y + y^x = 4A+1,\;\; 2019 = 4\text{B} + 3 \Rightarrow$ ziddiyat!

x va y lar simmetrik bo'lgani uchun x - toq, y - juft holida ham yechim yo'q.

Javob: $(x, y) \in \{(1; 2018), (2018; 1)\}$.

23-masala. Tenglamani butun sonlarda yeching: $1 + x + x^2 + x^3 = 2^y$

Yechish: Tenglikning o'ng tomonini ko'paytuvchilarga ajratamiz.

$$(1+x)(1+x^2) = 2^y \Rightarrow \begin{cases} 1+x = 2^\alpha \\ 1+x^2 = 2^\beta \end{cases} \Rightarrow \alpha + \beta = y$$

$$x = 2^\beta - 1 \Rightarrow 1 + \left(2^\beta - 1\right)^2 = 2^\alpha \Rightarrow 2^{2\beta} - 2^{\beta+1} + 2 = 2^\alpha \quad |:2 \;\Rightarrow$$

$$\Rightarrow 2^{2\beta-1} - 2^\beta + 1 = 2^{\alpha-1},\;\; \alpha, \beta \geq 0$$

1-hol. $x < 0 \Rightarrow \emptyset$

2-hol. $x = 0 \Rightarrow 1 = 2^y \Rightarrow y = 0$

3-hol. $x = 1 \Rightarrow y = 2$

4-hol. $x \geq 2 \Rightarrow \alpha, \beta \geq 2 \Rightarrow \underbrace{2^{2\beta-1} - 2^\beta + 1}_{toq} = \underbrace{2^{\alpha-1}}_{juft}$ ziddiyat!

Javob: $(x, y) \in \{(0; 0), (1; 2)\}$

24-masala. Tenglamani butun sonlarda yeching: $2^x + 7 = y^2$

Yechish: $x < 0$ bo'lsin. Bundan $2^x \in Q \Rightarrow y \notin Z$. $x = 1$ bo'lsa, $y = \pm 3$ bo'ladi.

$x \geq 2$ bo'lsin. Bundan esa y ning toqligi kelib chiqadi. $y = 2k+1, k \in N$

$$2^x + 7 = (2k+1)^2 \Rightarrow 2^x + 7 = 4k^2 + 4k + 1 \Rightarrow 4k(k+1) - 2^x = 6 \;\; |:2 \Rightarrow$$

$\Rightarrow 2k(k+1) - 2^{x-1} = 3$. Bu tenglikning chap qismi juft, o'ng qismi toq. Ziddiyat!

Javob: $(x, y) \in \{(1; 3), (1; -3)\}$

25-masala. Tenglamani butun sonlarda yeching: $xy^3 = 2019(y + 1)$

Yechish. $\underbrace{xy^3}_{\vdots y} = \underbrace{2019y}_{\vdots y} + \underbrace{2019}_{\vdots y} \Rightarrow 2019 \vdots y \Rightarrow \underbrace{xy^3}_{\vdots y^2} = \underbrace{2019y}_{\vdots y^2} + \underbrace{2019}_{\vdots y^2} \Rightarrow$

$\Rightarrow 2019 \vdots y^2 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1\,,\ y = -1 \Rightarrow x = 0\,,\ y = 1 \Rightarrow x = 4038$

Javob: $(x, y) \in \{(0; -1), (4038; 1)\}$.

26-masala. Tenglamani butun sonlarda yeching: $x^2 - y^3 = y^2$

Yechish. $x^2 = y^3 + y^2 \ \Rightarrow x^2 = y^2(y + 1)\,, y + 1 \geq 0 \ \Rightarrow y \geq -1$

$$x = |y|\sqrt{y + 1} \ \Rightarrow y + 1 = m^2\,, m \in Z \ \Rightarrow \ y = m^2 - 1, x = |m^2 - 1|m$$

Javob: $(x, y) \in \{(m|m^2 - 1|, m^2 - 1), m \in Z\}$.

27-masala. Tenglamani butun sonlarda cheksiz ko'p yechimi borligini isbotlang.

$$x^3 + y^3 + z^3 = x^2 + y^2 + z^2$$

Yechish. Quyidagicha almashtirish kiritamiz: $z = -y$

$$x^3 + y^3 - y^3 = x^2 + y^2 + y^2 \Rightarrow x^3 = x^2 + 2y^2 \Rightarrow x^3 - x^2 = 2y^2 \Rightarrow$$

$$\Rightarrow y \vdots x \Rightarrow y = mx, m \in Z$$

$$x^3 - x^2 = 2m^2x^2 \Rightarrow x - 1 = 2m^2 \Rightarrow x = 2m^2 + 1, m \in Z \ \Rightarrow$$

$$\Rightarrow y = m(2m^2 + 1), z = -m(2m^2 + 1)$$

Javob: $(x, y, z) \in \left\{\left(\ 2m^2 + 1, m(2m^2 + 1), -m(2m^2 + 1)\right), m \in Z\right\}$

28-masala. $m, n \ \in Z_+$ bo'lsa $x^2 + y^2 = (m^2 + n^2)^z$ tenglamaning butun musbat yechimi cheksiz ko'pligini isbotlang.

Yechish. Quyidagi hollarni ko'rib chiqamiz:

<u>1-hol:</u> $z = 2k + 1,\ k \in Z_+$ bo'lsin.

$$x^2 + y^2 = (m^2 + n^2)(m^2 + n^2)^{2k} = (m(m^2 + n^2)^k)^2 + (n(m^2 + n^2)^k)^2 \Rightarrow$$

$$\Rightarrow x_k = m(m^2 + n^2)^k, \qquad y_k = n(m^2 + n^2)^k, z_k = 2k + 1, k \in Z_+$$

<u>2-hol:</u> $z = 2k, k \in Z_+$

$$x^2 + y^2 = (m^2 + n^2)^{2k} = (m^2 + n^2)^2 \cdot (m^2 + n^2)^{2k-2} =$$

$$= ((m^2 - n^2)^2 + 4m^2n^2)(m^2 + n^2)^{2k-2} =$$

$$= [(m^2 - n^2)(m^2 + n^2)^{k-1}]^2 + [2mn(m^2 + n^2)^{k-1}]^2 \Rightarrow$$

$$\Rightarrow x_k = |m^2 - n^2|(m^2 + n^2)^{k-1}, y_k = 2mn(m^2 + n^2)^{k-1}, z_k = 2k, k \in Z^+$$

29-masala. $n \geq 2$ da $x^n + y^n = z^{n+1}$ tenglama cheksiz ko'p butun musbat yechimga ega emasligini isbotlang.

Yechish. $x^n + y^n = z \cdot z^n \Rightarrow z = \left(\frac{x}{z}\right)^n + \left(\frac{y}{z}\right)^n \Rightarrow \begin{cases} x = zm \\ y = zk \end{cases}, \ m, z \in Z^+ \Rightarrow$

$$\Rightarrow m^n + k^n = z \Rightarrow \begin{cases} x_n = m(m^n + k^n) \\ y_n = k(m^n + k^n) \\ z_n = m^n + k^n \end{cases}$$

30-masala. Tenglamani tub sonlarda yeching: $x^y + 1 = z$

Yechish. Quyidagi hollarni qaraymiz:

<u>1-hol:</u> $z = 2 \Rightarrow x = 1, \ y$ – tub $\Rightarrow \emptyset$

<u>2-hol:</u> $z \geq 2$ tub son bo'lsin. Bundan z – toq son, x – juft $\Rightarrow x = 2$.

$$2^y + 1 = z$$

a) $y = 2 \Rightarrow z = 5 \Rightarrow (x, y, z) \in (2,2,5)$

b) $y > 2$ tub son bo'lsin. Bundan y – toq son $y = 2k + 1, k \in N$

$2^{2k+1} + 1 = z \Rightarrow (2 + 1)(2^{2k} + \cdots + 1) = z \Rightarrow z \vdots 3, z \vdots 2^{2k} + \cdots + 1$ bu esa z – sonining tub ekanligiga zid!

31-masala. Quyidagi tenglikni qanoatlantiruvchi barcha butun x va tub p sonlarni toping.

$$x^8 + 2^{2^x + 2} = p$$

Yechish. $x = 1$ bo'lsa, $p = 17$

$x \geq 2$ bo'lsin. $2^x = 4k, k \in N$.

$$x^8 + 4 \cdot 2^{2^x} = p \Rightarrow x^8 + 4x^4 \cdot 2^{2k} + 4 \cdot 2^{4k} - 4x^4 \cdot 2^{2k} = p \Rightarrow$$

$$\Rightarrow (x^4 + 2 \cdot 2^{2k})^2 - (2x^2 \cdot 2^k)^2 = p \Rightarrow$$

$$\Rightarrow (x^4 - 2x^2 \cdot 2^k + 2 \cdot 2^k)(x^4 + 2x^2 \cdot 2^k + 2 \cdot 2^k) = p$$

Oxirgi tenglik p tub son ekanligiga zid!

Javob: $x = 1, p = 17$.

32-masala. Tenglamani natural sonlarda yeching: $x^3 - y^3 = xy + 61$

Yechish. $x > y$ ekanligi ma'lum.

$$(x - y)(x^2 + y^2 + xy) = xy + 61$$

$$(x - y)((x - y)^2 + 3xy) = xy + 61$$

$x - y = a, xy = b \Rightarrow a, b \in N$ belgilash kiritamiz.

$$a(a^2 + 3b) = b + 61 \Rightarrow (3a - 1)b = 61 - a^3 \Rightarrow b = \frac{61 - a^3}{3a - 1}$$

b natural sonligidan $61 - a^3 > 0 \Rightarrow a = \{1,2,3\}$ ekanligi kelib chiqadi.
1-hol: $a = 1 \Rightarrow b = 30 \Rightarrow x = 6, y = 5$

2-hol: $a = 2 \Rightarrow b \notin N$.

3-hol: $a = 3 \Rightarrow b \notin N$.

Javob: $x = 6, y = 5$.

Faollashtiruvchi savollar.

1. *Diofant tenglamalar deb nimaga aytiladi?*
2. *Tub sonlarning bo'luvchilari nechta?*
3. *Diofant tenglamalarni yechishning qanday usullari bor?*

Mustaqil yechish uchun masalalar

1. Tenglamani butun sonlarda yeching. $4x - 3y = 19$
2. Tenglamani butun sonlarda yeching. $4x + 5y = 17$
3. Tenglamani butun sonlarda yeching. $x^2 + xy - 5 = 0$
4. Tenglamani butun sonlarda yeching. $x^2 - 2xy - 3y^2 + 11 = 0$
5. Tenglamani butun sonlarda yeching. $x^2 - xy + 2x + 2 = 0$
6. Tenglamani butun sonlarda yeching. $xy - 2x^2 + 5x - 2y - 3 = 0$
7. Tenglamani butun sonlarda yechimga ega emasligini isbotlang. $x^2 + 1 = 3y$
8. Tenglamani butun sonlarda yechimga ega emasligini isbotlang.

$$x^2+4x=8y+11$$

9. Tenglamani tub sonlarda yeching. $p^2-2q^2=1$
10. Tenglamaning noldan farqli butun yechimi yo'qligini isbotlang.

$$x^3+2y^3+4z^3-6xyz=0$$

11. Tenglamani butun sonlarda yeching. $x^2=y^2+2y+13$
12. Tenglamani butun sonlarda yeching. $x^3+y^3=1984$
13. Tenglamani butun sonlarda yeching. $x^2-y^2=1987$
14. Tenglamani butun sonlarda yeching. $y^3-x^3=91$
15. Tenglamani butun sonlarda yeching. $x^8+x^7+x+1=0$
16. Tenglamani butun sonlarda yeching. $2x^3+xy-7=0$
17. Tenglamani natural sonlarda yechimga ega emasligini isbotlang. $y^2=5x^2+6$
18. Tenglamani natural sonlarda yechimga ega emasligini isbotlang. $x^3=2+3y^2$
19. Tenglamani natural sonlarda yechimga ega emasligini isbotlang.

$$x^4+4y^4=2(z^4+4t^4)$$

20. Tenglamani butun sonlarda yeching. $3x^2+4xy-7y^2=13$
21. Tenglamani butun sonlarda yeching. $127x-52y+1=0$
22. Tenglamani butun sonlarda yeching. $24x-17y=2$
23. Tenglamani butun sonlarda yeching. $y^2-xy-2x^2=13$
24. m va n o'zaro tub natural sonlar bo'lsin. $mx+ny=mn$ tenglamani qanoatlantiruvchi x va y natural sonlar mavjud emasligini isbotlang.
25. Tenglamani butun sonlarda yeching. $x^2-z^3=y^2$
26. Tenglamani tub sonlarda yeching. $p^6-q^2=0{,}5(p-q)^2$
27. Tenglamani tub sonlarda yeching. $p^3-q^5=(p+q)^2$
28. Tenglamani natural sonlarda yeching. $x^2+y^2+xy=\overline{aaa}^2$
29. Tenglamani butun sonlarda yeching. $2^x+1=y^2$
30. Tenglamani butun sonlarda yeching. $2^n-m^3=m^2-16$
31. Tenglamani butun sonlarda yeching. $x^2-xy+y-x=12$
32. Tenglamani butun sonlarda yeching. $x^2+5xy-y^2=6$
33. Tenglamani natural sonlarda yeching. $x^2+xy+y^2=x+y+9$
34. Tenglamani natural sonlarda yechimi yo'qligini isbotlang. $y^3=x^2+x$

35. Tenglamani natural sonlarda yechimi yo’qligini isbotlang. $x^2+xy+y^2=x^2y^2$

36. Tenglamani natural sonlarda yeching. $x^2-xy+2x-3y=11$

37. Tenglamani natural sonlarda yeching. $xy^2-xy-y^2+y=94$

38. Tenglamani natural sonlarda yeching. $x!+12=y^2$

39. Tenglamani natural sonlarda yechimi yo’qligini isbotlang.

$$(x+1)^2+(x+2)^2+...+(x+2001)^2=y^2$$

40. Tenglamani butun sonlarda yechimi yo’qligini isbotlang. $x^5-y^2=4$

41. Tenglamani butun sonlarda yeching. $x^2-2y^2+8z=3$

42. Tenglamani butun sonlarda yeching. $x^2-y^3=7$

43. Tenglamani butun sonlarda yeching. $x^3+x+10y=2014$

44. Berilgan x va y raqamlar uchun $(x+y)(x^x+y^y)=1981$ shart bajarilsa, bu raqamlarni toping.

45. Tenglamani butun sonlarda yeching. $6x^2+5y^2=7y$

46. Tenglamani butun sonlarda yeching. $3x^2+5xy-2y^2=24$

47. Tenglamani naturalsonlarda yechimi nechta. $2x+3y=100$

48. x ning barcha butun qiymatlarini topingki, bunda $y=\dfrac{3x^2+x+12}{x^2+5}$ ham butun son bo’lsin.

49. Tenglamani natural sonlarda yeching. $x+y+z=xyz$

50. Tenglamani natural sonlarda yeching. $x^2-y^2=2019$

51. Tenglamalar sistemasini yeching: $\begin{cases} x^2=13x+4y \\ y^2=4x+13y \end{cases}$

52. Tenglamalar sistemasini yeching: $\begin{cases} xyz=1 \\ xyt=2 \\ xtz=4 \\ yzt=27 \end{cases}$

53. Tenglamalar sistemasini yeching: $\begin{cases} x+y+z=2 \\ 2xy-z^2=4 \end{cases}$

54. Tenglamani yeching. $x^{19}+x^{95}=2x^{19+95}$

55. Tenglamalar sistemasini butun sonlarda yeching: $\begin{cases} xy+z=94 \\ x+yz=95 \end{cases}$

II BOB. TENGSIZLIKLAR

1-§. Klassik tengsizliklar

Tayanch so'zlar: *Teorema, lemma, Koshi tengsizligi, Koshi-Bunyakovskiy-Shvars tengsizligi, Gyugens tengsizligi, Bernulli tengsizligi, o'rta arifmetik, o'rta geometrik, o'rta kvadratik, o'rta garmonik, musbat sonlar, induksiya.*

O'rtacha qiymatlar.

$a=\{a_1,a_2,...,a_n\}$ musbat sonlar ketma-ketligi uchun

o'rta arifmetik qiymat $A(a)=A_n=\dfrac{a_1+a_2+...+a_n}{n}$,

o'rta geometrik qiymat $G(a)=G_n=\sqrt[n]{a_1a_2...a_n}$,

o'rta kvadratik qiymat $K(a)=K_n=\sqrt{\dfrac{a_1^2+a_2^2+...+a_n^2}{n}}$ va

o'rta garmonik qiymat $N(a)=N_n=\dfrac{n}{a_1^{-1}+a_2^{-1}+...+a_n^{-1}}$ larni aniqlaymiz.

Xususan x va y musbat sonlari uchun bu o'rta qiymatlar quyidagicha aniqlanadi:

$$A_2=\frac{x+y}{2};\ G_2=\sqrt{xy};\ K_2=\sqrt{\frac{x^2+y^2}{2}};\ N_2=\frac{2xy}{x+y}$$

O'rta arifmetik va o'rta geometrik qiymatlar haqida Koshi tengsizligi va uning turli isbotlari

1-teorema. $A_n\ge G_n$ va $A_n=G_n$ tenglik faqat va faqat $a_1=a_2=...=a_n$ tenglik o'rinli bo'lganida bajariladi.

1-isboti. $A_n\ge G_n$ ekanligini matematik induksiya metodidan foydalanib isbotlaymiz: $n=2$ da $\sqrt{a_1\cdot a_2}\le\dfrac{a_1+a_2}{2}\Rightarrow\left(\sqrt{a_1}-\sqrt{a_2}\right)^2\ge 0$. Berilgan tengsizlikni ixtiyoriy n ta natural sonlar uchun to'g'ri deb, $n+1$ ta natural sonlar uchun to'g'riligini isbotlaymiz. Bu sonlar $a_1,a_2,...,a_n,a_{n+1}$ bo'lib, a_{n+1} ularning orasida eng kattasi bo'lsin. Ya'ni, $a_{n+1}\ge a_1,\ a_{n+1}\ge a_2,...,a_{n+1}\ge a_n$. Shuning uchun $a_{n+1}\ge\dfrac{a_1+a_2+...+a_n}{n}$. Quyidagicha belgilash kiritamiz:

$$A_n=\frac{a_1+a_2+...+a_n}{n},\ A_{n+1}=\frac{a_1+a_2+...+a_n+a_{n+1}}{n+1}=\frac{n\cdot A_n+a_{n+1}}{n+1}$$

$a_{n+1} \ge A_n$ bo'lgani uchun $a_{n+1} = A_n + \alpha,\ \alpha \ge 0$ deb yozish mumkin. U holda

$$A_{n+1} = \frac{n \cdot A_n + a_{n+1}}{n+1} = A_n + \frac{\alpha}{n+1}.$$

Bu tenglikni ikkala qismini $n+1$ – darajaga ko'tarib, quyidagini topamiz:

$$(A_{n+1})^{n+1} = \left(A_n + \frac{\alpha}{n+1}\right)^{n+1} = (A_n)^{n+1} + (n+1)(A_n)^n \frac{\alpha}{n+1} + \ldots + \left(\frac{\alpha}{n+1}\right)^{n+1} \ge$$

$$\ge (A_n)^{n+1} + (A_n)^n \cdot \alpha = (A_n)^n \cdot (A_n + \alpha) = (A_n)^n \cdot a_{n+1}.$$

Farazga ko'ra, $(A_n)^n \ge a_1 a_2 \ldots a_n$. Buni e'tiborga olib, $(A_n)^{n+1} \ge (A_n)^n \cdot a_{n+1} \ge a_1 a_2 \ldots a_n a_{n+1}$.

Bundan $A_{n+1} \ge \sqrt[n+1]{a_1 a_2 \ldots a_n a_{n+1}}$. Tenglik $a_1 = a_2 = \ldots = a_n$ bo'lganda o'rinli bo'ladi.

2-isboti. Bu isbot ham matematik induksiya metodidan foydalaniladi.

$n = 2$ da $\sqrt{a_1 \cdot a_2} \le \frac{a_1 + a_2}{2} \Rightarrow \left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \ge 0$. Berilgan tengsizlikni ixtiyoriy n ta natural sonlar uchun $\frac{a_1 + a_2 + \ldots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \ldots a_n}$ to'g'ri deb, $n+1$ ta natural sonlar uchun $\frac{a_1 + a_2 + \ldots + a_n + a_{n+1}}{n+1} \ge \sqrt[n+1]{a_1 a_2 \ldots a_n a_{n+1}}$ to'g'riligini isbotlaymiz.

$$\frac{a_1 + a_2 + \ldots + a_n + a_{n+1}}{n+1} = \frac{n \cdot \frac{a_1 + a_2 + \ldots + a_n}{n} + a_{n+1}}{n+1} \ge \frac{n \cdot \sqrt[n]{a_1 a_2 \ldots a_n} + a_{n+1}}{n+1}.$$

Quyidagicha $a_1 a_2 a_3 \ldots a_n = x^{n(n+1)}, a_{n+1} = y^{n+1}$ belgilash kiritib, isbotlash talab etilgan tengsizlikning o'ng qismini chap tomonga o'tkazib ayirma nomanfiy ekanligini ko'rsatamiz:

$$\frac{a_1 + a_2 + \ldots + a_n + a_{n+1}}{n+1} - \sqrt[n+1]{a_1 a_2 \ldots a_n a_{n+1}} \ge \frac{n \cdot \sqrt[n]{a_1 a_2 \ldots a_n} + a_{n+1}}{n+1} - \sqrt[n+1]{a_1 a_2 \ldots a_n a_{n+1}} =$$

$$= \frac{nx^{n+1} + y^{n+1}}{n+1} - x^n y = \frac{nx^{n+1} + y^{n+1} - nx^n y - x^n y}{n+1} = \frac{1}{n+1}\left[nx^n(x-y) - y(x^n - y^n)\right] =$$

$$= \frac{x-y}{n+1}\left[nx^n - yx^{n-1} - y^2 x^{n-2} - \ldots - y^n\right] = \frac{x-y}{n+1}\left[x^n - yx^{n-1} + x^n - y^2 x^{n-2} + \ldots + x^n - y^n\right] =$$

$$= \frac{(x-y)^2}{n+1}\left[x^{n-1} + x^{n-2}(x+y) + x^{n-3}(x^2 + xy + y^2) + \ldots + (x^{n-1} + x^{n-2}y + \ldots + y^{n-1})\right] \ge 0$$

Bundan $A_{n+1} \ge G_{n+1}$ ekanligi kelib chiqadi. Teorema isbotlandi.

3-isboti. Teoremaning isboti quyidagi lemmaga asoslangan:

1-lemma. Agar $x_1, x_2, ..., x_n$ – ixtiyoriy musbat sonlar bo'lib, $x_1 x_2 ... x_n = 1$ bo'lsa, u holda

$$x_1 + x_2 + ... + x_n \geq n$$

tengsizlik o'rinli bo'ladi.

Isbot. Tengsizlikni matematik induksiya metodidan foydalanib isbotlaymiz. $n = 1$ da tengsizlikning to'g'riligi ravshan, $n = k$ da to'g'ri deb faraz qilamiz. $n = k + 1$ da ham tengsizlik o'rinli bo'lishini ko'rsatamiz, ya'ni

$x_1 x_2 ... x_k x_{k+1} = 1$ bo'lsa, $x_1 + x_2 + ... + x_k + x_{k+1} \geq k + 1$ ekanligini isbotlaymiz.

a) $x_1, x_2, ..., x_k, x_{k+1}$ hammasi 1 ga teng bo'lsin. U holda $x_1 + x_2 + ... + x_k + x_{k+1} = k + 1$ bo'ladi.

b) $x_1, x_2, ..., x_k, x_{k+1}$ – sonlarning ichida shunday ikkitasi borki, ulardan bittasi birdan qat'iy katta, bittasi birdan qat'iy kichik, chunki $x_1 x_2 ... x_k x_{k+1} = 1$. Umumiylikka ziyon yetkazmasdan $x_k < 1,\ x_{k+1} > 1$ bo'lsin. U holda

$$(x_{k+1} - 1)(1 - x_k) > 0 \Rightarrow 1 + x_k \cdot x_{k+1} < x_k + x_{k+1}.$$

$x_1 \cdot x_2 \cdot ... \cdot (x_k \cdot x_{k+1}) = 1$ bo'lgani uchun farazimizga ko'ra

$$x_1 + x_2 + ... + x_{k-1} + (x_k + x_{k+1}) \geq k$$

Shuning uchun

$$x_1 + x_2 + ... + x_{k-1} + (x_k + x_{k+1}) \geq x_1 + x_2 + ... + x_{k-1} + 1 + x_k \cdot x_{k+1} =$$

$$= (x_1 + x_2 + ... + x_{k-1} + x_k \cdot x_{k+1}) + 1 \geq k + 1$$

Demak, $x_1 + x_2 + ... + x_k + x_{k+1} \geq k + 1$. Tenglik sharti faqat $x_1 = x_2 = ... = x_k = x_{k+1}$ bo'lganda bajariladi.

Endi teoremaning isbotiga o'tamiz. Agar $a_1, a_2, ..., a_n$ – sonlaridan birortasi nolga teng bo'lsa, $\sqrt[n]{a_1 a_2 ... a_n} \leq \frac{a_1 + a_2 + ... + a_n}{n}$ tengsizlikning chap tomoni nolga aylanib, to'g'ri tengsizlik hosil bo'ladi. Shuning uchun $a_1 > 0,\ a_2 > 0, ...,\ a_n > 0$ deb hisoblashimiz mumkin. Belgilash kiritamiz:

$$x_1 = \frac{a_1}{\sqrt[n]{a_1 a_2 ... a_n}},\ x_2 = \frac{a_2}{\sqrt[n]{a_1 a_2 ... a_n}}, ..., x_n = \frac{a_n}{\sqrt[n]{a_1 a_2 ... a_n}}.$$

Ravshanki, $x_1, x_2, ..., x_n$ – sonlar musbat va $x_1 \cdot x_2 \cdot ... \cdot x_n = 1$. U holda isbotlangan lemmaga ko'ra $x_1 + x_2 + ... + x_n \geq n$ bo'ladi.

Demak, $\frac{a_1}{\sqrt[n]{a_1 a_2 ... a_n}} + \frac{a_2}{\sqrt[n]{a_1 a_2 ... a_n}} + ... + \frac{a_n}{\sqrt[n]{a_1 a_2 ... a_n}} \geq n$, ya'ni

$$\sqrt[n]{a_1 a_2 ... a_n} \leq \frac{a_1 + a_2 + ... + a_n}{n}$$

bo'ladi. Teorema isbotlandi.

4-isboti. Teoremaning isboti quyidagi lemmaga asoslangan.

2-lemma. $x \geq 1$ da $e^{x-1} \geq x$ tengsizlik o'rinli. Tenglik faqat $x = 1$ da bajariladi.

Isbot. Quyidagi $f(x) = e^{x-1} - x$ funksiyani qaraymiz. Bu funksiya $x \geq 1$ da o'suvchi. Haqiqatdan ham

$$f'(x) = e^{x-1} - 1 \geq 0 \Rightarrow e^{x-1} \geq e^0 \Rightarrow x \geq 1$$

Bundan

$$x \geq 1 \Rightarrow f(x) \geq f(1) \Rightarrow e^{x-1} - x \geq 0 \Rightarrow e^{x-1} \geq x$$

ekanligi kelib chiqadi. Lemma isbotlandi.

Endi tengsizlikni isbotiga o'tamiz:

$$1 = e^0 = \exp\left(\sum_{i=1}^{n} \frac{a_i}{A(a)} - 1\right) = \prod_{i=1}^{n} \exp\left(\frac{a_i}{A(a)} - 1\right) \geq \prod_{i=1}^{n} \frac{a_i}{A(a)} = \left(\frac{G(a)}{A(a)}\right)^n$$

Demak, $A(a) \geq G(a)$ va tenglik esa faqat $\frac{a_i}{A(a)} = 1,\ i = 1,2,...,n$ bo'lganda bajariladi. Bundan esa $a_1 = a_2 = ... = a_n = A(a)$ ekanligi kelib chiqadi.

Horijiy adabiyotlarda Koshi tengsizligi uchun "AM-GM tengsizligi" termini qo'llaniladi.

2-teorema. $G_n \geq N_n$ va $G_n = N_n$ tenglik faqat va faqat $a_1 = a_2 = ... = a_n$ tenglik o'rinli bo'lganida bajariladi.

Isbot. Bizga ma'lumki, kasrning maxrajini kichraytirsak kasrning qiymati kattalashadi. Yuqoridagi 1-teoremadan foydalanamiz:

$$N_n = \frac{n}{a_1^{-1} + a_2^{-1} + ... + a_n^{-1}} \le \frac{n}{n\sqrt[n]{a_1^{-1}a_2^{-1}...a_n^{-1}}} = \sqrt[n]{a_1a_2...a_n} = G_n$$

teorema isbotlandi.

3-teorema. $K_n \ge A_n$ va $K_n = A_n$ tenglik faqat va faqat $a_1 = a_2 = ... = a_n$ tenglik o'rinli bo'lganida bajariladi.

Isbot. $K_n \ge A_n$ ekanligini matematik induksiya metodidan foydalanib isbotlaymiz: $n = 2$ da

$$\sqrt{\frac{a_1^2 + a_2^2}{2}} = \sqrt{\frac{2a_1^2 + 2a_2^2}{4}} = \sqrt{\frac{a_1^2 + a_2^2 + a_1^2 + a_2^2}{4}} \ge \sqrt{\frac{a_1^2 + a_2^2 + 2a_1a_2}{4}} = \sqrt{\frac{(a_1 + a_2)^2}{4}} = \frac{a_1 + a_2}{2}.$$

$n = k$ da $\sqrt{\frac{a_1^2 + a_2^2 + ... + a_k^2}{k}} \ge \frac{a_1 + a_2 + ... + a_k}{k}$ o'rinli deb, $n = k + 1$ da

$$\sqrt{\frac{a_1^2 + a_2^2 + ... + a_k^2 + a_{k+1}^2}{k+1}} \ge \frac{a_1 + a_2 + ... + a_k + a_{k+1}}{k+1}$$

ekanligini isbotlaymiz.

$$K_n \ge A_n \Rightarrow K_k^2 \ge A_k^2 \Rightarrow (a_1 + a_2 + ... + a_k)^2 \le k \cdot \left(a_1^2 + a_2^2 + ... + a_k^2\right)$$

$$A_{k+1}^2 = \left(\frac{a_1 + a_2 + ... + a_k + a_{k+1}}{k+1}\right)^2 = \frac{(a_1 + a_2 + ... + a_k)^2 + 2a_{k+1}(a_1 + a_2 + ... + a_k) + a_{k+1}^2}{(k+1)^2} \le$$

$$\le \frac{k \cdot \left(a_1^2 + a_2^2 + ... + a_k^2\right) + 2a_{k+1}a_1 + 2a_{k+1}a_2 + ... + 2a_{k+1}a_k + a_{k+1}^2}{(k+1)^2} \le$$

$$\le \frac{k \cdot \left(a_1^2 + a_2^2 + ... + a_k^2\right) + a_1^2 + a_2^2 + ... + a_k^2 + (k+1)a_{k+1}^2}{(k+1)^2} =$$

$$= \frac{(k+1)\left(a_1^2 + a_2^2 + ... + a_k^2 + a_{k+1}^2\right)}{(k+1)^2} = \frac{a_1^2 + a_2^2 + ... + a_k^2 + a_{k+1}^2}{(k+1)} = K_{k+1}^2$$

Demak, $K_n \ge A_n$ bo'lar ekan.

4-teorema.(Bernulli tengsizligi) Agar $x_1, x_2, ..., x_n > -1$ va $x_1, x_2, ..., x_n$ sonlarning hammasi bir xil ishorali bo'lsa,

$$\left(1+x_1\right)\left(1+x_2\right)\ldots\left(1+x_n\right)\geq 1+x_1+x_2+\ldots+x_n$$

bo'ladi.

Isbot. Matematik induksiya metodidan foydalanib isbotlaymiz.

$n=1$ da $1+x_1\geq 1+x_1$ o'rinli.

$n=k$ da $\left(1+x_1\right)\left(1+x_2\right)\ldots\left(1+x_k\right)\geq 1+x_1+x_2+\ldots+x_k$ o'rinli deb,

$n=k+1$ da $\left(1+x_1\right)\left(1+x_2\right)\ldots\left(1+x_k\right)\left(1+x_{k+1}\right)\geq 1+x_1+x_2+\ldots+x_k+x_{k+1}$

ekanligini isbotlaymiz.

$$\left(1+x_1\right)\left(1+x_2\right)\ldots\left(1+x_k\right)\left(1+x_{k+1}\right)\geq\left(1+x_1+x_2+\ldots+x_k\right)\left(1+x_{k+1}\right)=$$

$$=1+x_1+x_2+\ldots+x_k+x_{k+1}\cdot\left(1+x_1+x_2+\ldots+x_k\right)=$$

$$=1+x_1+x_2+\ldots+x_k+x_{k+1}+\underbrace{x_1\cdot x_{k+1}}_{\geq 0}+\underbrace{x_2\cdot x_{k+1}}_{\geq 0}+\ldots+\underbrace{x_k\cdot x_{k+1}}_{\geq 0}\geq 1+x_1+x_2+\ldots+x_k+x_{k+1}.$$

Demak, $\left(1+x_1\right)\left(1+x_2\right)\ldots\left(1+x_n\right)\geq 1+x_1+x_2+\ldots+x_n$ bo'lar ekan.

1-natija. Agar $x_1=x_2=\ldots=x_n=x$ bo'lsa, u holda Bernulli tengsizligi quyidagi ko'rinishga keladi:

$$\left(1+x\right)^n\geq 1+nx$$

5-teorema. (Koshi-Bunyakovskiy-Shvars tengsizligi) Faraz qilaylik $\left(a_1,a_2,\ldots,a_n\right)$ va $\left(b_1,b_2,\ldots,b_n\right)$ – haqiqiy sonlarning istalgan ketma-ketliklari bo'lsin. Quyidagi tengsizlik o'rinli

$$\left(a_1^2+a_2^2+\ldots+a_n^2\right)\left(b_1^2+b_2^2+\ldots+b_n^2\right)\geq\left(a_1b_1+a_2b_2+\ldots+a_nb_n\right)^2.$$

Tenglik $\left(a_1,a_2,\ldots,a_n\right)$ va $\left(b_1,b_2,\ldots,b_n\right)$ o'zaro proporsional bo'lganda bajariladi (ya'ni, shunday haqiqiy k soni topilib, barcha $i\in\{1,2,\ldots,n\}$ lar uchun $a_i=kb_i$ tenglik bajarilsa).

1-isboti. Quyidagi funksiyani qaraylik

$$f(x)=\left(a_1^2+a_2^2+\ldots+a_n^2\right)x^2-2\left(a_1b_1+a_2b_2+\ldots+a_nb_n\right)x+\left(b_1^2+b_2^2+\ldots+b_n^2\right).$$

$\forall x\in R$ uchun $f(x)\geq 0$ bo'lganligi uchun $D\leq 0$ ya'ni

$$\left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^2+b_2^2+...+b_n^2\right)\geq\left(a_1b_1+a_2b_2+...+a_nb_n\right)^2$$

munosabat o'rinli. Tenglik $f(x)=0$ tenglama kamida bitta haqiqiy ildizga ega bo'lganida bajariladi, ya'ni $\left(a_1,a_2,...,a_n\right)$ va $\left(b_1,b_2,...,b_n\right)$ o'zaro proporsional bo'lganda bajariladi.

2-isboti. Ushbu Lagranj (ba'zi adabiyotlarda Koshi-Shvars) ayniyatidan

$$\left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^2+b_2^2+...+b_n^2\right)-\left(a_1b_1+a_2b_2+...+a_nb_n\right)^2=\sum_{i,j=1}^{n}\left(a_ib_j-a_jb_i\right)^2$$

darhol kelib chiqadi.

3-isboti. Bu isbot Gyolder tengsizligini isbotlashda ham qo'llaniladi. AM-GM tengsizligiga ko'ra, quyidagi tengsizlik o'rinli

$$\frac{a_i^2}{a_1^2+a_2^2+...+a_n^2}+\frac{b_i^2}{b_1^2+b_2^2+...+b_n^2}\geq\frac{2a_ib_i}{\sqrt{\left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^2+b_2^2+...+b_n^2\right)}}$$

i ning o'rniga 1 dan n gacha barcha natural sonlarni qo'yib, hosil bo'lgan barcha tengsizliklarni qo'shib, isbotlanishi lozim bo'lgan tengsizlikni hosil qilamiz.

4-isboti. Bu isbot vektorlardan foydalanib isbotlanadi. Quyidagi ikkita $\vec{a}\left(a_1,a_2,...,a_n\right)$ va $\vec{b}\left(b_1,b_2,...,b_n\right)$ vektorni olaylik. Bu vektorlarning skalyar ko'paytmasi va moduli mos ravishda $\vec{a}\vec{b}=a_1b_1+a_2b_2+...+a_nb_n$ va $|\vec{a}|=\sqrt{a_1^2+a_2^2+...+a_n^2}$, $|\vec{b}|=\sqrt{b_1^2+b_2^2+...+b_n^2}$ orqali topiladi. Boshqa tomondan skalyar ko'paytma $\vec{a}\vec{b}=|\vec{a}||\vec{b}|\cos\varphi$ kabi aniqlanadi. Bu yerda φ – $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak. Bundan $\cos\varphi=\dfrac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|}$. $\cos\varphi\leq 1\Rightarrow\vec{a}\vec{b}\leq|\vec{a}||\vec{b}|$ tengsizlik hosil bo'ladi. Oxirgi tengsizlikdan

$$\left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^2+b_2^2+...+b_n^2\right)\geq\left(a_1b_1+a_2b_2+...+a_nb_n\right)^2$$

ekanligi kelib chiqadi.

2-natija. Haqiqiy sonlarning istalgan ikki $\left(a_1,a_2,...,a_n\right)$ va $\left(b_1,b_2,...,b_n\right)$, $b_i>0,\ i=1,2,...,n$ ketma-ketligi uchun quyidagi tengsizlik o'rinli

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+...+\frac{a_n^2}{b_n}\geq\frac{\left(a_1+a_2+...+a_n\right)^2}{b_1+b_2+...+b_n}.$$

3-natija. Haqiqiy sonlarning istalgan ikki $\left(a_1,a_2,...,a_n\right)$ va $\left(b_1,b_2,...,b_n\right)$ ketma-ketligi uchun quyidagi tengsizlik o'rinli

$$\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}+...+\sqrt{a_n^2+b_n^2}\geq\sqrt{(a_1+a_2+...+a_n)^2+(b_1+b_2+...+b_n)^2}\ .$$

4-natija. Haqiqiy sonlarning istalgan $\left(a_1,a_2,...,a_n\right)$ ketma-ketligi uchun quyidagi tengsizlik o'rinli

$$\left(a_1+a_2+...+a_n\right)^2\leq n\cdot\left(a_1^2+a_2^2+...+a_n^2\right).$$

6-teorema.(Gyugens tengsizligi) Agar $a_1,a_2,...,a_n$ lar haqiqiy musbat sonlar bo'lsa, quyidagi tengsizlikni isbotlang

$$(1+a_1)(1+a_2)...(1+a_n)\geq\left(1+\sqrt[n]{a_1a_2...a_n}\right)^n .$$

Isbot. AM-GM tengsizligini qo'llab quyidagi tengsizliklarni yozishimiz mumkin:

$$\frac{1}{1+a_1}+\frac{1}{1+a_2}+...+\frac{1}{1+a_n}\geq\frac{n}{\sqrt[n]{(1+a_1)(1+a_2)...(1+a_n)}}$$

$$\frac{a_1}{1+a_1}+\frac{a_2}{1+a_2}+...+\frac{a_n}{1+a_n}\geq\frac{n\sqrt[n]{a_1a_2...a_n}}{\sqrt[n]{(1+a_1)(1+a_2)...(1+a_n)}}$$

Bu ikki tengsizliklarni qo'shib, kerakli natijani olamiz.

3-lemma. Agar $x,y>0$ bo'lsa, u holda quyidagi tengsizlik o'rinli

$$\frac{1}{x}+\frac{1}{y}\geq\frac{4}{x+y}.$$

Isbot. $\frac{1}{x}+\frac{1}{y}\geq\frac{4}{x+y}\Rightarrow\frac{x+y}{xy}\geq\frac{4}{x+y}\Rightarrow\left(x+y\right)^2\geq4xy\Rightarrow\left(x-y\right)^2\geq0$.

4-lemma. Agar $x,y>0,\ \ xy\geq1$ bo'lsa, u holda quyidagi tengsizlik o'rinli

$$\frac{1}{x^2+1}+\frac{1}{y^2+1}\geq\frac{2}{xy+1}.$$

Isbot. Bizga ma'lumki, yuqoridagi shartdan $xy-1\geq0$ o'rinli.

$$\frac{x^2+y^2+2}{\left(x^2+1\right)\left(y^2+1\right)}\geq\frac{2}{xy+1}\Rightarrow2+xy\left(x^2+y^2\right)+2xy+x^2+y^2\geq2\left(x^2+y^2+x^2y^2+1\right)\Rightarrow$$

$$\Rightarrow xy\left(x^2+y^2\right)+2xy+x^2+y^2 \geq 2x^2+2y^2+2x^2y^2 \Rightarrow xy\left(x^2+y^2\right)+2xy \geq 2x^2y^2+x^2+y^2 \Rightarrow$$

$$xy\left(x^2+y^2\right)-2x^2y^2+2xy-x^2-y^2 \geq 0 \Rightarrow xy\left(x^2+y^2-2xy\right)-\left(x^2+y^2-2xy\right) \geq 0 \Rightarrow$$

$$\Rightarrow (xy-1)(x-y)^2 \geq 0.$$

5-lemma. Agar $x, y > 0$ bo'lsa, u holda quyidagi tengsizlik o'rinli

$$\frac{x^2}{x+y} \geq \frac{3x-y}{4}.$$

Isbot. Tengsizlikni o'ng qismini chap tomonga o'tkazib, nomanfiyligini ko'rsatamiz: $\frac{x^2}{x+y} \geq \frac{3x-y}{4} \Rightarrow 4x^2 \geq 3x^2+2xy-y^2 \Rightarrow x^2+y^2 \geq 2xy \Rightarrow (x-y)^2 \geq 0$.

Faollashtiruvchi savollar.

1. *Musbat n ta sonning o'rta arifmetigi qanday topiladi?*
2. *Musbat n ta sonning o'rta geometrigi qanday topiladi?*
3. *Musbat n ta sonning o'rta kvadartigi qanday topiladi?*
4. *Musbat n ta sonning o'rta garmonigi qanday topiladi?*

2-§. Tengsizliklarni isbotlashning usullari haqida.

Tayanch so'zlar: *Koshi tengsizligi, Koshi-Bunyakovskiy-Shvars tengsizligi, Gyugens tengsizligi, Bernulli tengsizligi, o'rta arifmetik, o'rta geometrik, o'rta kvadratik, o'rta garmonik, musbat sonlar, induksiya.*

1-masala. Istalgan a, b, c sonlari uchun $2a^2+b^2+c^2 \geq 2a(b+c)$ ekanligini isbotlang.

Isbot. Istalgan a, b, c sonlari uchun $a^2+b^2 \geq 2ab,\ a^2+c^2 \geq 2ac$ tengsizliklar o'rinli. Bu tengsizliklarni hadma-had qo'shib isbotlash talab etilgan tengsizlikni hosil qilamiz.

2-masala. Musbat a, b, c sonlari uchun $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} \geq 6$ tengsizlikni isbotlang.

Isbot. Tengsizlikning chap qismida shakl almashtirish bajarib, uni quyidagi ko'rinishda yozamiz:

$$\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)\geq 6$$

Ikkita musbat son uchun o'rta arifmetik va o'rta geometrik qiymatlar orasidagi Koshi tengsizligidan foydalanamiz:

$$\frac{a}{b}+\frac{b}{a}\geq 2,\quad \frac{a}{c}+\frac{c}{a}\geq 2,\quad \frac{b}{c}+\frac{c}{b}\geq 2$$

Bu tengsizliklarni hadma-had qo'shib, isbotlash talab etilgan tengsizlikni hosil qilamiz.

3-masala. $a^2+b^2+c^2\geq ab+bc+ac$ tengsizlikni isbotlang, bu yerda a,b,c – musbat sonlar.

Isbot. Ma'lumki, $a^2+b^2\geq 2ab,\ b^2+c^2\geq 2bc,\ a^2+c^2\geq 2ac$ tengsizliklarni qo'shib, ushbu $a^2+b^2+c^2\geq ab+bc+ac$ tengsizlikni hosil qilamiz.

4-masala. $a^4+b^4+c^4\geq abc(a+b+c)$ tengsizlikni isbotlang, bu yerda a,b,c – musbat sonlar.

Isbot. 3-masalaga ko'ra $a^4+b^4+c^4=\left(a^2\right)^2+\left(b^2\right)^2+\left(c^2\right)^2\geq a^2b^2+b^2c^2+c^2a^2$ ga egamiz. Bu yerdan esa $a^2b^2+b^2c^2+c^2a^2\geq ab\cdot bc+bc\cdot ca+ca\cdot ab=abc(a+b+c)$.

5-masala. $a^4+b^4+c^4+d^4\geq 4abcd$ tengsizlikni isbotlang, bu yerda a,b,c – musbat sonlar.

Isbot. $a^4+b^4+c^4+d^4\geq 4\sqrt[4]{a^4b^4c^4d^4}=4abcd$.

6-masala. $\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)\geq a\sqrt{b}+b\sqrt{a}$ tengsizlikni isbotlang, bu yerda a,b – musbat sonlar.

Isbot. $\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)=\frac{1}{2}(a+b)\left(a+b+\frac{1}{2}\right)$. Ikkinchidan, $\frac{1}{2}(a+b)\geq\sqrt{ab}$, $a+b+\frac{1}{2}=a+\frac{1}{4}+b+\frac{1}{4}\geq\sqrt{a}+\sqrt{b}$. Bu tengsizliklarni hadma-had ko'paytirib isbotlash talab etilgan tengsizlikni hosil qilamiz.

7-masala. Agar $a+b+c=1$ bo'lsa, $a^2+b^2+c^2\geq\frac{1}{3}$ ni isbotlang.

Isbot. Shartdan $a^2+b^2+c^2+2ab+2bc+2ac=1$ tenglikni hosil qilamiz. Yuqoridagi 3-masaladagi tengsizlikdan foydalanmamiz, ya'ni

$$1=a^2+b^2+c^2+2(ab+bc+ac)\le 3(a^2+b^2+c^2)\Rightarrow a^2+b^2+c^2\ge\frac{1}{3}.$$

8-masala. Musbat a,b,c sonlari $abc=1$ shartni qanoatlantirsa, $a+b+c\le a^2+b^2+c^2$ tengsizlikni isbotlang.

Isbot. Bizga ma'lumki $a^2+1\ge 2a,\ b^2+1\ge 2b,\ c^2+1\ge 2c$.

$$a^2+b^2+c^2+3=a^2+1+b^2+1+c^2+1\ge 2(a+b+c)=a+b+c+a+b+c\ge$$

$$\ge a+b+c+3\sqrt[3]{abc}=a+b+c+3\Rightarrow a+b+c\le a^2+b^2+c^2.$$

9-masala. Musbat a,b,c sonlari $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge a+b+c$ shartni qanoatlantirsa, $a+b+c\ge 3abc$ tengsizlikni isbotlang.

Isbot. Yuqoridagi shartni quyidagicha yozib olamiz:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge a+b+c\Rightarrow ab+bc+ac\ge abc(a+b+c).$$

Endi quyidagi ifodani baholaymiz:

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ac)\ge 3(ab+bc+ac)\ge 3abc(a+b+c)\Rightarrow$$

$$\Rightarrow (a+b+c)^2\ge 3abc(a+b+c)\Rightarrow a+b+c\ge 3abc.$$

10-masala. Musbat a,b,c sonlari $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le a+b+c$ shartni qanoatlantirsa, $ab+bc+ac\ge 3$ tengsizlikni isbotlang.

Isbot. Shartni quyidagicha yozib olamiz:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le a+b+c\Rightarrow ab+bc+ac\le abc(a+b+c)$$

Quyidagi ifodani qaraymiz:

$$(ab+bc+ac)^2=a^2b^2+b^2c^2+c^2a^2+2(a^2bc+ab^2c+abc^2)\ge$$

$$\ge a^2bc+ab^2c+abc^2+2(a^2bc+ab^2c+abc^2)=3abc(ab+bc+ac)\ge$$

$$\ge 3(ab+bc+ac)\Rightarrow (ab+bc+ac)^2\ge 3(ab+bc+ac)\Rightarrow ab+bc+ac\ge 3.$$

11-masala. Musbat a,b,c sonlari $a+b+c\ge abc$ shartni qanoatlantirsa, $a^2+b^2+c^2\ge\sqrt{3}abc$ tengsizlikni isbotlang.

Isbot. Yuqoridagi $a^2+b^2+c^2 \ge ab+bc+ac$ tengsizlikdan foydalanamiz.

$$\left(a^2+b^2+c^2\right)^2 \ge a^4+b^4+c^4+2\left(a^2b^2+b^2c^2+c^2a^2\right) \ge 3\left(a^2b^2+b^2c^2+c^2a^2\right) \ge$$

$$\ge 3\left(a^2bc+b^2ac+c^2ab\right) = 3abc\left(a+b+c\right) \ge 3abc\cdot abc = 3\left(abc\right)^2 \Rightarrow$$

$$\Rightarrow \left(a^2+b^2+c^2\right)^2 \ge 3\left(abc\right)^2 \Rightarrow a^2+b^2+c^2 \ge \sqrt{3}abc.$$

12-masala. Agar $a^5-a^3+a=2$ bo'lsa, $3<a^6<4$ tengsizlikni isbotlang

Isbot. Shartdan $a^5-a^3+a=2 \Rightarrow a\left(a^4-a^2+1\right)=2 \Rightarrow a^4-a^2+1=\dfrac{2}{a}$ ekanligi kelib chiqadi. Bu yerda $a>0, a\ne 1$. Dastlab $3<a^6$ ekanligini isbotlaymiz:

$$a^6+1=\left(a^2+1\right)\left(a^4-a^2+1\right)=\left(a^2+1\right)\cdot\frac{2}{a}=2\left(a+\frac{1}{a}\right)>4 \Rightarrow 3<a^6.$$

Endi $a^6<4$ ekanligini isbotlaymiz:

$$2=a^5+a-a^3>2a^3-a^3=a^3 \Rightarrow a^3<2 \Rightarrow a^6<4.$$

13-masala. Aytaylik x, y haqiqiy sonlar bo'lsin. $\dfrac{x+y}{2}\cdot\dfrac{x^2+y^2}{2}\cdot\dfrac{x^3+y^3}{2} \le \dfrac{x^6+y^6}{2}$ tengsizlikni isbotlang.

Isbot. Dastlab quyidagi ifodani baholaymiz:

$$\begin{cases} x^4+x^4+x^4+y^4 \ge 4x^3y \\ y^4+y^4+y^4+x^4 \ge 4xy^3 \end{cases} \Rightarrow x^4+y^4 \ge x^3y+xy^3$$

$$2\left(x^4+y^4\right)=x^4+y^4+x^4+y^4 \ge x^4+y^4+x^3y+xy^3=\left(x+y\right)\left(x^3+y^3\right) \Rightarrow$$

$$\Rightarrow \frac{x^4+y^4}{2} \ge \frac{x+y}{2}\cdot\frac{x^3+y^3}{2} \qquad (*)$$

Xuddi shunga o'xshash

$$\begin{cases} x^6+x^6+x^6+x^6+y^6+y^6 \ge 6x^4y^2 \\ y^6+y^6+y^6+y^6+x^6+x^6 \ge 6x^2y^4 \end{cases} \Rightarrow x^6+y^6 \ge x^4y^2+x^2y^4$$

$$2\left(x^6+y^6\right)=x^6+y^6+x^6+y^6 \ge x^6+y^6+x^4y^2+x^2y^4 \ge \left(x^2+y^2\right)\left(x^4+y^4\right) \Rightarrow$$

$$\Rightarrow \frac{x^6+y^6}{2} \ge \frac{x^2+y^2}{2}\cdot\frac{x^4+y^4}{2} \qquad (**)$$

$(*)$ va $(**)$ ga ko'ra

$$\frac{x+y}{2}\cdot\frac{x^2+y^2}{2}\cdot\frac{x^3+y^3}{2}\le\frac{x^4+y^4}{2}\cdot\frac{x^2+y^2}{2}\le\frac{x^6+y^6}{2}.$$

14-masala. Agar a,b,c musbat sonlar uchun $ab+bc+ac=abc$ tenglikni qanoatlantirsa, $\frac{a^4+b^4}{ab(a^3+b^3)}+\frac{b^4+c^4}{bc(b^3+c^3)}+\frac{c^4+a^4}{ac(c^3+b^3)}\ge 1$ tengsizlikni isbotlang.

Isbot. Koshiga asosan quyidagi tengsizlik o'rinli: $a^4+b^4\ge\frac{1}{2}(a+b)(a^3+b^3)$.

Shartdan $ab+bc+ac=abc\Rightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ ni hosil qilamiz.

$$\frac{a^4+b^4}{ab(a^3+b^3)}\ge\frac{(a+b)(a^3+b^3)}{2ab(a^3+b^3)}=\frac{a+b}{2ab}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$$

Xuddi shunday

$$\frac{b^4+c^4}{bc(b^3+c^3)}\ge\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right),\ \frac{c^4+a^4}{ca(c^3+a^3)}\ge\frac{1}{2}\left(\frac{1}{c}+\frac{1}{a}\right)$$

tengsizliklarni hosil qilamiz. Bu tengsizliklarni hadma-had qo'shib, isbotlash talab qilingan tengsizlikni hosil qilamiz.

15-masala. $a,b,c>0$ bo'lsa, $\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\le\frac{1}{abc}$ tengsizlikni isbotlang.

Isbot. $a^3+b^3\ge ab(a+b)$ tengsizlikdan foydalanamiz:

$$\frac{1}{a^3+b^3+abc}\le\frac{1}{ab(a+b)+abc}=\frac{1}{ab(a+b+c)}$$

Xuddi shunday

$$\frac{1}{b^3+c^3+abc}\le\frac{1}{bc(b+c)+abc}=\frac{1}{bc(a+b+c)}$$

$$\frac{1}{c^3+a^3+abc}\le\frac{1}{ac(a+c)+abc}=\frac{1}{ac(a+b+c)}.$$

Bu tengsizliklarni hadma-had qo'shib

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\le$$

$$\le\frac{1}{ab(a+b+c)}+\frac{1}{bc(a+b+c)}+\frac{1}{ac(a+b+c)}=\frac{a+b+c}{abc(a+b+c)}=\frac{1}{abc}$$

isbotlash talab etilgan tengsizlikni hosil qilamiz.

16-masala. a,b,c nomanfiy sonlar uchun $\frac{(a+b+c)^2}{3} \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$ tengsizlikni isbotlang.

Isbot. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ac) \geq 3(ab+bc+ac)$

$$\begin{cases} ab+bc \geq 2b\sqrt{ac} \\ ab+ac \geq 2a\sqrt{bc} \\ ac+bc \geq 2c\sqrt{ab} \end{cases} \Rightarrow ab+bc+ac \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$$

$$(a+b+c)^2 \geq 3(ab+bc+ac) \geq 3\left(a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}\right) \Rightarrow$$

$$\Rightarrow \frac{(a+b+c)^2}{3} \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$$

17-masala. a,b,c musbat sonlar uchun $abc=1$ bo'lsa, $\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1$ tengsizlikni isbotlang.

Isbot. Quyidagicha belgilash qilamiz: $a = x^3,\ b = y^3,\ c = z^3 \Rightarrow abc = 1 \Rightarrow xyz = 1$.

Endi $\frac{1}{x^3+y^3+xyz} + \frac{1}{y^3+z^3+xyz} + \frac{1}{z^3+x^3+xyz} \leq 1$ tengsizlikni isbotlash yetarli.

$x^3 + y^3 \geq xy(x+y)$ tengsizlikka asosan

$$\frac{1}{x^3+y^3+xyz} \leq \frac{1}{xy(x+y)+xyz} = \frac{1}{xy(x+y+z)} \cdot \frac{z}{z} = \frac{z}{x+y+z}$$

Xuddi shunday

$$\frac{1}{y^3+z^3+xyz} \leq \frac{y}{x+y+z},\ \frac{1}{z^3+x^3+xyz} \leq \frac{z}{x+y+z}$$

Bu tengsizliklarni hadlab qo'shib yuborib, talab qilingan tengsizlikning isbotini hosil qilamiz

18-masala. $0 \leq x,y,z \leq 1$ bo'lsa, $\frac{x}{7+y^3+z^3} + \frac{y}{7+x^3+z^3} + \frac{z}{7+x^3+y^3} \leq \frac{1}{3}$ tengsizlikni isbotlang.

Isbot. Shartdan $0 \leq x,y,z \leq 1 \Rightarrow x^3 \leq 1,\ y^3 \leq 1,\ z^3 \leq 1$

$$\frac{x}{7+y^3+z^3} = \frac{x}{6+1+y^3+z^3} \leq \frac{x}{x^3+y^3+z^3} =$$

$$= \frac{x}{x^3+1+1+y^3+1+1+z^3+1+1} \leq \frac{x}{3(x+y+z)}$$

Xuddi shunday

$$\frac{y}{7+x^3+z^3} \leq \frac{y}{3(x+y+z)},\ \frac{z}{7+x^3+y^3} \leq \frac{z}{3(x+y+z)}$$

Bu tengsizliklarni hadlab qo'shib

$$\frac{x}{7+y^3+z^3}+\frac{y}{7+x^3+z^3}+\frac{z}{7+x^3+y^3}\le\frac{1}{3}$$

tengsizlikni hosil qilamiz.

19-masala. a,b,c,d musbat sonlar uchun $a+b+c+d=1$ bo'lsa, $\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\ge\frac{1}{2}$ tengsizlikni isbotlang.

Isbot. Yuqoridagi K.B.S tengsizligining natijasidan foydalanamiz, ya'ni

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+...+\frac{a_n^2}{b_n}\ge\frac{(a_1+a_2+...+a_n)^2}{b_1+b_2+...+b_n}$$

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\ge\frac{(a+b+c+d)^2}{2(a+b+c+d)}=\frac{1}{2}$$

20-masala. $a,b,c>0$ bo'lsa, $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\ge 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$ tengsizlikni isbotlang.

Isbot. Yuqoridagi 3-lemmadan foydalanamiz, ya'ni $\frac{4}{x+y}\le\frac{1}{x}+\frac{1}{y}$.

$$\frac{4a}{b+c}=a\cdot\frac{4}{b+c}\le a\left(\frac{1}{b}+\frac{1}{c}\right),\quad \frac{4b}{c+a}\le b\left(\frac{1}{a}+\frac{1}{c}\right),\quad \frac{4c}{a+b}\le c\left(\frac{1}{a}+\frac{1}{b}\right)$$

Bu tengsizliklarni hadlab qo'shib, isbotlash talab etilgan tengsizlikni hosil qilamiz.

21-masala. a,b,c,d musbat sonlar uchun $abcd\ge 1$ shart bajarilsa, $\frac{1}{a^4+1}+\frac{1}{b^4+1}+\frac{1}{c^4+1}+\frac{1}{d^4+1}\ge\frac{4}{abcd+1}$ tengsizlikni isbotlang.

Isbot. Yuqoridagi 4-lemmadan foydalanamiz, ya'ni: $\frac{1}{x^2+1}+\frac{1}{y^2+1}\ge\frac{2}{xy+1}$.

$$\frac{1}{a^4+1}+\frac{1}{b^4+1}+\frac{1}{c^4+1}+\frac{1}{d^4+1}\ge\frac{2}{a^2b^2+1}+\frac{2}{c^2d^2+1}\ge\frac{4}{abcd+1}$$

22-masala. $x\ne y\ne z\ne 1,\ xyz=1$ bo'lsa, $\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\ge 1$ tengsizlikni isbotlang.

Isbot. Belgilash kiritamiz: $\frac{x}{x-1}=a,\ \frac{y}{y-1}=b,\ \frac{z}{z-1}=c$. Bundan esa

$$x=\frac{a}{a-1},\ y=\frac{b}{b-1},\ z=\frac{c}{c-1}\Rightarrow\frac{abc}{(a-1)(b-1)(c-1)}=1\Rightarrow a+b+c=ab+bc+ac+1$$

shartni hosil qilamiz. Belgilashga ko'ra isbotlashimiz kerak $a^2+b^2+c^2\ge 1$.

$$a+b+c=ab+bc+ac+1 \Rightarrow 2(a+b+c)=2(ab+bc+ac)+2 \qquad (*)$$

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca) \qquad (**)$$

$(**)$ dan $(*)$ ni ayiramiz:

$$(a+b+c)^2-2(a+b+c)=a^2+b^2+c^2-2 \Rightarrow (a+b+c)^2-2(a+b+c)+1=a^2+b^2+c^2-1 \Rightarrow$$

$$(a+b+c-1)^2=a^2+b^2+c^2-1\ge 0 \Rightarrow a^2+b^2+c^2\ge 1$$

23-masala. $x_1x_2x_3...x_n=1,\ n\ge 3$ tenglikni qanoatlantiradigan sonlar uchun $\frac{x_1^8}{(x_1^4+x_2^4)x_2}+\frac{x_2^8}{(x_2^4+x_3^4)x_3}+...+\frac{x_n^8}{(x_n^4+x_1^4)x_1}\ge\frac{n}{2}$ tengsizlikni isbotlang.

Isbot. $\frac{x_1^8}{(x_1^4+x_2^4)x_2}=\frac{x_1^8}{x_1^4x_2+x_2^5}\cdot\frac{x_1}{x_1}=\frac{x_1^9}{x_1^5x_2+x_1x_2^5}$. Bundan quyidagi tengsizlikni isbotlash yetarli $\frac{x_1^9}{x_1^5x_2+x_1x_2^5}+\frac{x_2^9}{x_2^5x_3+x_2x_3^5}+...+\frac{x_n^9}{x_n^5x_1+x_nx_1^5}\ge\frac{n}{2}$. Quyidagi tengsizlikdan foydalanamiz: $x_1^6+x_2^6\ge x_1^5x_2+x_1x_2^5$. Tengsizlikning chap qismini S orqali belgilab, $S\ge\frac{n}{2}$ ekanligini ko'rsatamiz:

$$S\ge\sum_{cyc}\frac{x_1^9}{x_1^6+x_2^6}=\sum_{cyc}\left(x_1^3-\frac{x_1^3x_2^6}{x_1^6+x_2^6}\right)\ge\sum_{cyc}\left(x_1^3-\frac{x_1^3x_2^6}{2x_1^3x_2^3}\right)=\sum_{cyc}\left(x_1^3-\frac{x_2^3}{2}\right)=\sum_{cyc}\frac{x_1^3}{2}\ge\frac{n}{2}$$

24-masala. $x,y,z>0,\ x+y+z=3$ tenglikni qanoatlantiradigan sonlar uchun $\frac{x^3}{y^3+8}+\frac{y^3}{z^3+8}+\frac{z^3}{x^3+8}\ge\frac{1}{9}+\frac{2}{27}(xy+yz+zx)$ tengsizlikni isbotlang.

Isbot.Tengsizlikning chap qismini M deb belgilaylik. Bizga ma'lumki Koshi tengsizligiga asosan quyidagi tengsizlik o'rinli:

$$\frac{x^3}{y^3+8}+\frac{y+2}{27}+\frac{y^2-2y+4}{27}\ge 3\sqrt[3]{\frac{x^3}{y^3+8}\cdot\frac{y+2}{27}\cdot\frac{y^2-2y+4}{27}}=\frac{x}{3}.$$

Xuddi shunday

$$\frac{y^3}{z^3+8}+\frac{z+2}{27}+\frac{z^2-2z+4}{27}\ge\frac{y}{3},\quad \frac{z^3}{x^3+8}+\frac{x+2}{27}+\frac{x^2-2x+4}{27}\ge\frac{z}{3}.$$

Bu tengsizliklarni hadma-had qo'shib

$$M+\frac{x+y+z+6}{27}+\frac{x^2+y^2+z^2-2(x+y+z)+12}{27}\geq\frac{x+y+z}{3}\Rightarrow$$

$$\Rightarrow M+\frac{1}{3}+\frac{x^2+y^2+z^2}{27}+\frac{2}{9}\geq 1\Rightarrow M+\frac{x^2+y^2+z^2}{27}\geq\frac{4}{9}\Rightarrow$$

$$\Rightarrow M+\frac{x^2+y^2+z^2}{27}+\frac{2(xy+yz+zx)}{27}\geq\frac{4}{9}+\frac{2(xy+yz+zx)}{27}\Rightarrow$$

$$\Rightarrow M\geq\frac{4}{9}-\frac{1}{3}+\frac{2(xy+yz+zx)}{27}\Rightarrow M\geq\frac{1}{9}+\frac{2(xy+yz+zx)}{27}$$

tengsizlikni hosil qilamiz.

25-masala. $a,b,c>0$ sonlar $abc=1$ tenglikni qanoatlantirsa, $\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$ tengsizlikni isbotlang.

Isbot. $\frac{1}{a}=x,\ \frac{1}{b}=y,\ \frac{1}{c}=z$ belgilash kiritamiz. U holda $xyz=1$ va

$$a+b=\frac{x+y}{xy},b+c=\frac{y+z}{yz},\ a+c=\frac{x+z}{xz}$$

larni hosil qilamiz. Bulardan

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}=\frac{x^3yz}{y+z}+\frac{y^3xz}{x+z}+\frac{z^3xy}{x+y}=$$

$$=\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\geq\frac{(x+y+z)^2}{2(x+y+z)}=\frac{x+y+z}{2}\geq\frac{3\sqrt[3]{xyz}}{2}=\frac{3}{2}.$$

26-masala. $x,y,z>0$ sonlar $x+y+z=3$ tenglikni qanoatlantirsa, $\sqrt{x}+\sqrt{y}+\sqrt{z}\geq xy+yz+zx$ tengsizlikni isbotlang.

Isbot. Quyidagi ifodani baholaymiz:

$$x^2+y^2+z^2+2\sqrt{x}+2\sqrt{y}+2\sqrt{z}=\left(x^2+\sqrt{x}+\sqrt{x}\right)+\left(y^2+\sqrt{y}+\sqrt{y}\right)+\left(z^2+\sqrt{z}+\sqrt{z}\right)\geq$$

$$\geq 3\sqrt[3]{x^2\sqrt{x}\sqrt{x}}+3\sqrt[3]{y^2\sqrt{y}\sqrt{y}}+3\sqrt[3]{z^2\sqrt{z}\sqrt{z}}=3(x+y+z)=(x+y+z)^2=$$

$$=x^2+y^2+z^2+2(xy+yz+zx)\Rightarrow\sqrt{x}+\sqrt{y}+\sqrt{z}\geq xy+yz+zx.$$

27-masala. Agar $x^p y^q z^r = 1$ va $p+q+r=1$ bo'lsa, $\frac{(px)^2}{qy+rz}+\frac{(qy)^2}{rz+px}+\frac{(rz)^2}{px+qy}\ge\frac{1}{2}$ tengsizlikni isbotlang.

Isbot. Quyidagicha belgilash qilamiz:

$$\begin{cases} px=\alpha \\ qy=\beta \\ rz=\gamma \end{cases} \Rightarrow \begin{cases} p^p x^p = \alpha^p \\ q^q y^q = \beta^q \\ r^r z^r = \gamma^r \end{cases} \Rightarrow p^p q^q r^r = \alpha^p \beta^q \gamma^r \Rightarrow \begin{cases} p=\alpha \\ q=\beta \\ r=\gamma \end{cases} \Rightarrow \alpha+\beta+\gamma=1$$

Bu belgilashlarga ko'ra $\frac{\alpha^2}{\beta+\gamma}+\frac{\beta^2}{\alpha+\gamma}+\frac{\gamma^2}{\alpha+\beta}\ge\frac{1}{2}$ tengsizlikni isbotlash yetarli.

$$\frac{\alpha^2}{\beta+\gamma}+\frac{\beta^2}{\alpha+\gamma}+\frac{\gamma^2}{\alpha+\beta}\ge\frac{(\alpha+\beta+\gamma)^2}{2(\alpha+\beta+\gamma)}=\frac{\alpha+\beta+\gamma}{2}=\frac{1}{2}.$$

28-masala. Agar $0<abc\le 1,\ a,b,c\in R^+$ bo'lsa, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 1+\frac{6}{a+b+c}$ tengsizlikni isbotlang.

Isbot. Bizga ma'lumki $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 9 \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{9}{a+b+c}$ va $abc\le 1$. Endi quyidagi ifodani baholaymiz:

$$\frac{3}{a}+\frac{3}{b}+\frac{3}{c}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge\frac{3}{\sqrt[3]{abc}}+2\cdot\frac{9}{a+b+c}\ge 3+\frac{18}{a+b+c}\Rightarrow$$

$$\Rightarrow\frac{3}{a}+\frac{3}{b}+\frac{3}{c}\ge 3+\frac{18}{a+b+c}\Rightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 1+\frac{6}{a+b+c}.$$

29-masala. Barcha musbat haqiqiy a,b,c sonlar uchun $(a^2+2)(b^2+2)(c^2+2)\ge 9(ab+bc+ac)$ tengsizlikni isbotlang.

Isbot. Dastlab ikkita ko'paytuvchini baholaymiz:

$$(a^2+2)(b^2+2)=(a^2+1+1)(b^2+1+1)=(a^2+1)(b^2+1)+a^2+b^2+3\ge$$

$$\ge(a+b)^2+\frac{1}{2}(a+b)^2+3=\frac{3}{2}(a+b)^2+3=3\left((a+b)^2+2\right).$$

Endi umumiy qismini isbotlaymiz.

$$(a^2+2)(b^2+2)(c^2+2)\ge\frac{3}{2}\left((a+b)^2+2\right)(c^2+2)=\frac{3}{2}\left((a+b)^2+(\sqrt{2})^2\right)\left((\sqrt{2})^2+c^2\right)\ge$$

$$\geq \frac{3}{2}\left(\sqrt{2}(a+b)+\sqrt{2}c\right)^2 = 3(a+b+c)^2 \geq 3\left(a^2+b^2+c^2+2ab+2bc+2ac\right) \geq 9(ab+bc+ac).$$

30-masala. Agar $x_1, x_2, ..., x_n > 0$ va $x_1 + x_2 + ... + x_n = 1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang

$$\frac{x_1}{\sqrt{x_1+x_2+...+x_n}} + \frac{x_2}{\sqrt{(1+x_1)(x_2+...+x_n)}} + ... + \frac{x_n}{\sqrt{(1+x_1+x_2+...+x_{n-1})x_n}} \geq 1.$$

Isbot. $\sqrt{ab} \leq \frac{a+b}{2}$, $(a,b \geq 0)$ tengsizlikni kasrning maxrajiga qo'llaymiz:

$$\frac{x_1}{\sqrt{x_1+x_2+...+x_n}} = \frac{x_1}{\sqrt{1\cdot(x_1+x_2+...+x_n)}} \geq \frac{x_1}{\frac{1+x_1+x_2+...+x_n}{2}} = \frac{2x_1}{1+x_1+x_2+...+x_n} = x_1$$

Xuddi shunday qolgan kasrlarga ham qo'llaymiz:

$$\frac{x_2}{\sqrt{(1+x_1)(x_2+...+x_n)}} \geq x_2, \frac{x_3}{\sqrt{(1+x_1+x_2)(x_3+...+x_n)}} \geq x_3, ..., \frac{x_n}{\sqrt{(1+x_1+...+x_{n-1})x_n}} \geq x_n$$

Bu tengsizliklarni hadlab qo'shib, isbotlash talab qilingan tengsizlikni hosil qilamiz.

31-masala. Agar $a,b,c > 0,\ p \in N$ bo'lsa, $a^{p+2} + b^{p+2} + c^{p+2} \geq a^p bc + ab^p c + abc^p$ tengsizlikni isbotlang.

Isbot. Koshiga asosan quyidagi tengsizlik o'rinli:

$$\underbrace{a^{p+2} + a^{p+2} + ... + a^{p+2}}_{p+2\ ta} + b^{p+2} + c^{p+2} \geq (p+2)\sqrt[p+2]{\left(a^p\right)^{p+2} b^{p+2} c^{p+2}} = (p+2)a^p bc$$

Xuddi shunday

$$\underbrace{b^{p+2} + b^{p+2} + ... + b^{p+2}}_{p+2\ ta} + a^{p+2} + c^{p+2} \geq (p+2)ab^p c$$

$$\underbrace{c^{p+2} + c^{p+2} + ... + c^{p+2}}_{p+2\ ta} + a^{p+2} + b^{p+2} \geq (p+2)abc^p$$

Bu tengsizliklarni hadma-had qo'shib

$$(p+2)a^{p+2} + (p+2)b^{p+2} + (p+2)c^{p+2} \geq (p+2)\left(a^p bc + ab^p c + abc^p\right) \Rightarrow$$

$$\Rightarrow a^{p+2} + b^{p+2} + c^{p+2} \geq a^p bc + ab^p c + abc^p$$

tengsizlikni hosil qilamiz.

32-masala. Musbat a,b,c sonlar $a+b+c=1$ shartni qanoatlantirsa, $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\ge\frac{2}{1+a}+\frac{2}{1+b}+\frac{2}{1+c}$ tengsizlikni isbotlang.

Isbot. Yuqoridagi 3-lemmadan foydalanamiz: $\frac{1}{x}+\frac{1}{y}\ge\frac{4}{x+y}$

$$\frac{2}{1-a}+\frac{2}{1-b}+\frac{2}{1-c}\ge\frac{2}{b+c}+\frac{2}{a+c}+\frac{2}{a+b}=\left(\frac{1}{b+c}+\frac{1}{a+b}\right)+\left(\frac{1}{b+c}+\frac{1}{a+c}\right)+$$

$$+\left(\frac{1}{a+c}+\frac{1}{a+b}\right)\ge\frac{4}{2b+a+c}+\frac{4}{2c+a+b}+\frac{4}{2a+b+c}=\frac{4}{1+a}+\frac{4}{1+b}+\frac{4}{1+c}\Rightarrow$$

$$\Rightarrow\frac{2}{1-a}+\frac{2}{1-b}+\frac{2}{1-c}\ge\frac{4}{1+a}+\frac{4}{1+b}+\frac{4}{1+c}\Rightarrow\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\ge\frac{2}{1+a}+\frac{2}{1+b}+\frac{2}{1+c}.$$

33-masala. $x,y,z>0$ va $xy+yz+zx\le xyz$ bo'lsa, $\sqrt{x}+\sqrt{y}+\sqrt{z}\le\sqrt{xyz}$ tengsizlikni isbotlang.

Isbot. $xy+yz+zx\le xyz\Rightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le1$ (1) shartni hosil qilamiz.

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\le\sqrt{xyz}\Rightarrow\frac{1}{\sqrt{xy}}+\frac{1}{\sqrt{yz}}+\frac{1}{\sqrt{zx}}\le1\quad(2)$$

(2) tengsizlikni (1) shart bilan isbotlash yetarli. Belgilash kiritamiz: $\frac{1}{x}=a,\ \frac{1}{y}=b,\ \frac{1}{z}=c.$ Bundan $a+b+c\le1$ shart va $\sqrt{ab}+\sqrt{bc}+\sqrt{ac}\le1$ ni isbotlaymiz.

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ac}\le\frac{a+b}{2}+\frac{b+c}{2}+\frac{a+c}{2}=a+b+c\le1\Rightarrow\sqrt{ab}+\sqrt{bc}+\sqrt{ac}\le1.$$

34-masala. $a,b,c,d>0$ va $a+b+c+d=3$ bo'lsa, $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\le\frac{1}{a^2b^2c^2d^2}$ tengsizlikni isbotlang.

Isbot. Umumiylikka ziyon yetkazmasdan $a\ge b\ge c\ge d$ deb olamiz. Bundan esa $a^2\ge cd$ va $b^2\ge cd$ (1) shartni hosil qilamiz. Quyidagicha tengsizlikni yozib olamiz:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\le\frac{1}{a^2b^2c^2d^2}\Rightarrow a^2b^2c^2+a^2b^2d^2+a^2c^2d^2+b^2c^2d^2\le1$$

ekanligini isbotlaymiz.

$$ab(c+d)\le\left(\frac{a+b+c+d}{3}\right)^3=1\Rightarrow a^2b^2(c+d)^2\le1\Rightarrow a^2b^2\left(c^2+d^2+2cd\right)\le1\Rightarrow$$

$$\Rightarrow a^2b^2c^2 + a^2b^2d^2 + 2a^2b^2cd \le 1$$

shartni hosil qilamiz. Endi (1) shartdan foydalanamiz:

$$a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2 = a^2b^2c^2 + a^2b^2d^2 + a^2 \cdot cd \cdot cd + b^2 \cdot cd \cdot cd \le$$

$$\le a^2b^2c^2 + a^2b^2d^2 + a^2b^2 \cdot cd + b^2a^2 \cdot cd = a^2b^2c^2 + a^2b^2d^2 + 2a^2b^2cd \le 1 \Rightarrow$$

$$\Rightarrow a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2 \le 1.$$

35-masala. $a,b,c > 0$ bo’lsa, $a^2(b+c-a) + b^2(a+c-b) + c^2(a+b-c) \le 3abc$ tengsizlikni isbotlang.

Isbot. Qavslarni ochib o’ng tarafga o’tkazib, hosil bo’lgan ifoda nomanfiy ekanligini ko’rsatish yetarli.

$$a^2b + a^2c - a^3 + b^2a + b^2c - b^3 + c^2a + c^2b - c^3 \le 3abc \Rightarrow$$

$$\Rightarrow a^3 + b^3 + c^3 + 3abc - \left(a^2(b+c) + b^2(a+c) + c^2(a+b)\right) \ge 0 \Rightarrow$$

$$\Rightarrow a^3 + abc - a^2(b+c) + b^3 + abc - b^2(a+c) + c^3 + abc - c^2(a+b) \ge 0 \Rightarrow$$

$$\Rightarrow a\left(a^2 + bc - ab - ac\right) + b\left(b^2 + ac - ab - bc\right) + c\left(c^2 + ab - ac - bc\right) \ge 0 \Rightarrow$$

$$\Rightarrow a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \ge 0$$

a,b,c larni umumiylikka ziyon yetkazmasdan $a \ge b \ge c$ deb olaylik.

$$(a-b)\left(a^2 - ac - b^2 + bc\right) + c(c-a)(c-b) \ge 0 \Rightarrow$$

$$\Rightarrow (a-b)\left((a-b)(a+b) - c(a-b)\right) + c(c-a)(c-b) \ge 0 \Rightarrow$$

$$\Rightarrow (a-b)^2(a+b-c) + c(c-a)(c-b) \ge 0$$

$$\Rightarrow (a-b)^2 \ge 0,\ (a+b-c) \ge 0,\ \ c(c-a)(c-b) \ge 0.$$

36-masala. $x_1, x_2, ..., x_n$ musbat haqiqiy sonlar bo’lsa, u holda $\frac{x_1^3}{x_1^2 + x_1x_2 + x_2^2} + \frac{x_2^3}{x_2^2 + x_2x_3 + x_3^2} + ... + \frac{x_n^3}{x_n^2 + x_nx_1 + x_1^2} \ge \frac{x_1 + x_2 + ... + x_n}{3}$ tengsizlikni isbotlang.

Isbot. Bizga ma’lumki quyidagi tengsizlik o’rinli:

$$\frac{x_1^3}{x_1^2 + x_1x_2 + x_2^2} = x_1 - \frac{x_1x_2(x_1 + x_2)}{x_1^2 + x_2^2 + x_1x_2} \ge x_1 - \frac{x_1x_2(x_1 + x_2)}{3x_1x_2} = x_1 - \frac{x_1 + x_2}{3}.$$

Xuddi shunday

$$\frac{x_2^3}{x_2^2+x_2x_3+x_3^2} \ge x_2 - \frac{x_2+x_3}{3}, \ldots, \frac{x_n^3}{x_n^2+x_nx_1+x_1^2} \ge x_n - \frac{x_n+x_1}{3}$$

tengsizliklarni yozishimiz mumkin. Bu tengsizliklarni hadlab qo'shib isbotlash talab etilgan tengsizlikni hosil qilamiz.

Bu usulda isbotlash Koshiga teskari usul ham deb yuritiladi. Quyida shu usulga tushadigan bir nechta tengsizliklarni isbotini ko'rib chiqamiz.

37-masala. $a,b,c>0$ va $a+b+c=3$ bo'lsa, $\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+a^2} \ge \frac{3}{2}$ tengsizlikni isbotlang.

Isbot. $a+b+c=3$ shartdan $ab+bc+ca \le 3$ ni hosil qilamiz. Har bir qo'shiluvchining maxrajiga Koshi tengsizligini quyidagicha qo'llaymiz:

$$\frac{a}{1+b^2} = a - \frac{ab^2}{1+b^2} \ge a - \frac{ab^2}{2b} = a - \frac{ab}{2}.$$

Xuddi shunday qolganlarini ham

$$\frac{b}{1+c^2} \ge b - \frac{bc}{2}, \ \frac{c}{1+a^2} \ge c - \frac{ac}{2}$$

yozishimiz mumkin. Bu tengsizliklarni hadma-had qo'shib

$$\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+a^2} \ge a+b+c-\frac{ab+bc+ca}{2} \ge 3-\frac{3}{2}=\frac{3}{2}$$

tengsizlikni hosil qilamiz.

38-masala. $a,b,c>0$ va $a+b+c=3$ bo'lsa, $\frac{a+1}{b^2+1}+\frac{b+1}{c^2+1}+\frac{c+1}{a^2+1} \ge 3$ tengsizlikni isbotlang.

Isbot. $a+b+c=3$ shartdan $ab+bc+ca \le 3$ ni hosil qilamiz. Quyidagi bahodan foydalanamiz:

$$\frac{a+1}{b^2+1} = a+1-\frac{b^2(a+1)}{b^2+1} \ge a+1-\frac{b^2(a+1)}{2b} = a+1-\frac{ab+b}{2}.$$

Xuddi shunday qolganlarini ham

$$\frac{b+1}{c^2+1} \ge b+1-\frac{bc+c}{2}, \ \frac{c+1}{a^2+1} \ge c+1-\frac{ac+a}{2}$$

yozishimiz mumkin. Bu tengsizliklarni hadma-had qo'shib isbotlash talab qilingan tengsizlikni hosil qilamiz.

39-masala. $a,b,c>0$ va $a+b+c=3$ bo'lsa, $\frac{1}{1+2b^2c}+\frac{1}{1+2c^2a}+\frac{1}{1+2a^2b}\geq 1$ tengsizlikni isbotlang.

Isbot. Quyidagi baholashdan foydalanish yetarli:

$$\frac{1}{1+2b^2c}=1-\frac{2b^2c}{1+2b^2c}=1-\frac{2b^2c}{1+b^2c+b^2c}\geq 1-\frac{2\sqrt[3]{b^2c}}{3}\geq 1-\frac{2(2b+c)}{9}.$$

Xuddi shunday qolganlarini ham

$$\frac{1}{1+2c^2a}\geq 1-\frac{2(2c+a)}{9},\ \frac{1}{1+2a^2b}\geq 1-\frac{2(2a+b)}{9}$$

yozishimiz mumkin. Bu tengsizliklarni hadma-had qo'shib isbotlash talab qilingan tengsizlikni hosil qilamiz.

40-masala. $S_n=a_1+a_2+...+a_n,\ a_i>0\ (i=1,2,...,n)$ yig'indini olamiz. Ixtiyoriy natural n uchun $\frac{a_1}{S_1^2}+\frac{a_2}{S_2^2}+...+......+\frac{a_n}{S_n^2}<\frac{2}{a_1}$ tengsizlikni isbotlang.

Isbot. $k\geq 2$ bo'lganda $\frac{1}{S_{k-1}}-\frac{1}{S_k}=\frac{a_k}{S_{k-1}S_k}>\frac{a_k}{S_k^2}$ bo'ladi.

$$\sum_{k=2}^{n}\frac{a_k}{S_k^2}<\sum_{k=2}^{n}\left(\frac{1}{S_{k-1}}-\frac{1}{S_k}\right)=\frac{1}{S_1}-\frac{1}{S_n}<\frac{1}{S_1}=\frac{1}{a_1}$$

$S_1=a_1$ ekanligini hisobga olib, oxirgi tengsizlikning har ikkala tomoniga $\frac{a_1}{S_1^2}$ ni qo'shsak

$$\frac{a_1}{S_1^2}+\frac{a_2}{S_2^2}+...+\frac{a_n}{S_n^2}<\frac{1}{a_1}+\frac{a_1}{S_1^2}=\frac{1}{a_1}+\frac{1}{a_1}=\frac{2}{a_1}$$

kelib chiqadi.

41-masala. $a,b,c>0$ va $abc=1$ bo'lsa, $\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1$ tengsizlikni isbotlang.

Isbot. $abc=1$ shartdan foydalanib quyidagicha belgilash qilamiz: $a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x}.$

$$\left(\frac{x}{y}+\frac{z}{y}-1\right)\left(\frac{y}{z}+\frac{x}{z}-1\right)\left(\frac{z}{x}+\frac{y}{x}-1\right)\le 1 \Rightarrow xyz \ge (x+y-z)(y+z-x)(z+x-y)$$

tengsizlikni isbotlash yetarli. Bu tengsizlikning isboti esa quyidagi tengsizliklarni hadma-had ko'paytirishdan kelib chiqadi:

$$\begin{cases} x^2 \ge x^2-(z-y)^2=(x+y-z)(x+z-y) \\ y^2 \ge y^2-(x-z)^2=(x+y-z)(y+z-x) \\ z^2 \ge z^2-(x-y)^2=(x+z-y)(z+y-x) \end{cases}$$

42-masala. Agar $a,b,c\ge 0$ va $a+b+c=3$ bo'lsa, u holda $\frac{a}{1+b}+\frac{b}{1+c}+\frac{c}{1+a}\ge\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+a}$ tengsizlikni isbotlang.

Isbot. Biz $\frac{a-1}{1+b}+\frac{b-1}{1+c}+\frac{c-1}{1+a}\ge 0$ tengsizlikni isbotlaymiz.

$$(a-1)(1+c)(1+a)+(b-1)(1+b)(1+a)+(c-1)(1+b)(1+c)\ge 0 \Rightarrow$$

$$\Rightarrow a^2+b^2+c^2+a^2c+ab^2+bc^2-(a+b+c)-3\ge 0 \Rightarrow$$

$$\Rightarrow a^2+b^2+c^2+a^2c+ab^2+bc^2\ge 6 \Rightarrow a^2+b^2+c^2+a^2c+ab^2+bc^2+a+b+c\ge 9$$

tengsizlini isbotlash yetarli. $ab^2+a\ge 2ab,\ ca^2+c\ge 2ac,\ bc^2+b\ge 2bc$ tengsizliklarni hadlab qo'shib, $ab^2+ca^2+bc^2+a+b+c\ge 2(ab+bc+ac)$ tengsizlikni hosil qilamiz.

$$a^2+b^2+c^2+a^2c+ab^2+bc^2+a+b+c\ge a^2+b^2+c^2+2(ab+bc+ac)=(a+b+c)^2=9.$$

43-masala. Agar $a,b,c\ge 0$ bo'lsa, $(a^3+1)(b^3+1)(c^3+1)\ge(abc+1)^3$ tengsizlikni isbotlang.

Isbot. Bizga ma'lumki quyidagi $\frac{a^3}{a^3+1}+\frac{b^3}{b^3+1}+\frac{c^3}{c^3+1}\ge\frac{3abc}{\sqrt[3]{(a^3+1)(b^3+1)(c^3+1)}}$, $\frac{1}{a^3+1}+\frac{1}{b^3+1}+\frac{1}{c^3+1}\ge\frac{3}{\sqrt[3]{(a^3+1)(b^3+1)(c^3+1)}}$ tengsizliklar o'rinli. Bu tengsizliklarni hadma-had qo'shib, $3\ge\frac{3(abc+1)}{\sqrt[3]{(a^3+1)(b^3+1)(c^3+1)}} \Rightarrow (a^3+1)(b^3+1)(c^3+1)\ge(abc+1)^3$ isbotlash talab etilgan tengsizlikni hosil qilamiz.

44-masala. Agar $a\geq 0$ bo'lsa, $2(a^2+1)^3\geq (a+1)^3(a^3+1)$ tengsizlikni isbotlang.

Isbot. Qavslarni ochib,

$$2(a^6+3a^4+3a^2+1)\geq (a^3+3a^2+3a+1)(a^3+1)\Rightarrow a^6-3a^5+3a^4-2a^3+3a^2-3a+1\geq 0$$

hosil bo'lgan ifodani nomanfiy ekanligini ko'rsatamiz.

$$a^6-a^5+2a^5+2a^4+a^4-a^3-a^3+a^2+2a^2-2a-a+1\geq 0\Rightarrow$$

$$\Rightarrow a^5(a-1)-2a^4(a-1)+a^3(a-1)-a^2(a-1)+2a(a-1)-(a-1)\geq 0$$

$$\Rightarrow (a-1)(a^5-2a^4+a^3-a^2+2a-1)\geq 0\Rightarrow$$

$$\Rightarrow (a-1)(a^5-a^4-a^4+a^3-a^2+a+a-1)\geq 0\Rightarrow$$

$$\Rightarrow (a-1)(a^4(a-1)-a^3(a-1)-a(a-1)+(a-1))\geq 0\Rightarrow$$

$$\Rightarrow (a-1)^2(a^4-a^3-a+1)\geq 0\Rightarrow (a-1)^2(a^3(a-1)-(a-1))\geq 0\Rightarrow$$

$$\Rightarrow (a-1)^3(a^3-1)\geq 0\Rightarrow (a-1)^4(a^2+a+1)\geq 0$$

45-masala. Agar $a,b,c\geq 0$ bo'lsa, $2(a^2+1)(b^2+1)(c^2+1)\geq (a+1)(b+1)(c+1)(abc+1)$ tengsizlikni isbotlang.

Isbot. Yuqoridagi 43-44 masaladagi tengsizliklarga asosan

$$\begin{cases} 2(a^2+1)^3\geq (a+1)^3(a^3+1) \\ 2(b^2+1)^3\geq (b+1)^3(b^3+1) \\ 2(c^2+1)^3\geq (c+1)^3(c^3+1) \end{cases}$$

tengsizliklarni hadma-had ko'paytirib

$$[2(a^2+1)(b^2+1)(c^2+1)]^3\geq [(a+1)(b+1)(c+1)]^3\cdot \underbrace{(a^3+1)(b^3+1)(c^3+1)}_{\geq (abc+1)^3}\geq$$

$$\geq [(a+1)(b+1)(c+1)(abc+1)]^3\Rightarrow 2(a^2+1)(b^2+1)(c^2+1)\geq (a+1)(b+1)(c+1)(abc+1)$$

tengsizlikni hosil qilamiz.

46-masala. Agar $a,b,c,d,e>1$ bo'lsa, $\frac{a^2}{c-1}+\frac{b^2}{d-1}+\frac{c^2}{e-1}+\frac{d^2}{a-1}+\frac{e^2}{b-1}\geq 20$ tengsizlikni isbotlang.

Isbot. Belgilash kiritamiz:

$$a = x+1,\ b = y+1,\ c = z+1,\ d = n+1,\ e = m+1,\ a,b,c,d,e > 1 \Rightarrow x,y,z,n,m > 0.$$

Isbotlash kerak: $\frac{(x+1)^2}{z} + \frac{(y+1)^2}{n} + \frac{(z+1)^2}{m} + \frac{(n+1)^2}{x} + \frac{(m+1)^2}{y} \geq 20$

$$\frac{x^2}{z} + \frac{y^2}{n} + \frac{z^2}{m} + \frac{n^2}{x} + \frac{m^2}{y} + \frac{1}{z} + \frac{1}{n} + \frac{1}{m} + \frac{1}{x} + \frac{1}{y} \geq 10\sqrt[10]{\frac{x^2}{z}\cdot\frac{y^2}{n}\cdot\frac{z^2}{m}\cdot\frac{n^2}{x}\cdot\frac{m^2}{y}\cdot\frac{1}{z}\cdot\frac{1}{n}\cdot\frac{1}{m}\cdot\frac{1}{x}\cdot\frac{1}{y}} = 10$$

$$2\left(\frac{x}{z} + \frac{y}{n} + \frac{z}{m} + \frac{n}{x} + \frac{m}{y}\right) \geq 2\cdot 5\sqrt[5]{\frac{x}{z}\cdot\frac{y}{n}\cdot\frac{z}{m}\cdot\frac{n}{x}\cdot\frac{m}{y}} = 10$$

tengsizliklarni hadma-had qo'shib, isbotlash talab qilingan tengsizlikni hosil qilamiz.

47-masala. Musbat a,b,c sonlari uchun $\frac{ab}{c} + \frac{bc}{a} + \frac{ac}{b} \leq 3$ bo'lsa, $\frac{a^2}{b^3} + \frac{b^2}{c^3} + \frac{c^2}{a^3} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ tengsizlikni isbotlang.

Isbot. Bizga ma'lumki, $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c)$ tengsizlik o'rinli. Shartdan quyidagi tengsizlikni hosil qilamiz:

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ac}{b} \leq 3 \Rightarrow a^2b^2 + b^2c^2 + c^2a^2 \leq 3abc \Rightarrow a+b+c \leq 3.$$

Endi quyidagi ifodaga KBS tengsizligini qo'llaymiz:

$$(a+b+c)\left(\frac{a^2}{b^3} + \frac{b^2}{c^3} + \frac{c^2}{a^3}\right) = \left(\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 + \left(\sqrt{b}\right)^2\right)\left(\left(\frac{c}{a\sqrt{a}}\right)^2 + \left(\frac{a}{b\sqrt{b}}\right)^2 + \left(\frac{b}{c\sqrt{c}}\right)^2\right) \geq$$

$$\geq \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 \Rightarrow \frac{a^2}{b^3} + \frac{b^2}{c^3} + \frac{c^2}{a^3} \geq \frac{\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2}{a+b+c} = \frac{\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)}{a+b+c} \geq$$

$$\geq \frac{3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)}{3} = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

48-masala. Musbat a,b,c sonlari uchun $a+b+c=1$ bo'lsa, u holda

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \geq 64$$

tengsizlikni isbotlang.

Isbot. $a+b+c=1$ dan $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}=\frac{1}{abc}$ va $1=a+b+c\geq 3\sqrt[3]{abc}\Rightarrow\frac{1}{abc}\geq 27$ tengsizlikni hosil qilamiz.

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=1+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{abc}=4+\frac{2}{abc}+\frac{b}{a}+\frac{c}{a}+\frac{a}{b}+\frac{c}{b}+\frac{a}{c}+\frac{b}{c}\geq$$

$$\geq 4+2\cdot 27+2\sqrt{\frac{a}{b}\cdot\frac{b}{a}}+2\sqrt{\frac{c}{b}\cdot\frac{b}{c}}+2\sqrt{\frac{a}{c}\cdot\frac{c}{a}}=4+54+2+2+2=64$$

49-masala. Agar a,b,c musbat haqiqiy sonlar bo'lsa,

$$\frac{a+b}{c^2}+\frac{a+c}{b^2}+\frac{b+c}{a^2}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

tengsizlikni isbotlang.

Isbot. a,b,c musbat ekanligidan $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>0$ ekanligi kelib chiqadi.

$$\frac{a+b}{c^2}+\frac{a+c}{b^2}+\frac{b+c}{a^2}>\frac{a}{c^2}+\frac{a}{b^2}+\frac{b}{a^2}+\frac{b}{c^2}+\frac{c}{b^2}+\frac{c}{a^2}=a\left(\frac{1}{c^2}+\frac{1}{b^2}\right)+b\left(\frac{1}{a^2}+\frac{1}{c^2}\right)+c\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\geq$$

$$\geq\frac{2a}{bc}+\frac{2b}{ac}+\frac{2c}{ab}=2\left(\frac{a^2+b^2+c^2}{abc}\right)\geq\frac{2(ab+bc+ac)}{abc}=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

50-masala. Agar a,b,c musbat haqiqiy sonlar bo'lsa,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{a+c}\geq\frac{9}{a+b+c}$$

tengsizlikni isbotlang.

Isbot. Dastlab tengsizlikni chap qismini so'ngra o'ng qismini isbotlaymiz.

1) $\frac{1}{a}+\frac{1}{b}\geq\frac{2}{\sqrt{ab}},\ \frac{1}{b}+\frac{1}{c}\geq\frac{2}{\sqrt{bc}},\ \frac{1}{a}+\frac{1}{c}\geq\frac{2}{\sqrt{ac}}$ tengsizliklarni hadlab qo'shamiz:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{ac}}+\frac{1}{\sqrt{bc}}\geq\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{a+c}\Rightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{a+c}.$$

2) $$\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{a+c}\geq\frac{6}{\sqrt[3]{(a+b)(b+c)(a+c)}}=\frac{2}{\sqrt[3]{(a+b)(b+c)(a+c)}}+\frac{2}{\sqrt[3]{(a+b)(b+c)(a+c)}}+$$

$$+\frac{2}{\sqrt[3]{(a+b)(b+c)(a+c)}} \geq \frac{2\cdot 3}{2(a+b+c)}+\frac{2\cdot 3}{2(a+b+c)}+\frac{2\cdot 3}{2(a+b+c)}=\frac{9}{a+b+c} \Rightarrow$$

$$\Rightarrow \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{a+c} \geq \frac{9}{a+b+c}.$$

51-masala. Agar a,b,c musbat haqiqiy sonlar bo'lsa,

$$\frac{a^8+b^8+c^8}{a^3b^3c^3} \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

tengsizlikni isbotlang.

Isbot. 8 ta son uchun Koshi tengsizligini qo'llaymiz:

$$a^8+a^8+a^8+b^8+b^8+b^8+c^8+c^8 \geq 8\sqrt[8]{a^{24}\cdot b^{24}\cdot c^{16}}=8a^3b^3c^2$$

$$a^8+a^8+a^8+b^8+b^8+c^8+c^8+c^8 \geq 8a^3b^2c^3$$

$$a^8+a^8+b^8+b^8+b^8+c^8+c^8+c^8 \geq 8a^2b^3c^3$$

Hosil bo'lgan tengsizliklarni hadma-had qo'shib

$$a^8+b^8+c^8 \geq a^3b^3c^2+a^3b^2c^3+a^2b^3c^3$$

tengsizlikni hosil qilamiz. Bu tengsizlikdan esa

$$\frac{a^8+b^8+c^8}{a^3b^3c^3} \geq \frac{a^3b^3c^2+a^3b^2c^3+a^2b^3c^3}{a^3b^3c^3}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

ekanligi kelib chiqadi.

52-masala. $\forall n \geq 2,\ a_1,a_2,...,a_n \geq 0$ sonlari uchun quyidagi tengsizlikni isbotlang:

$$a_1+a_2+...+a_n-n\sqrt[n]{a_1a_2...a_n} \geq \left(\sqrt{a_1}-\sqrt{a_2}\right)^2$$

Isbot. Koshi tengsizligidan foydalanamiz, ya'ni $x_1+x_2+...+x_n \geq n\sqrt[n]{x_1x_2...x_n}$.

$$a_1+a_2+...+a_n-\left(\sqrt{a_1}-\sqrt{a_2}\right)^2 \geq n\sqrt[n]{a_1a_2...a_n} \Rightarrow a_3+a_4+...+a_n+2\sqrt{a_1a_2} \geq n\sqrt[n]{a_1a_2...a_n} \Rightarrow$$

$$\Rightarrow \sqrt{a_1a_2}+\sqrt{a_1a_2}+a_3+a_4+...+a_n \geq n\sqrt[n]{a_1a_2...a_n}$$

Tengsizlikni isbotlash yetarli. Bu tengsizlik esa Koshi tengsizligiga asosan o'rinli.

$$\sqrt{a_1a_2}+\sqrt{a_1a_2}+a_3+a_4+...+a_n \geq n\sqrt[n]{\sqrt{a_1a_2}\sqrt{a_1a_2}a_3a_4...a_n}=n\sqrt[n]{a_1a_2...a_n}$$

53-masala. $a,b,c>0$ va $a^2+b^2+c^2=\frac{5}{3}$ bo'lsa, $\frac{1}{a}+\frac{1}{b}-\frac{1}{c}<\frac{1}{abc}$ tengsizlikni isbotlang.

Isbot. Tengsizlikni $\frac{1}{a}+\frac{1}{b}<\frac{1}{abc}+\frac{1}{c}\Rightarrow (a+b)c<1+ab$ ko'rinishga keltirib isbotlaymiz:

$$(a+b)c\le\frac{(a+b)^2+c^2}{2}=\frac{2ab+a^2+b^2+c^2}{2}=ab+\frac{\frac{5}{3}}{2}=ab+\frac{5}{6}<ab+1.$$

54-masala. Agar $a,b>0$ va $x_i\in[a,b],\ (i=\overline{1,n})$ bo'lsa, quyidagi tengsizlikni isbotlang:

$$(x_1+x_2+...+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}\right)\le\frac{b}{a}\cdot n^2$$

Isbot. Shartga ko'ra $a\le x_i\le b,\ (i=\overline{1,n})$ ekanligidan foydalanamiz:

$$(x_1+x_2+...+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}\right)\le\left(\underbrace{b+b+...+b}_{n\ ta}\right)\left(\underbrace{\frac{1}{a}+\frac{1}{a}+...+\frac{1}{a}}_{n\ ta}\right)=b\cdot n\cdot\frac{1}{a}\cdot n=\frac{b}{a}\cdot n^2.$$

55-masala. Agar $x,y\ge 0$ bo'lsa, quyidagi tengsizlikni isbotlang:

$$(x+y)\left(1+\sqrt{xy}\right)\ge 2\sqrt{xy(1+x)(1+y)}$$

Isbot: Tengsizlikning o'ng qismini kichikligini ko'rsatamiz:

$$2\sqrt{xy(1+x)(1+y)}\le\sqrt{xy}(1+x+1+y)=\left(\sqrt{xy}+1-1\right)(x+y+2)=$$

$$=\left(\sqrt{xy}+1\right)(x+y)+2\sqrt{xy}-x-y\le\left(\sqrt{xy}+1\right)(x+y)+x+y-x-y=\left(\sqrt{xy}+1\right)(x+y).$$

56-masala: Agar $x,y,z>0$ bo'lsa, quyidagi tengsizlikni isbotlang:

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}+\frac{2}{x+y+z}\ge 2$$

Isbot: Titu lemmadan foydalanamiz, ya'ni musbat sonlar uchun

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n}\ge\frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n}$$

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}+\frac{2}{x+y+z}\geq\frac{(x+y+z)^2}{2(x+y+z)}+\frac{2}{x+y+z}\geq 2$$

57-masala. Agar $x,y,z>0$ bo'lsa, quyidagi tengsizlikni isbotlang:

$$\frac{x^4+y^4+z^4}{x+y+z}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 4$$

Isbot: $a^2+b^2+c^2\geq ab+bc+ac$ tengsizlikdan foydalanamiz:

$$\frac{x^4+y^4+z^4}{x+y+z}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{x^2y^2+z^2y^2+x^2z^2}{x+y+z}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq$$

$$\geq\frac{x^2yz+xy^2z+xyz^2}{x+y+z}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{xyz(x+y+z)}{x+y+z}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=$$

$$=xyz+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 4\sqrt[4]{xyz\cdot\frac{1}{xyz}}=4$$

58-masala. Agar $a,b,c>0$ bo'lsa, quyidagi tengsizlikni isbotlang:

$$\frac{a^5}{b^2}+\frac{b^5}{c^2}+\frac{c^5}{a^2}\geq a^3+b^3+c^3$$

Isbot: Koshi tengsizligidan quyidagicha foydalanamiz:

$$\frac{a^5}{b^2}+\frac{a^5}{b^2}+\frac{a^5}{b^2}+b^3+b^3\geq 5\sqrt[5]{\frac{(a^3)^5}{b^6}\cdot b^6}=5a^3$$

$$\frac{b^5}{c^2}+\frac{b^5}{c^2}+\frac{b^5}{c^2}+c^3+c^3\geq 5b^3$$

$$\frac{c^5}{a^2}+\frac{c^5}{a^2}+\frac{c^5}{a^2}+a^3+a^3\geq 5c^3$$

Hosil bo'lgan tengsizliklarni hadma-had qo'shib, isbotlash talab etilgan tengsizlikni hosil qilamiz.

Faollashtiruvchi savollar.

1. *O'zaro teskari sonlarni ta'rifini ayting?*
2. *2 ta son uchun o'rta arifmetik qanday topiladi?*
3. *2 ta son uchun o'rta geometrik qanday topiladi?*
4. *O'rta arifmetik va o'rta geometrik orasidagi munosabatni ayting?*

5. *O'rta arifmetik va o'rta geometrik orasidagi munosabat manfiy sonlar uchun ham o'rinlimi?*

Mustaqil yechish uchun masalalar

1. Agar $a,b>0$ bo'lsa, $\frac{a}{b}+\frac{b}{a}\geq 2$ tengsizlikni isbotlang.
2. Agar $a,b,c>0$ bo'lsa, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$ tengsizlikni isbotlang.
3. Agar $a,b,c,d>0$ bo'lsa, $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\geq 4$ tengsizlikni isbotlang.
4. Agar $a,b>0$ va birdan farqli sonlar bo'lsa, $|\log_a b|+|\log_b a|\geq 2$ tengsizlikni isbotlang.
5. $\frac{1}{\log_2 \pi}+\frac{1}{\log_\pi 2}>2$ tengsizlikni isbotlang.
6. $a,b,c\geq 0$ bo'lsa, $(a+b)(b+c)(c+a)\geq 8abc$ tengsizlikni isbotlang.
7. $a,b>0$ bo'lsa, $(a+1)(b+1)(a+b)\geq 8ab$ tengsizlikni isbotlang.
8. Agar $a,b,c\geq 0$ bo'lsa, $(a+1)(b+1)(a+c)(b+c)\geq 16abc$ tengsizlikni isbotlang.
9. Agar $a,b,c\geq 0$ va $a+b+c=1$ bo'lsa, $(1-a)(1-b)(1-c)\geq 8abc$ tengsizlikni isbotlang.
10. Agar $a,b,c,d\geq 0$ bo'lsa, $a+b+c+d\geq 2\sqrt{(a+b)(c+d)}$ tengsizlikni isbotlang.
11. $a,b>0$ bo'lsa, $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$ tengsizlikni isbotlang.
12. $a,b,c>0$ bo'lsa, $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$ tengsizlikni isbotlang.
13. $a_i>0,\ i=1,2,...,n$ bo'lsa, $(a_1+a_2+...+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\right)\geq n^2$ tengsizlikni isbotlang.
14. Agar $a_1,a_2,...,a_n>0$ va $a_1a_2...a_n=1$ bo'lsa, $(1+a_1)(1+a_2)...(1+a_n)\geq 2^n$ tengsizlikni isbotlang.
15. Agar $a,b>0$ bo'lsa, $a^4+b^4+8\geq 8ab$ tengsizlikni isbotlang.
16. $a,b>0$ bo'lsa, $a^2+b^2+1\geq ab+a+b$ tengsizlikni isbotlang.
17. $x_1,x_2,x_3,x_4,x_5>0$ bo'lsa, $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2\geq x_1(x_2+x_3+x_4+x_5)$ tengsizlikni isbotlang.
18. Agar $a,b,c>0$ bo'lsa, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ac}}$ tengsizlikni isbotlang.
19. $a,b,c,d\geq 0$ bo'lsa, $\frac{a+b+c+d}{4}\geq \sqrt[4]{abcd}$ tengsizlikni isbotlang.

20. $a,b,c \geq 0$ bo'lsa, $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ tengsizlikni isbotlang.

21. $a,b,c > 0$ bo'lsa, $\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} \geq a+b+c$ tengsizlikni isbotlang.

22. $a,b,c \in R$ bo'lsa, $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ tengsizlikni isbotlang.

23. $a,b,c \in R$ bo'lsa, $(a+b+c)^2 \geq 3(ab+bc+ac)$ tengsizlikni isbotlang.

24. $a,b,c \in R$ va $a^2+b^2+c^2=1$ bo'lsa, $ab+bc+ac \leq 1$ tengsizlikni isbotlang.

25. $a^2+b^2+c^2=1$ bo'lsa, $\frac{1}{2} \leq ab+bc+ac \leq 1$ tengsizlikni isbotlang.

26. Agar $a+b+c=1$ bo'lsa, $a^2+b^2+c^2 \geq \frac{1}{3}$ tengsizlikni isbotlang.

27. $a,b,c \geq 0$ bo'lsa, $ab+bc+ac \geq \sqrt{3abc(a+b+c)}$ tengsizlikni isbotlang.

28. $a,b > 0$ va $a+b=1$ bo'lsa, $a^2+b^2 \geq \frac{1}{2}$ tengsizlikni isbotlang.

29. Agar $a+b \geq 1$ bo'lsa, $a^4+b^4 \geq \frac{1}{8}$ tengsizlikni isbotlang.

30. Agar $a,b,c \geq 0$ va $a^2+b^2+c^2=1$ bo'lsa, $a+b+c \leq \sqrt{3}$ tengsizlikni isbotlang.

31. $a \in R$ da $\frac{a}{a^2-4a+9} \leq \frac{1}{2}$ tengsizlikni isbotlang.

32. $a \in R$ da $\frac{a^4+16}{a^2+4} \geq 2a$ tengsizlikni isbotlang.

33. $a \in R$ da $\frac{a}{a^2+a+1} \leq \frac{1}{3}$ tengsizlikni isbotlang.

34. $a \in R$ bo'lsa, $\frac{16a^2}{(1+a^2)(9a^2+1)} \leq 1$ tengsizlikni isbotlang.

35. $a \in R$ da $\frac{a^2+a+2}{\sqrt{a^2+a+1}} \geq 2$ tengsizlikni isbotlang.

36. $a \in R$ bo'lsa, $a^2+1+\frac{1}{a^2+1} \geq 2$ tengsizlikni isbotlang.

37. Agar $a > 0$ bo'lsa, $a^{10} + \frac{3}{a^2} + \frac{4}{a} \geq 8$ tengsizlikni isbotlang.

38. Agar $a > 0$ bo'lsa, $a^{40} + \frac{1}{a^{16}} + \frac{2}{a^4} + \frac{4}{a^2} + \frac{8}{a} \geq 16$ tengsizlikni isbotlang.

39. Agar $a > 0$ bo'lsa, $a^4 + \frac{1}{a^2} + \frac{2}{a} \geq 4$ tengsizlikni isbotlang.

40. $a \neq 0$ bo'lsa, $a^2 + \frac{1}{a^2} \geq a + \frac{1}{a}$ tengsizlikni isbotlang.

41. $a,b > 0$ va $a+b=1$ bo'lsa, $\left(\frac{1}{a^2}-1\right)\left(\frac{1}{b^2}-1\right) \geq 9$ tengsizlikni isbotlang.

42. Agar $a,b,c \geq 0$ bo'lsa, $(a+b+c)(ab+bc+ac) \geq 9abc$ tengsizlikni isbotlang.

43. Agar $abc=1$ bo'lsa, $ab+bc+ac+a+b+c \geq 6$ tengsizlikni isbotlang.

44. $a,b,c \ge 0$ va $abc=1$ bo'lsa, $(3a+2b+c)(3b+2c+a)(3c+2a+b) \ge 216$ tengsizlikni isbotlang.

45. $a,b,c \ge 0$ va $abc(a+b+c)=1$ bo'lsa, $(a+b)(b+c) \ge 2$ tengsizlikni isbotlang.

46. $a,b \ge 0$ bo'lsa, $a^3+b^3 \ge ab(a+b)$ tengsizlikni isbotlang.

47. $a,b \ge 0$ bo'lsa, $a^5+b^5 \ge a^4b+ab^4$ tengsizlikni isbotlang.

48. $a^4+b^4 \ge a^3b+ab^3$ tengsizlikni isbotlang.

49. $a^6+b^6 \ge a^5b+ab^5$ tengsizlikni isbotlang.

50. $a^6+b^6 \ge a^4b^2+a^2b^4$ tengsizlikni isbotlang.

51. Agar $a,b \ge 0$ bo'lsa, $(a+b)^3 \le 4(a^3+b^3)$ tengsizlikni isbotlang.

52. Agar $a,b \ge 0$ bo'lsa, $a^5+b^5 \ge a^3b^2+a^2b^3$ tengsizlikni isbotlang.

53. $a,b,c \ge 0$ bo'lsa, $a^3+b^3+c^3 \ge a^2\sqrt{bc}+b^2\sqrt{ac}+c^2\sqrt{ab}$ tengsizlikni isbotlang.

54. $a,b \ge 0$ bo'lsa, $\frac{a^7+b^7}{2} \ge \frac{a^5+b^5}{2} \cdot \frac{a^2+b^2}{2}$ tengsizlikni isbotlang.

55. $a,b \ge 0$ bo'lsa, $2\sqrt{a}+3\sqrt[3]{b} \ge 5\sqrt[5]{ab}$ tengsizlikni isbotlang.

56. $a,b,c > 0$ bo'lsa, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{a^8+b^8+c^8}{a^3b^3c^3}$ tengsizlikni isbotlang.

57. Agar $a,b,c > 0$ bo'lsa, $\frac{a^5}{b^2}+\frac{b^5}{c^2}+\frac{c^5}{a^2} \ge a^3+b^3+c^3$ tengsizlikni isbotlang.

58. Agar $a,b,c > 0,\ n,k \in N$ bo'lsa, $\frac{a^{n+k}}{b^k}+\frac{b^{n+k}}{c^k}+\frac{c^{n+k}}{a^k} \ge a^n+b^n+c^n$ tengsizlikni isbotlang.

59.

60. $a \in R$ da $9a^2-30|a|+25 \ge 0$ tengsizlikni isbotlang.

61. $a \in R$ da $a^2-2a+10 \ge 6|a-1|$ tengsizlikni isbotlang.

62. $a \in R$ da $a^2-4a+5 \ge 2|a-2|$ tengsizlikni isbotlang.

63. Agar $a \ge b \ge 0$ bo'lsa, $a^3-b^3 \ge a^2b-ab^2$ tengsizlikni isbotlang.

64. Agar $a \ge b \ge 0$ bo'lsa, $a^3-b^3 \ge 3ab(a-b)$ tengsizlikni isbotlang.

65. Agar $a \ge b \ge 0$ bo'lsa, $a^5-b^5 \ge a^4b-ab^4$ tengsizlikni isbotlang.

66. Agar $ab \ge 0$ bo'lsa, $(a^2-b^2)^2 \ge (a-b)^4$ tengsizlikni isbotlang.

67. $a_1,a_2,...,a_n > 0$ uchun $a_1a_2^2a_3^3...a_n^n=1$ bo'lsa, $\frac{1}{a_1}+\frac{2}{a_2}+\frac{3}{a_3}+...+\frac{n}{a_n} \ge \frac{n(n+1)}{2}$ tengsizlikni isbotlang.

68. $n \ge 2(n \in N)$ bo'lsa, $n! < \left(\frac{n+1}{2}\right)^n$ tengsizlikni isbotlang.

69. $\sqrt{1\cdot 2}+\sqrt{2\cdot 3}+\sqrt{3\cdot 4}+...+\sqrt{99\cdot 100}<\frac{9999}{2}$ tengsizlikni isbotlang.

70. $\sqrt{42+\sqrt{42+\sqrt{42+...+\sqrt{42}}}}+\sqrt{56+\sqrt{56+\sqrt{56+...+\sqrt{56}}}}<15$ tengsizlikni isbotlang.

71. $\sqrt{20+\sqrt{20+\sqrt{20+...+\sqrt{20}}}}<5$ tengsizlikni isbotlang.

72. Agar $a,b,c>0$ va $a+b+c\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ bo'lsa, $a^3+b^3+c^3\geq a+b+c$ tengsizlikni isbotlang.

73. $a,b>0$ va $a+b=1$ bo'lsa, $\frac{a^2}{a+1}+\frac{b^2}{b+1}\geq\frac{1}{3}$ tengsizlikni isbotlang.

74. $a,b,c>0$ va $a+b+c=1$ bo'lsa, $\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{1}{2}$ tengsizlikni isbotlang.

75. $a_i>0,\ i=1,2,...,2019$ bo'lsa, $\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+...+\frac{a_{2019}^2}{a_{2019}+a_1}\geq\frac{1}{2}(a_1+a_2+...+a_{2019})$ tengsizlikni isbotlang.

76. x,y,z musbat sonlar uchun $\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}+\frac{2}{x+y+z}\geq 2$ tengsizlikni isbotlang.

77. Agar $a,b,c,d>0$ bo'lsa, $\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\geq 2$ tengsizlikni isbotlang.

78. x,y,z musbat sonlar uchun $\frac{x^4+y^4+z^4}{x+y+z}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 4$ tengsizlikni isbotlang.

79. $a,b,c>0$ va $abc=1$ bo'lsa, $\frac{b+c}{\sqrt{a}}+\frac{c+a}{\sqrt{b}}+\frac{a+b}{\sqrt{c}}\geq\sqrt{a}+\sqrt{b}+\sqrt{c}+3$ tengsizlikni isbotlang.

80. Agar $a,b,c>0$ bo'lsa, $\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{a+c}}+\sqrt{\frac{c}{a+b}}\geq 2$ tengsizlikni isbotlang.

81. Agar $a,b,c>0$ va $a+b+c=1$ bo'lsa, $\frac{a^2+3ab}{a+b}+\frac{b^2+3bc}{b+c}+\frac{c^2+3ac}{a+c}\leq 2$ tengsizlikni isbotlang.

82. Agar $a,b,c>0$ va $a+b+c=1$ bo'lsa, $\sqrt{\frac{ab}{ab+c}}+\sqrt{\frac{bc}{bc+a}}+\sqrt{\frac{ac}{ac+b}}\leq\frac{3}{2}$ tengsizlikni isbotlang.

83. Agar $a,b,c\geq 0$ bo'lsa, $abc\geq(a+b-c)(a+c-b)(b+c-a)$ tengsizlikni isbotlang.

84. $a+2b+3c=14$ bo'lsa, $a^2+b^2+c^2\geq 14$ tengsizlikni isbotlang.

85. $a,b,c\geq 0$ va $a^2+b^2+c^2=1$ bo'lsa, $\frac{ab}{c}+\frac{ac}{b}+\frac{bc}{a}\geq\sqrt{3}$ tengsizlikni isbotlang.

86. $a,b>0$ bo'lsa, $\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}}\le\sqrt[3]{2(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}$ tengsizlikni isbotlang.

87. $a,b\in R$ va $ab>0$ bo'lsa, $\sqrt[3]{\frac{a^2b^2(a+b)^2}{4}}\le\frac{a^2+10ab+b^2}{12}$ tengsizlikni isbotlang.

88. $a,b,c>0$ bo'lsa, $(a+b)^2\le(1+c)a^2+\left(1+\frac{1}{c}\right)b^2$ tengsizlikni isbotlang.

89. $a,b,c,d\ge 0$ bo'lsa, $16(abc+bcd+cda+dab)\le(a+b+c+d)^3$ tengsizlikni isbotlang.

90. $a,b,c>0$ bo'lsa, $\frac{a^2-ab}{a+b}+\frac{b^2-bc}{b+c}+\frac{c^2-ac}{a+c}\ge 0$ tengsizlikni isbotlang.

91. Agar $a_1,a_2,...a_n\in[a,b],0<a<b$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$n^2\le(a_1+a_2+...+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\right)\le\frac{(a+b)^2}{4ab}n^2$$

92. $a_i>0,i=1,2,...,n$ bo'lsa, $\frac{a_1}{a_2+a_3}+\frac{a_2}{a_3+a_4}+...+\frac{a_{n-2}}{a_{n-1}+a_n}+\frac{a_{n-1}}{a_n+a_1}+\frac{a_n}{a_1+a_2}>\frac{n}{2}$ tengsizlikni isbotlang.

93. $a,b,c\ge 0$ bo'lsa, $4(a+b+c)^3\ge 27(a^2b+b^2c+c^2a)$ tengsizlikni isbotlang.

94. $a,b\ge 0$ va $a+b=2$ bo'lsa, $a^2b^2(a^2+b^2)\le 2$ tengsizlikni isbotlang.

95. $a,b>0$ bo'lsa, $(a+b)(1+\sqrt{ab})\ge 2\sqrt{ab(1+a)(1+b)}$ tengsizlikni isbotlang.

96. $a>b>0$ bo'lsa, $\frac{a^3-b^3}{2}>\left(\frac{a-b}{2}\right)^3$ tengsizlikni isbotlang.

97. $a\in R,a\ne 0$ bo'lsa, $1+\frac{1}{a^2}>\frac{2}{a}-\frac{11}{25a^2}+\frac{2}{5a}$ tengsizlikni isbotlang.

98. $a,b,c>0$ va $a+b+c=3$ bo'lsa, $\frac{1}{\sqrt{a+b-c}}+\frac{1}{\sqrt{b+c-a}}+\frac{1}{\sqrt{a+c-b}}\ge\frac{9}{ab+bc+ac}$ tengsizlikni isbotlang.

99. $0<a,b,c<1$ bo'lsa, $\sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)}<1$ tengsizlikni isbotlang.

100. $a,b,c\in R$ va $a^2+b^2+c^2=3$ bo'lsa, $|a|+|b|+|c|-abc\le 4$ tengsizlikni isbotlang.

101. $\overline{abc}$ uch xonali son bo'lsa, $\overline{abc}\cdot\overline{bca}\cdot\overline{cab}\ge\overline{aaa}\cdot\overline{bbb}\cdot\overline{ccc}$ tengsizlikni isbotlang.

102. Agar $a,b,c>0$ bo'lsa, $\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\ge\frac{a+b+c}{3}$ tengsizlikni isbotlang.

103. Agar $a,b,c,d>0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+cd+d^2}+\frac{d^3}{d^2+da+a^2}\geq\frac{a+b+c+d}{3}$$

104. $a,b,c>0$ va $a+b+c=1$ bo'lsa, $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\geq\frac{9}{2}$ tengsizlikni isbotlang.

105. $a,b,c>0$ bo'lsa, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{c+a}{c+b}+\frac{b+a}{c+a}+\frac{c+b}{a+b}$ tengsizlikni isbotlang.

106. Agar $a,b,c>0$ va $a+b+c=\frac{1982}{3}$ bo'lsa, $\sqrt{a+3}+\sqrt{b+3}+\sqrt{c+3}\leq\sqrt{2009}$ tengsizlikni isbotlang.

107. Agar $ac-bc+c^2<0$ bo'lsa, $b^2>4ac$ tengsizlikni isbotlang.

108. $n\in N(n\geq 2)$ da $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...+\frac{1}{n^2}<\frac{n-1}{n}$ tengsizlikni isbotlang.

109. $a_2,a_3,...,a_{10}$ shunday musbat haqiqiy sonlarki, $a_2a_3...a_{10}=1$. $(1+a_2)^2(1+a_3)^3...(1+a_{10})^{10}>10^{10}$ tengsizlikni isbotlang.

110. $a,b,c>0$ va $a+b+c=1$ bo'lsa, $\frac{a^2+b}{b+c}+\frac{b^2+c}{c+a}+\frac{c^2+a}{a+b}\geq 2$ tengsizlikni isbotlang.

111. $a,b,c,d>0$ va $a+b+c+d=4$ bo'lsa, $\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+d^2}+\frac{d}{1+a^2}\geq 2$ tengsizlikni isbotlang.

112. $a,b,c,d>0$ va $a+b+c+d=4$ bo'lsa, $\frac{a}{1+b^2c}+\frac{b}{1+c^2d}+\frac{c}{1+d^2a}+\frac{d}{1+a^2b}\geq 2$ tengsizlikni isbotlang.

113. $a,b,c,d>0$ va $a+b+c+d=4$ bo'lsa, $\frac{a+1}{1+b^2}+\frac{b+1}{1+c^2}+\frac{c+1}{1+d^2}+\frac{d+1}{1+a^2}\geq 4$ tengsizlikni isbotlang.

114. $a,b,c,d>0$ va $a+b+c+d=4$ bo'lsa, $\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}+\frac{1}{1+d^2}\geq 2$ tengsizlikni isbotlang.

115. $a,b,c>0$ bo'lsa, $\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+a^2}\geq\frac{a+b+c}{2}$ tengsizlikni isbotlang.

116. $a,b,c,d>0$ bo'lsa, $\frac{a^4}{a^3+2b^3}+\frac{b^4}{b^3+2c^3}+\frac{c^4}{c^3+2d^3}+\frac{d^4}{d^3+2a^3}\geq\frac{a+b+c+d}{2}$ tengsizlikni isbotlang.

117. $a,b,c>0$ va $abc=1$ bo'lsa, $\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1$ tengsizlikni isbotlang.

118. $S_n=\sum_{k=1}^{n}\frac{1}{k}$ bo'lsa, $\sum_{k=1}^{n}\frac{1}{k\cdot S_k^2}<2$ tengsizlikni isbotlang.

119. $x, y, z \in R$ bo'lsa, $\frac{x^2+y^2+1}{2x^2+1}+\frac{y^2+z^2+1}{2y^2+1}+\frac{z^2+x^2+1}{2z^2+1}\geq 3$ tengsizlikni isbotlang.

120. $a>0, b>0, c>0$ bo'lsa, $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\leq\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ tengsizlikni isbotlang.

121. Agar $a>0, b>0, c>0$ bo'lsa, .$(3a+2b+c)\left(\frac{3}{a}+\frac{2}{b}+\frac{1}{c}\right)\geq 36$ tengsizlikni isbotlang.

122. Agar $a>0, b>0, c>0$ bo'lsa,

$$\sqrt{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\geq\frac{1}{2\sqrt{2}}(a+b)(b+c)(c+a)$$

tengsizlikni isbotlang.

123. Agar $a>0, b>0, c>0$ bo'lsa, $a^3+b^3+c^3\geq a^2b+b^2c+c^2a$ tengsizlikni isbotlang.

124. Agar $a>0, b>0, c>0$ bo'lsa, $2(a^3+b^3+c^3)\geq ab(a+b)+bc(b+c)+ca(c+a)$ tengsizlikni isbotlang.

125. Agar $a>0, b>0, c>0$ bo'lsa, $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{a+b+c}{2}$ tengsizlikni isbotlang.

126. Agar $a>0, b>0, c>0$ bo'lsa, $\frac{a^2+b^2}{2c}+\frac{b^2+c^2}{2a}+\frac{c^2+a^2}{2b}\geq a+b+c$ tengsizlikni isbotlang.

127. Agar $a>0, b>0, c>0$ bo'lsa, $\frac{a^2+b^2}{2c}+\frac{b^2+c^2}{2a}+\frac{c^2+a^2}{2b}\leq\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}$ tengsizlikni isbotlang.

128. Agar $a>0, b>0, c>0$ bo'lsa, $\frac{a^3+b^3+c^3}{a+b+c}\geq\frac{a^2+b^2+c^2}{3}$ tengsizlikni isbotlang.

129. Agar $a\neq 0, b\neq 0$ bo'lsa, $a^4+b^4\leq\frac{a^6}{b^2}+\frac{b^6}{a^2}$ tengsizlikni isbotlang.

130. Agar $a>0, b>0$ bo'lsa, $\frac{1}{a^3}+\frac{1}{b^3}\leq\frac{1}{a^3}\sqrt{\frac{b}{a}}+\frac{1}{b^3}\sqrt{\frac{a}{b}}$ tengsizlikni isbotlang.

131. Agar $a>0, b>0$ bo'lsa, $\frac{a^3}{b}+\frac{b^3}{a}\geq a^2+b^2$ tengsizlikni isbotlang.

132. Agar $a_1, a_2, ..., a_n$ yig'indisi birga teng bo'lgan ixtiyoriy musbat sonlar bo'lsa, $\frac{a_1}{\sqrt{1-a_1}}+\frac{a_2}{\sqrt{1-a_2}}+...+\frac{a_n}{\sqrt{1-a_n}} \geq \sqrt{\frac{n}{n-1}}$ tengsizlikni isbotlang.

133. Agar a, b, c musbat haqiqiy sonlar bo'lsa, $\frac{a^3+b^3+c^3}{a^2+b^2+c^2} \geq \frac{a+b+c}{3}$ tengsizlikni isbotlang.

134. Agar a, b, c musbat haqiqiy sonlar bo'lsa, $a^7+b^7+c^7 \geq a^2b^2c^2(a+b+c)$ tengsizlikni isbotlang.

135. Agar a, b, c musbat haqiqiy sonlar bo'lsa, $9(a^3+b^3+c^3) \geq (a+b+c)^3$ tengsizlikni isbotlang.

136. Agar $a>0, b>0, c>0$ bo'lsa, $(a^ab^bc^c)^2 \geq a^{b+c} \cdot b^{a+c} \cdot c^{a+b}$ tengsizlikni isbotlang.

137. Agar $a \geq 0, b \geq 0$ bo'lsa, $a^3(b+1)+b^3(a+1) \geq a^2(b+b^2)+b^2(a+a^2)$ tengsizlikni isbotlang.

138. Agar $a>b>c>0$ haqiqiy sonlar bo'lsa, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}<\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$ tengsizlikni isbotlang.

139. Agar a, b haqiqiy sonlar bo'lsa, $2a^4+2b^4 \geq ab(a+b)^2$ tengsizlikni isbotlang.

140. Agar a, b, c musbat haqiqiy sonlar bo'lsa,

$$\frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(b+c)(b+a)}+\frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}$$

tengsizlikni isbotlang.

141. Agar $a^2+b^2+ab+bc+ac<0$ bo'lsa, $a^2+b^2<c^2$ tengsizlikni isbotlang.

142. Agar ko'phad $P(t)=t^2-4t$ ko'rinishda va $x \geq 1, y \geq 1$ bo'lsa, $P(x^2+y^2) \geq P(2xy)$ tengsizlikni isbotlang.

3-§. Geometrik mazmundagi tengsizliklarni o'rta qiymatlar orasidagi munosabatlardan foydalanib isbotlash

Tayanch so'zlar: *Koshi tengsizligi, Koshi-Bunyakovskiy-Shvars tengsizligi, Gyugens tengsizligi, Bernulli tengsizligi, o'rta arifmetik, o'rta geometrik, o'rta kvadratik, musbat sonlar, yuza, uchburchak, perimetr, radius, aylana, Geron formulasi, kosinuslar teoremasi, sinuslar teoremasi, Pifagor teoremasi, Menelay teoremasi, Cheva teoremasi, to'g'ri chiziq, gipotenuza, katet, mediana, bissektrisa, balandlik, burchak.*

Mazkur paragrafda uchburchaklarda uchraydigan tengsizliklarni AM-GM tengsizligi yordamida isbotlashning o'quvchilarga tez va tushunarli bo'ladigan usulini keltiramiz.

1-masala. Agar ΔABC da $p = a + b + c$ bo'lsa, $a^2 + b^2 + c^2 \geq \frac{p^2}{3}$ tengsizlikni isbotlang.

Isbot. $2ab \leq a^2 + b^2$ tengsizlikdan fodalanamiz.

$$\frac{p^2}{3} = \frac{(a+b+c)^2}{3} = \frac{a^2 + b^2 + c^2 + 2ab + 2bc + 2ac}{3} \leq \frac{a^2 + b^2 + c^2 + 2(a^2 + b^2 + c^2)}{3} = a^2 + b^2 + c^2$$

2-masala. Agar ΔABC da $p = a + b + c$ bo'lsa, $a^3 + b^3 + c^3 \geq \frac{p^3}{9}$ tengsizlikni isbotlang.

Isbot. Yuqoridagi 1-masaladagi va $3abc \leq a^3 + b^3 + c^3$ tengsizliklardan fodalanamiz.

$$\frac{p^3}{9} = \frac{p}{3} \cdot \frac{p^2}{3} \leq \frac{p}{3} \cdot \left(a^2 + b^2 + c^2\right) = \frac{(a+b+c)\left(a^2 + b^2 + c^2\right)}{3} =$$

$$= \frac{a^3 + b^3 + c^3 + abb + acc + baa + bcc + caa + cbb}{3} \leq \frac{a^3 + b^3 + c^3 + 2\left(a^3 + b^3 + c^3\right)}{3} = a^3 + b^3 + c^3$$

3-masala. Agar p – uchburchakning yarim perimetri, S – yuzasi bo'lsa, $a^3 + b^3 + c^3 \geq \frac{8\sqrt{3}}{3} Sp$ tengsizlikni isbotlang.

Isbot. 2-masalada p – yarim perimetr deb olsak, $a^3 + b^3 + c^3 \geq \frac{8p^3}{9}$ o'rinli bo'ladi. Endi quyidagi tengsizlikdan ham foydalanamiz:

$x + y + z \geq 3\sqrt[3]{xyz},\ x = p - a,\ y = p - b,\ z = p - c$ desak, quyidagini hosil qilamiz:

$$p - a + p - b + p - c \geq 3\sqrt[3]{(p-a)(p-b)(p-c)} \Rightarrow \frac{p^3}{27} \geq (p-a)(p-b)(p-c).$$

$$\frac{8\sqrt{3}}{3} Sp = \frac{8\sqrt{3}}{3} p\sqrt{p(p-a)(p-b)(p-c)} \leq \frac{8\sqrt{3}}{3} p\sqrt{p \cdot \frac{p^3}{27}} = \frac{8\sqrt{3}}{3} \cdot \frac{p^3}{3\sqrt{3}} = \frac{8p^3}{9} \leq a^3 + b^3 + c^3$$

4-masala. Ixtiyoriy uchburchak uchun $a^3 + b^3 + c^3 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 4p$ tengsizlikni isbotlang. Bu yerda p – yarim perimetr.

Isbot. Bizga ma'lumki $a^3+\frac{1}{a}\geq 2a,\ b^3+\frac{1}{b}\geq 2b,\ c^3+\frac{1}{c}\geq 2c$ tengsizliklar o'rinli. Bu tengsizliklarni hadma-had qo'shib isbotlash talab etilgan tengsizlikni hosil qilamiz.

5-masala. ΔABC da R – tashqi, r – ichki chizilgan aylana radiuslari va p – yarim perimetri bo'lsa, $\frac{1}{R^2}\leq\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\leq\frac{1}{4r^2}$ tengsizlikni isbotlang

Isbot. Dastlab quyidagi $(p-a)(p-b)(p-c)\leq\frac{abc}{8}$ tengsizlikni isbotlaylik.

$$(p-a)(p-b)(p-c)=\frac{2\sqrt{(p-a)(p-b)}\cdot 2\sqrt{(p-b)(p-c)}\cdot 2\sqrt{(p-c)(p-a)}}{8}\leq$$

$$\leq\frac{(p-a+p-b)(p-b+p-c)(p-c+p-a)}{8}=\frac{abc}{8}$$

Avval tengsizlikning chap qismini isbotlaymiz:

$$\frac{1}{R^2}\leq\frac{16S^2}{(abc)^2}=\frac{16p(p-a)(p-b)(p-c)}{(abc)^2}\leq\frac{16p\cdot\frac{abc}{8}}{(abc)^2}=\frac{2p}{abc}=\frac{a+b+c}{abc}=\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$$

Endi o'ng qismini isbotlaymiz, ya'ni

$$\frac{1}{4r^2}=\frac{p^2}{4S^2}=\frac{p^2}{4p(p-a)(p-b)(p-c)}\geq\frac{p}{4\cdot\frac{abc}{8}}=\frac{2p}{abc}=\frac{a+b+c}{abc}=\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$$

6-masala.Uchburchakka tashqi chizilgan aylananing radiusi ichki chizilgan aylananing radiusini ikkilanganidan kichikmas ekanligini isbotlang, ya'ni $R\geq 2r$.

Isbot. Yuqoridagi 5-masaladagi tengsizlikdan foydalanamiz.

$$\frac{1}{R^2}\leq\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\leq\frac{1}{4r^2}\Rightarrow\frac{1}{R^2}\leq\frac{1}{4r^2}\Rightarrow R\geq 2r$$

7-masala. a,b,c lar to'g'ri burchakli uchburchakning tomonlari bo'lsa, (c gipotenuza) $ab(a+b+c)<\frac{5}{4}c^3$ tengsizlikning isbotlang.

Isbot. $ab\leq\frac{a^2+b^2}{2}=\frac{c^2}{2}$ va $a+b=\sqrt{a^2+b^2+2ab}\leq\sqrt{c^2+2\cdot\frac{c^2}{2}}=\sqrt{2}c$

Endi

$$ab(a+b+c)<\frac{c^2}{2}\cdot\left(\sqrt{2}c+c\right)=\frac{\sqrt{2}+1}{2}\cdot c^3=\frac{2\sqrt{2}+2}{4}\cdot c^3<\frac{3+2}{4}\cdot c^3=\frac{5}{4}c^3$$

8-masala. a,b,c lar uchburchakning tomonlari bo'lsa, $a^3+b^3+3abc>c^3$ tengsizlikning isbotlang.

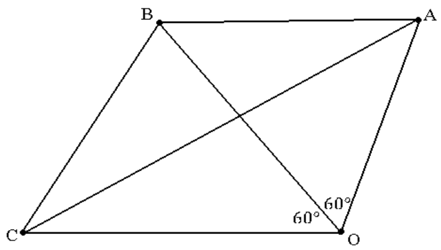
Isbot. Tengsizlikni isbotlash uchun $a+b>c$ va $a^2-ab+b^2>0$ lardan foydalanamiz:

$$a^3+b^3+3abc=(a+b)(a^2-ab+b^2)+3abc>c(a^2-ab+b^2)+3abc=$$

$$=a^2c-abc+b^2c+3abc=c(a^2+2ab+b^2)=c(a+b)^2>c\cdot c^2=c^3$$

9-masala. Agar $a,b,c>0$ bo'lsa, $\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}\geq\sqrt{a^2+ac+c^2}$ tengsizlikni isbotlang.

Isbot. $OA=a,\ OB=b,\ OC=c$ deylik. $\angle AOB=60°,\ \angle BOC=60°$ qilib olamiz, u holda $\angle AOC=120°$. $\Delta OAB,\ \Delta OBC$ va ΔOAC larda kosinuslar teoremasiga binoan:



$$AB^2=OA^2+OB^2-2\cdot OA\cdot OB\cdot\cos 60°=a^2+b^2-ab,$$

$$BC^2=OB^2+OC^2-2\cdot OB\cdot OC\cdot\cos 60°=b^2+c^2-bc,$$

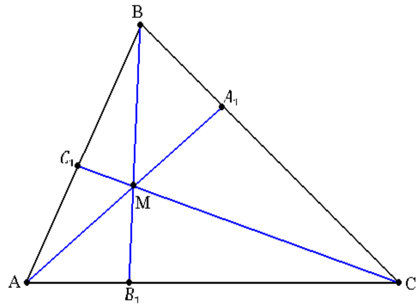
$$AC^2=OA^2+OC^2-2\cdot OA\cdot OC\cdot\cos 120°=a^2+c^2+ac.$$

Bundan $AB=\sqrt{a^2-ab+b^2},\ BC=\sqrt{b^2-bc+c^2},\ AC=\sqrt{a^2+ac+c^2}$. ΔABC da $AB+BC>AC$. Agar B nuqta AC ustida bo'lib qolsa, u holda $AB+BC=AC$ bo'ladi.

Demak, $AB+BC\geq AC$. Bu tengsizlikka AB, BC, AC ning yuqorida topilgan qiymatlarini qo'ysak, isbot qilinishi talab etilgan tengsizlikni hosil qilamiz.

10-masala. ABC uchburchakning ichida ixtiyoriy M nuqta olingan va bu nuqtadan AM, BM, CM to'g'ri chiziqlar o'tkazilgan. Bu to'g'ri chiziqlar uchburchak tomonlarini mos ravishda A_1, B_1, C_1 nuqtalarda kesib o'tadi. $\frac{AM}{A_1M}+\frac{BM}{B_1M}+\frac{CM}{C_1M}\geq 6$ tengsizlikni isbotlang.

Isbot. Menelay teoremasiga asosan:



$$\Delta ABB_1 \text{ da } \frac{AC_1}{C_1B}\cdot\frac{BM}{B_1M}\cdot\frac{B_1C}{AC}=1 \Rightarrow \frac{BM}{B_1M}=\frac{AC\cdot C_1B}{AC_1\cdot B_1C}$$

$$\Delta ABA_1 \text{ da } \frac{BC_1}{AC_1}\cdot\frac{AM}{A_1M}\cdot\frac{A_1C}{BC}=1 \Rightarrow \frac{AM}{A_1M}=\frac{BC\cdot AC_1}{BC_1\cdot A_1C}$$

$$\Delta ACC_1 \text{ da } \frac{AB_1}{B_1C}\cdot\frac{CM}{C_1M}\cdot\frac{C_1B}{AB}=1 \Rightarrow \frac{CM}{C_1M}=\frac{AB\cdot B_1C}{AB_1\cdot C_1B}$$

tengliklar o'rinli. Endi Cheva teoremasiga ko'ra $\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}=1$ tenglik ham o'rinli. Yuqoridagi tengliklarni hadlab qo'shib

$$\frac{AM}{A_1M}+\frac{BM}{B_1M}+\frac{CM}{C_1M}=\frac{BC\cdot AC_1}{BC_1\cdot A_1C}+\frac{AC\cdot C_1B}{AC_1\cdot B_1C}+\frac{AB\cdot B_1C}{AB_1\cdot C_1B}\geq 3\sqrt[3]{\frac{BC\cdot AC_1}{BC_1\cdot A_1C}\cdot\frac{AC\cdot C_1B}{AC_1\cdot B_1C}\cdot\frac{AB\cdot B_1C}{AB_1\cdot C_1B}}=$$

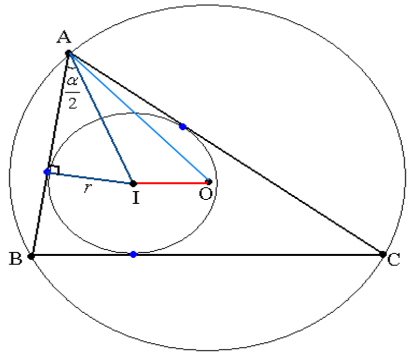
$$=3\sqrt[3]{\frac{AC\cdot BC\cdot AB}{A_1C\cdot AB_1\cdot BC_1}}=3\sqrt[3]{\frac{AB_1+B_1C}{AB_1}\cdot\frac{BA_1+A_1C}{AC_1}\cdot\frac{AC_1+C_1B}{BC_1}}=3\sqrt[3]{\left(1+\frac{B_1C}{AB_1}\right)\left(1+\frac{BA_1}{AC_1}\right)\left(1+\frac{AC_1}{BC_1}\right)}\geq$$

$$\geq 3\sqrt[3]{8\sqrt{\frac{B_1C}{AB_1}\cdot\frac{BA_1}{AC_1}\cdot\frac{AC_1}{BC_1}}}=3\cdot 2=6$$

isbotlash talab etilgan tengsizlikni hosil qilamiz.

11-masala. Ixtiyoriy uchburchak uchun $\frac{2R}{r}\geq\frac{1}{\sin\frac{\alpha}{2}\left(1-\sin\frac{\alpha}{2}\right)}$ tengsizlikni isbotlang.

Isbot. Chizmadan quyidagilarni aniqlaymiz: $AI=\dfrac{r}{\sin\dfrac{\alpha}{2}}\Rightarrow\sin\dfrac{\alpha}{2}=\dfrac{r}{AI}$.



ΔAIO dan: $OI\ge|AI-AO|\Rightarrow OI^2\ge(AI-R)^2\Rightarrow R^2-2Rr\ge AI^2-2R\cdot AI+R^2\Rightarrow$

$$\Rightarrow 2R(AI-r)\ge AI^2\Rightarrow 2R\ge\frac{AI^2}{AI-r}\Rightarrow\frac{2R}{r}\ge\frac{AI^2}{AI\cdot r-r^2}\Rightarrow\frac{2R}{r}\ge\frac{1}{\dfrac{r}{AI}-\left(\dfrac{r}{AI}\right)^2}=$$

$$=\frac{1}{\sin\dfrac{\alpha}{2}-\sin^2\dfrac{\alpha}{2}}\Rightarrow\frac{2R}{r}\ge\frac{1}{\sin\dfrac{\alpha}{2}\left(1-\sin\dfrac{\alpha}{2}\right)}$$

12-masala. Ixtiyoriy uchburchak uchun quyidagi tengsizlikni isbotlang.

$$h_a+h_b+h_c\le\frac{9R}{2}$$

Isbot. Bizga ma'lumki, quyidagi tengliklar o'rinli:

$$a=2R\sin\alpha,\quad h_a=c\sin\beta=2R\sin\beta\sin\gamma$$
$$b=2R\sin\beta,\quad h_b=a\sin\gamma=2R\sin\alpha\sin\gamma$$
$$c=2R\sin\gamma,\quad h_c=b\sin\alpha=2R\sin\alpha\sin\beta$$

$$h_a+h_b+h_c\le\frac{9R}{2}\Rightarrow\frac{h_a}{R}+\frac{h_b}{R}+\frac{h_c}{R}\le\frac{9}{2}\Leftrightarrow 2\sin\gamma\sin\beta+2\sin\alpha\sin\gamma+2\sin\alpha\sin\beta\le\frac{9}{2}\Rightarrow$$

$$\Rightarrow\sin\gamma\sin\beta+\sin\alpha\sin\gamma+\sin\alpha\sin\beta\le\frac{9}{4}.$$

$$\sin\gamma\sin\beta+\sin\alpha\sin\gamma+\sin\alpha\sin\beta\le\sin^2\alpha+\sin^2\beta+\sin^2\gamma\le\frac{9}{4}.$$

Oxirgi tengsizlik esa o'rinli.

13-masala. Ixtiyoriy uchburchak uchun quyidagi tengsizlikni isbotlang.

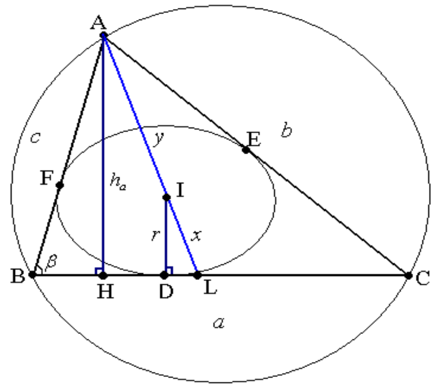
$$9 \le \frac{h_a}{r} + \frac{h_b}{r} + \frac{h_c}{r}$$

Isbot. Bizga ma’lumki, uchburchakning bissektrisalar kesishish nuqtasi unga ichki chizilgan aylana markazida bo’ladi. Chizmadan quyidagilarni aniqlaymiz:

BL – bissektrisa. $BL = \frac{ac}{b+c} \Rightarrow \frac{y}{x} = \frac{c}{BL} = \frac{b+c}{a}$.

$$\Delta AHL \sim \Delta IDL \Rightarrow \frac{h_a}{r} = \frac{x+y}{x} = 1 + \frac{y}{x} = 1 + \frac{b+c}{a}$$

Xuddi shunday, $\frac{h_b}{r} = 1 + \frac{a+c}{b}$, $\frac{h_c}{r} = 1 + \frac{a+b}{c}$.



$$9 \le \frac{h_a}{r} + \frac{h_b}{r} + \frac{h_c}{r} \Leftrightarrow 6 \le \frac{a+c}{b} + \frac{a+b}{c} + \frac{b+c}{a} \Leftrightarrow \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) \ge 2 + 2 + 2 = 6.$$

14-masala. Ixtiyoriy uchburchak uchun quyidagi tengsizlikni isbotlang.

$$9r^2 \le (p-a)(p-b) + (p-b)(p-c) + (p-c)(p-a)$$

Isbot. Belgilashlar kiritamiz: $a = y+z,\ b = x+z,\ c = x+y$. Bundan esa

$p-a = x,\ p-b = y,\ p-c = z,\ S = \sqrt{(x+y+z)xyz},\ r = \frac{S}{p} = \sqrt{\frac{xyz}{x+y+z}}$ ekanligi kelib chiqadi.

$$9r^2 \le (p-a)(p-b) + (p-b)(p-c) + (p-c)(p-a) \Leftrightarrow \frac{9xyz}{x+y+z} \le xy + yz + zx \Leftrightarrow$$

$$\Leftrightarrow 9xyz \le (x+y+z)(xy+yz+zx)$$

Oxirgi tengsizlikning isboti esa $x+y+z\geq 3\sqrt[3]{xyz}$ va $xy+yz+zx\geq 3\sqrt[3]{x^2y^2z^2}$ tengsizliklarni hadma-had ko'paytirishdan kelib chiqadi.

15-masala. Ixtiyoriy uchburchak uchun quyidagi tengsizlikni isbotlang.

$$(p-a)(p-b)+(p-b)(p-c)+(p-c)(p-a)\leq\frac{9R^2}{4}$$

Isbot. Bizga ma'lumki $\sin\gamma\sin\beta+\sin\alpha\sin\gamma+\sin\alpha\sin\beta\leq\frac{9}{4}$ tengsizlik o'rinli.

$$(p-a)(p-b)+(p-b)(p-c)+(p-c)(p-a)\leq\frac{9R^2}{4}\Leftrightarrow$$

$$\Leftrightarrow\frac{(b+c-a)(a+c-b)}{4}+\frac{(a+c-b)(a+b-c)}{4}+\frac{(a+b-c)(b+c-a)}{4}\leq\frac{9R^2}{4}\Leftrightarrow$$

$$\Leftrightarrow\left(\frac{b}{2R}+\frac{c}{2R}-\frac{a}{2R}\right)\left(\frac{a}{2R}+\frac{c}{2R}-\frac{b}{2R}\right)+\left(\frac{a}{2R}+\frac{c}{2R}-\frac{b}{2R}\right)\left(\frac{a}{2R}+\frac{b}{2R}-\frac{c}{2R}\right)+$$

$$+\left(\frac{a}{2R}+\frac{b}{2R}-\frac{c}{2R}\right)\left(\frac{b}{2R}+\frac{c}{2R}-\frac{a}{2R}\right)\leq\frac{9}{4}\Leftrightarrow\sum_{cyc}(\sin\beta+\sin\gamma-\sin\alpha)(\sin\alpha+\sin\gamma-\sin\beta)\leq\frac{9}{4}\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\left(\sin^2\gamma-\sin^2\alpha-\sin^2\beta+2\sin\alpha\sin\beta\right)\leq\frac{9}{4}\Leftrightarrow$$

$$\Leftrightarrow 2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha\leq\frac{9}{4}+\sin^2\alpha+\sin^2\beta+\sin^2\gamma$$

tengsizlikni isbotlash yetarli. Bu tengsizlikni isboti esa

$$\begin{cases}\sin\gamma\sin\beta+\sin\alpha\sin\gamma+\sin\alpha\sin\beta\leq\frac{9}{4}\\ \sin\gamma\sin\beta+\sin\alpha\sin\gamma+\sin\alpha\sin\beta\leq\sin^2\alpha+\sin^2\beta+\sin^2\gamma\end{cases}$$

tengsizliklardan kelib chiqadi.

16-masala. Ixtiyoriy uchburchak uchun quyidagi tengsizlikni isbotlang.

$$27r^2\leq h_ah_b+h_bh_c+h_ch_a$$

Isbot. Quyidagi tengliklardan o'rinli: $h_a=\frac{2S}{a},h_b=\frac{2S}{b},h_c=\frac{2S}{c},r=\frac{2S}{a+b+c}$.

$$27\cdot\frac{4S^2}{(a+b+c)^2}\leq\sum_{cyc}\frac{4S^2}{ab}\Leftrightarrow 27\cdot\frac{4S^2}{(a+b+c)^2}\leq\frac{4S^2(a+b+c)}{abc}\Leftrightarrow 27abc\leq(a+b+c)^3.$$

Oxirgi tengsizlik esa *AM-GM* tengsizligiga ko'ra o'rinli.

17-masala. Ixtiyoriy uchburchak uchun quyidagi tengsizlikni isbotlang.

$$h_a h_b + h_b h_c + h_c h_a \le \frac{27R^2}{4}$$

Isbot. Quyidagi tengliklardan o'rinli:

$$h_a = c\sin\beta = 2R\sin\beta\sin\gamma,\ h_b = 2R\sin\alpha\sin\gamma,\ h_c = 2R\sin\alpha\sin\beta.$$

$$\sum_{cyc} h_a h_b = \sum_{cyc} 4R^2 \sin\alpha\sin\beta\sin^2\gamma \le \frac{27R^2}{4} \Leftrightarrow$$

$$\Leftrightarrow 4R^2 \sin\alpha\cdot\sin\beta\cdot\sin\gamma(\sin\alpha+\sin\beta+\sin\gamma) \le \frac{27R^2}{4} \Leftrightarrow$$

$$\Leftrightarrow \sin\alpha\cdot\sin\beta\cdot\sin\gamma(\sin\alpha+\sin\beta+\sin\gamma) \le \frac{27}{16}$$

tengsizlikni isbotlash yetarli. Biz bilamizki $\sin^2\alpha+\sin^2\beta+\sin^2\gamma \le \frac{9}{4}$ tengsizlik o'rinli.

$$\sin^2\alpha\cdot\sin^2\beta\cdot\sin^2\gamma \le \frac{(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)^3}{27} \le \frac{\frac{9^3}{4^3}}{27} = \frac{27}{64} \qquad (1)$$

$$(\sin\alpha+\sin\beta+\sin\gamma)^2 \le 3(\sin^2\alpha+\sin^2\beta+\sin^2\gamma) \le \frac{27}{4} \qquad (2)$$

(1) va (2) ni hadma-had ko'paytirsak:

$$\sin^2\alpha\cdot\sin^2\beta\cdot\sin^2\gamma(\sin\alpha+\sin\beta+\sin\gamma)^2 \le \frac{27^2}{16^2} \Rightarrow \sin\alpha\cdot\sin\beta\cdot\sin\gamma(\sin\alpha+\sin\beta+\sin\gamma) \le \frac{27}{16}$$

18-masala. Ixtiyoriy uchburchak uchun quyidagi tengsizlikni isbotlang.

$$\frac{1}{p-a}+\frac{1}{p-b}+\frac{1}{p-c} \ge \frac{2\sqrt{3}}{R}$$

Isbot. Belgilashlar kiritamiz: $a = y+z,\ b = x+z,\ c = x+y$. Bundan esa

$$p-a = x,\ p-b = y,\ p-c = z,\ S = \sqrt{(x+y+z)xyz},\ R = \frac{abc}{4S} = \frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}$$

ekanligi kelib chiqadi. Biz bilamizki, $(p-a)(p-b)+(p-b)(p-c)+(p-c)(p-a)\leq\frac{9R^2}{4}$ o'rinli. Budan

$$xy+yz+xz\leq\frac{9(x+y)^2(y+z)^2(z+x)^2}{64xyz(x+y+z)}\Leftrightarrow$$

$$\Leftrightarrow(x+y)^2(y+z)^2(z+x)^2\geq\frac{64}{9}(xy+yz+zx)(x+y+z)xyz \qquad (1)$$

$$\frac{2\sqrt{3}}{R}\leq\frac{1}{p-a}+\frac{1}{p-b}+\frac{1}{p-c}\Leftrightarrow\frac{8\sqrt{3xyz(x+y+z)}}{(x+y)(y+z)(z+x)}\leq\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Rightarrow$$

$$\Rightarrow\frac{8\sqrt{3x^3y^3z^3(x+y+z)}}{xy+yz+zx}\leq(x+y)(y+z)(z+x)\Leftrightarrow\frac{64\cdot3x^3y^3z^3(x+y+z)}{(xy+yz+zx)^2}\leq(x+y)^2(y+z)^2(z+x)^2$$

$$(xy+yz+zx)^3\geq27x^2y^2z^2\Rightarrow\frac{27x^2y^2z^2}{(xy+yz+zx)^3}\leq1 \qquad (2)$$

(2) dan

$$\frac{64}{9}(x+y+z)(xy+yz+zx)xyz\cdot\frac{27x^2y^2z^2}{(xy+yz+zx)^3}\leq\frac{64}{9}(x+y+z)(xy+yz+zx)xyz$$

ni hosil qilamiz. Bundan esa

$$\frac{64}{9}(x+y+z)(xy+yz+zx)xyz\leq(x+y)^2(y+z)^2(z+x)^2$$

tengsizlikni isbotlash yetarli. Bu tengsizlikning (1) asosan o'rinli.

19-masala. Ixtiyoriy uchburchak uchun quyidagi tengsizlikni isbotlang.

$$\frac{1}{p-a}+\frac{1}{p-b}+\frac{1}{p-c}\geq\frac{3R}{2\sqrt{3}r^2}$$

Isbot. 18-masaladagi belgilashlarga ko'ra tengsizlikni quyidagicha yozamiz:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\leq\frac{\sqrt{3}\cdot\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}}{2\cdot\frac{xyz}{x+y+z}}\Leftrightarrow8\sqrt{xyz}\cdot(xy+yz+zx)\leq\sqrt{3}(x+y)(y+z)(z+x)\sqrt{x+y+z}$$

$$64(xy+yz+zx)^2\cdot xyz\leq3(x+y)^2(y+z)^2(z+x)^2(x+y+z)\Rightarrow$$

$$\Rightarrow \frac{64}{3} \cdot \frac{(xy+yz+zx)^2}{x+y+z} \cdot xyz \le (x+y)^2(y+z)^2(z+x)^2$$

ekanligini isbotlaymiz.

$$3(xy+yz+zx) \le (x+y+z)^2 \Rightarrow \frac{3(xy+yz+zx)}{(x+y+z)^2} \le 1$$

tengsizlikni qo'llab

$$\frac{64}{9} \cdot (xy+yz+zx)(x+y+z) \cdot xyz \cdot \frac{3(xy+yz+zx)}{(x+y+z)^2} \le (x+y)^2(y+z)^2(z+x)^2 \Rightarrow$$

$$\frac{64}{9}(x+y+z)(xy+yz+zx)xyz \le (x+y)^2(y+z)^2(z+x)^2$$

tengsizlikni hosil qilamiz. Bu tengsizlik esa doim o'rinli.

20-masala. Agar a,b,c lar to'g'ri burchakli uchburchakning tomonlari bo'lsa, (c – gipotenuza) ushbu $ab(a+b+c) < \frac{5}{4}c^3$ tengsizlikni isbotlang.

Isbot. Bizga ma'lumki $\sin\alpha = \frac{a}{c}$, $\cos\alpha = \frac{b}{c}$, $\sin 2\alpha \le 1$ o'rinli. Bularga asosan tengsizligimiz

$$ab(a+b+c) < \frac{5}{4}c^3 \Rightarrow 4 \cdot \frac{a}{c} \cdot \frac{b}{c}\left(\frac{a}{c} + \frac{b}{c} + 1\right) < 5 \Rightarrow 4\sin\alpha\cos\alpha(\sin\alpha + \cos\alpha + 1) < 5$$

ko'rinishga keladi. Hosil bo'lgan tengsizlikni isbotlaymiz:

$$4\sin\alpha\cos\alpha(\sin\alpha+\cos\alpha+1) = 2\sin 2\alpha\left(\sqrt{1+\sin 2\alpha}+1\right) \le 2\left(\sqrt{2}+1\right) = 2\sqrt{2}+2 < 3+2 = 5.$$

Faollashtiruvchi savollar.

1. *Pifagor teoremasini ayting?*
2. *Kosinuslar teoremasini ayting?*
3. *Sinuslar teoremasini ayting?*
4. *Geron formulasini nima uchun qo'llaniladi?*
5. *Uchburchakning medianasi deb nimaga aytiladi?*
6. *Uchburchakning bissektrisasi deb nimaga aytiladi?*
7. *Uchburchakning balandligi deb nimaga aytiladi?*
8. *Uchburchakning ichki burchaklari yig'indisi nechchiga teng?*

Mustaqil yechish uchun masalalar

Ixtiyoriy uchbuchak uchun quyidagi tengsizliklarni isbotlang. Bu yerda a, b, c uchburchak tomonlari, α, β, γ – uchburchak ichki burchaklari, $h_a, h_b, h_c, m_a, m_b, m_c, l_a, l_b, l_c$ - mos ravishda balandlik, mediana va bissektrissa, R, r – mos ravishda uchburchakka tashqi va ichki chizilgan aylana radisulari, P, p – perimetr va yarim perimetr, S – yuzasi.

1. $h_a \le \sqrt{p(p-a)},\ h_b \le \sqrt{p(p-b)},\ h_c \le \sqrt{p(p-c)}$
2. $2\sqrt{(p-a)(p-b)} \le c,\ 2\sqrt{(p-b)(p-c)} \le a,\ 2\sqrt{(p-c)(p-a)} \le b$
3. $h_a^2 + h_b^2 + h_c^2 \le p^2$
4. $\sqrt{(p-a)(p-b)} + \sqrt{(p-b)(p-c)} + \sqrt{(p-c)(p-a)} \le p$
5. $p^2 \le \frac{27}{4}R^2$
6. $\frac{1}{p-a} + \frac{1}{p-b} + \frac{1}{p-c} \ge 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$
7. $R \ge \sqrt{\frac{abc}{a+b+c}}$
8. $\frac{h_a^2}{a^2} + \frac{h_b^2}{b^2} + \frac{h_c^2}{c^2} \ge \frac{9}{4}$
9. $\frac{2}{p-a} + \frac{2}{p-b} + \frac{2}{p-c} \ge \frac{4}{a} + \frac{4}{b} + \frac{4}{c}$
10. $x, y, z > 0$ bo’lsa, quyidagi tengsizlikni isbotlang:
$$\sqrt{x^2 + xy + y^2} + \sqrt{x^2 + xz + z^2} > \sqrt{y^2 + yz + z^2}$$
11. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge \frac{9}{P}$
12. $9r \le \sqrt{3}p$
13. $S \ge \sqrt{\frac{27}{2}Rr^3}$
14. $S > 2r\sqrt{Rr}$
15. $r^2 \le \frac{\sqrt{3}}{9}S$
16. $(p-a)(p-b) + (p-b)(p-c) + (p-c)(p-a) \ge \sqrt{3}S$
17. $a^2 + b^2 + c^2 \ge 4\sqrt{3}S$
18. $P^2 \ge 12\sqrt{3}S$
19. $a^4 + b^4 + c^4 \ge 16S^2$
20. $\frac{\sqrt{3}}{R} \le \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le \frac{\sqrt{3}R}{4r^2}$
21. $\sqrt{p-a} + \sqrt{p-b} + \sqrt{p-c} \le \sqrt{3p}$
22. $(a^2 + b^2 + c^2)(h_a^2 + h_b^2 + h_c^2) \ge 36S^2$

23. $\frac{1}{\sqrt{sin\alpha}}+\frac{1}{\sqrt{sin\beta}}+\frac{1}{\sqrt{sin\gamma}}\leq\frac{P}{\sqrt{2S}}$

24. $m_a^2+m_b^2+m_c^2\leq\frac{27R^2}{4}$

25. $a^2+b^2+c^2\leq 9R^2$

26. $\frac{a}{b+c-a}+\frac{b}{a+c-b}+\frac{c}{a+b-c}\geq 3$

27. $a^4+b^4+c^4+abc(a+b+c)\geq 2(a^2b^2+a^2c^2+b^2c^2)$

28. $sin\frac{\alpha}{2}\cdot sin\frac{\beta}{2}\cdot sin\frac{\gamma}{2}\leq\frac{1}{8}$

29. $sin\alpha\cdot sin\beta\cdot sin\gamma\leq\frac{3\sqrt{3}}{8}$

30. $cos\alpha+cos\beta+cos\gamma\leq\frac{3}{2}$

31. $cos\alpha\cdot cos\beta\cdot cos\gamma\leq\frac{1}{8}$

32. $cos^2\alpha+cos^2\beta+cos^2\gamma\geq\frac{3}{4}$

33. $h_a\leq\sqrt{bc}\cdot cos\frac{\alpha}{2}$

34. $S\geq\frac{l_al_bl_c}{p}$

35. $\frac{ab+bc+ac}{4S}\leq\sqrt{3}$

36. Agar *ABC* uchburchakning bissektrissalari *AD*, *BE*, va *CF* bo’lsa,

$$S_{DEF}\leq\frac{1}{4}\cdot S_{ABC}$$

tengsizlikni isbotlang.

37. $a,b,c>0$ bo’lsa, quyidagi tengsizlikni isbotlang:

$$\sqrt{(a+c)^2+b^2}+\sqrt{(a-c)^2+b^2}\geq 2\sqrt{a^2+b^2}$$

38. Agar $x,y,z>0$ bo’lsa, quyidagi tengsizlikni isbotlang.

$$\sqrt{x^2+y^2-xy}+\sqrt{x^2+z^2}\geq\sqrt{y^2+z^2+\sqrt{3}yz}$$

39. a,b,c uchburchak tomonlari bo’lsa, $\left|\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}\right|<\frac{1}{8}$ tengsizlikni isbotlang.

40. *ABC* uchburchakning *A*, *B*, *C* burchaklarining bissektrissalari qarama-qarshi tomonni, mos ravishda A_1,B_1,C_1 nuqtalarda kesib o’tadi. Uchburchakka tashqi chizilgan aylanani A_2,B_2,C_2 nuqtalarda kesadi. Quyidagi tengsizlikni isbotlang (tenglik qachon bajariladi?):

$$\frac{AA_1}{AA_2}+\frac{BB_1}{BB_2}+\frac{CC_1}{CC_2}\leq\frac{9}{4}$$

41. *ABC* uchburchakning AL_A, BL_B, CL_C burchaklarining bissektrissalari tushirilgan. Ularni davom ettirsak, ABC uchburchakka tashqi chizilgan aylana bilan mos ravishda A_1, B_1, C_1 nuqtalarda kesib o'tadi. Bissektrisalar kesishish nuqtai L bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$LA_1 + LB_1 + LC_1 \geq 6r$$

42. *ABC* uchburchakning AL_A, BL_B, CL_C burchaklarining bissektrissalari tushirilgan. Ularni davom ettirsak, ABC uchburchakka tashqi chizilgan aylana bilan mos ravishda A_1, B_1, C_1 nuqtalarda kesib o'tadi. Bissektrisalar kesishish nuqtasi L bo'lsa, u holda quyidagi tengsizliklarni isbotlang:

$$LA + LB + LC \geq 6r$$

43. $m_a \geq \frac{1}{2}\sqrt{a(8p-9a)}$
44. Uchburchakning ichidan olingan ixtiyoriy nuqtadan uning uchlarigacha bo'lgan masofalarning yig'indisi uchburchak yarim perimetridan katta ekanligini isbotlang.
45. Uchburchakning a, b, c tomonlari qarshisidagi burchaklari mos ravishda α, β, γ bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$2(cos^2\alpha + cos^2\beta + cos^2\gamma) \geq \frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2}$$

46. Qavariq $ABCD$ to'rtburchak uchun $AB + BD \leq AC + CD$ tengsizlik o'rinli bo'lsa, u holda $AB < AC$ ni isbotlang.

4-§. Vektorlarning tengsizliklarni isbotlashda qo'llanilishi

Tayanch so'zlar: *Koshi tengsizligi, Koshi-Bunyakovskiy-Shvars tengsizligi, uchburchak tengsizligi, perimetr, kosinuslar teoremasi, sinuslar teoremasi, Pifagor teoremasi, vector, skalyar ko'paytma, burchak, gradus, koordinata, modul.*

Bu paragrafda dastlab vektorlarning bir nechta xossalarini keltirib o'tamiz, so'ngra bu xossalarini tengsizliklarni isbotlashda tadbiqlaymiz.

Bizga uch o'lchovli fazoda $\vec{a}(a_1, a_2, a_3)$ va $\vec{b}(b_1, b_2, b_3)$ vektorlar berilgan bo'lsin. Bu vektorlarning moduli (uzunligi) quyidagi formula bilan hisoblanadi. $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$. Ularning yig'indisi (ayirmasi) $|\vec{c}|(c_1, c_2, c_3)$ quyidagicha hisoblanadi:

$c_1 = a_1 + b_1$, $c_2 = a_2 + b_2$, $c_3 = a_3 + b_3$ (mos ravishda $c_1 = a_1 - b_1$, $c_2 = a_2 - b_2$, $c_3 = a_3 - b_3$).

Agar noldan farqli ikkita vektor parallel to'g'ri chiziqlarda yoki bitta to'g'ri chiziqda joylashgan bo'lsa, kollinear vektorlar deyiladi.

Kollinear vektorlarda mos koordinatalari proporsional bo'ladi.

Ikkita $\vec{a}(a_1, a_2, a_3)$ va $\vec{b}(b_1, b_2, b_3)$ vektorlar uchun quyidagi tengsizliklar o'rinli: $|\vec{a}| + |\vec{b}| \geq |\vec{a} \pm \vec{b}|$, ya'ni

$$\sqrt{a_1^2 + a_2^2 + a_3^2} + \sqrt{b_1^2 + b_2^2 + b_3^2} \geq \sqrt{(a_1 \pm b_1)^2 + (a_2 \pm b_2)^2 + (a_3 \pm b_3)^2} \quad (1)$$

Bu formula koordinatalari *n* ta bo'lgan vektorlar uchun ham o'rinli:

$\vec{a}$ va $\vec{b}$ vektorlarning skalyar ko'paytmasi $\vec{a} \cdot \vec{b}$ o'zgarmas miqdor bo'lib, agar $\vec{a}(a_1, a_2, a_3)$ va $\vec{b}(b_1, b_2, b_3)$ koordinatalari bilan berilgan bo'lsa quyidagi formula bilan hisoblanadi:

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| cos\varphi \qquad (2)$$

Bu yerda φ - $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak.

(2) formuladan quyidagi tengsizlik kelib chiqadi:

$$\vec{a} \cdot \vec{b} \leq |\vec{a}| \cdot |\vec{b}| \qquad (3)$$

Agar $\vec{a}(a_1, a_2, a_3)$ va $\vec{b}(b_1, b_2, b_3)$ koordinatalari bilan berilgan bo'lsa, $\vec{a}$ va $\vec{b}$ vektorlarning skalyar ko'paytmasi yana bir formuladan foydalanib topishimiz mumkin:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad (4)$$

(2) va (4) formuladan osongina $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak kosinusini topishimiz mumkin:

$$cos\varphi = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}} \qquad (5)$$

(2) formuladan, $\vec{a}$ va $\vec{b}$ vektorlar faqatgina $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}|$ bo'lgandagina kollinear bo'ladi.

Tengsizliklarni isbotlashda vektorlarning xossalaridan foydalanish.

1-masala. Agar $x^2+y^2+z^2=4$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$\sqrt{x^4+1}+\sqrt{y^4+1}+\sqrt{z^4+1}\geq 5$$

Isbot. Tekislikda koordinatalari bilan berilgan uchta vektorlarni qaraymiz: $\vec{a}(x^2;1),\vec{b}(y^2;1),\vec{c}(z^2;1)$ u holda, $|\vec{a}|=\sqrt{x^4+1}$, $|\vec{b}|=\sqrt{y^4+1}$, $|\vec{c}|=\sqrt{z^4+1}$, $\vec{d}=\vec{a}+\vec{b}+\vec{c}$ ekanligidan foydalansak, u holda

$$|\vec{d}|=\sqrt{(x^2+y^2+z^2)^2+(1+1+1)^2}=\sqrt{4^2+3^2}=5$$

U holda $|\vec{a}|+|\vec{b}|+|\vec{c}|\geq|\vec{d}|$ ekanligidan tengsizlikdan $|\vec{a}|,|\vec{b}|,|\vec{c}|$ va $|\vec{d}|$ larning o'rniga qiymatlarni qo'yganimizda $\sqrt{x^4+1}+\sqrt{y^4+1}+\sqrt{z^4+1}\geq 5$ tengsizligi hosil bo'ladi. Tenglik vektorlar kollinear bo'lganda bajariladi.

2-masala. Agar $a+b+c=1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$\sqrt{7a+1}+\sqrt{7b+1}+\sqrt{7c+1}\leq\sqrt{30}$$

Isbot. Aytaylik ildiz ostidagi ifoda nomanfiy bo'lsin, quyidagi ikki vektorni qaraymiz: $\vec{p}(1;1;1)$ va $\vec{q}\left(\sqrt{7a+1};\sqrt{7b+1};\sqrt{7c+1}\right)$ bo'lsin. $\vec{p}$ va $\vec{q}$ vektorlar uchun $(\vec{p}\cdot\vec{q})^2\leq\vec{p}^2\cdot\vec{q}^2$ o'rinli. Bundan

$$\left(\sqrt{7a+1}+\sqrt{7b+1}+\sqrt{7c+1}\right)^2\leq(1+1+1)(7a+1+7b+1+7c+1)=$$

$$=3(7(a+b+c)+3)=3\cdot 10=30$$

yoki

$$\sqrt{7a+1}+\sqrt{7b+1}+\sqrt{7c+1}\leq\sqrt{30}$$

ekanligi kelib chiqadi.

3-masala. Agar $x_1+x_2+\cdots+x_n=sin\varphi$ va $y_1+y_2+\cdots+y_n=cos\varphi$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}+\cdots+\sqrt{x_n^2+y_n^2}\geq 1$$

Isbot. Koordinatalari

$$\vec{a}_1(x_1;y_1),\qquad\vec{a}_2(x_2;y_2),\ldots,\vec{a}_n(x_n;y_n)$$

bo'lgan vektorlar berilgan bo'lsin, u holda

$$|\vec{a}_1| = \sqrt{x_1^2 + y_1^2}, \qquad |\vec{a}_2| = \sqrt{x_2^2 + y_2^2}\,, \dots, |\vec{a}_n| = \sqrt{x_n^2 + y_n^2}$$

bo'ladi. Quyidagi vektorni olamiz. $\vec{a} = \vec{a}_1 + \vec{a}_2 + \dots + \vec{a}_n$ bundan

$$\vec{a}(x_1 + x_2 + \dots + x_n; y_1 + y_2 + \dots + y_n) \Rightarrow |\vec{a}| = \sqrt{sin^2\varphi + cos^2\varphi} = 1$$

Bizga ma'lumki,

$$|\vec{a}_1| + |\vec{a}_2| + \dots + |\vec{a}_n| \geq |\vec{a}_1 + \vec{a}_2 + \dots + \vec{a}_n| = |\vec{a}|$$

tengsizlik o'rinli. Bu tengsizlikdan esa

$$\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} + \dots + \sqrt{x_n^2 + y_n^2} \geq 1$$

ekanligi kelib chiqadi.

4-masala. Agar $m^2 + n^2 = a^2 + b^2 + c^2 = 1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$|ma + nb + c| \leq \sqrt{2}$$

Isbot. Koordinatalari $\vec{x}(m; n; 1)$ va $\vec{y}(a; b; c)$ bo'lgan vektorlarni olamiz. Bu vektorlarning skalyar ko'paytmasi $\vec{x} \cdot \vec{y} = ma + mb + c$ va modullari mos ravishda $|\vec{x}| = \sqrt{m^2 + n^2 + 1} = \sqrt{2}$ va $|\vec{y}| = \sqrt{a^2 + b^2 + c^2} = 1$ bo'ladi. $\vec{x} \cdot \vec{y} \leq |\vec{x}||\vec{y}|$ tengsizlikka asosan

$$|ma + nb + c| \leq \sqrt{2}$$

ekanligi kelib chiqadi.

5-masala. Agar $x_1 + x_2 + \dots + x_n = 6$ va $y_1 + y_2 + \dots + y_n = 8$ va $z_1 + z_2 + \dots + z_n = 10$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$\sqrt{x_1^2 + y_1^2 + z_1^2} + \sqrt{x_2^2 + y_2^2 + z_2^2} + \dots + \sqrt{x_n^2 + y_n^2 + z_n^2} \geq 10\sqrt{2}$$

Isbot. Uch o'lchovli fazoda *n* ta vektorlarni olamiz:

$$\vec{a}_1(x_1, y_1, z_1), \qquad \vec{a}_2(x_2, y_2, z_2), \dots, \vec{a}_n(x_n, y_n, z_n)$$

Endi ularning yig'indisidan iborat $\vec{a} = \vec{a}_1 + \vec{a}_2 + \dots + \vec{a}_n$ vektorni hosil qilamiz. Bu vektorning koordinatalari

$$\vec{a}(x_1 + x_2 + \dots + x_n; y_1 + y_2 + \dots + y_n; z_1 + z_2 + \dots + z_n) = (6,8,10)$$

dan iborat bo'ladi. Bizga ma'lumki

$$|\vec{a}_1| + |\vec{a}_2| + \dots + |\vec{a}_n| \geq |\vec{a}_1 + \vec{a}_2 + \dots + \vec{a}_n| = |\vec{a}|$$

tengsizlik o'rinli. Bu tengsizlikdan esa

$$\sqrt{x_1^2 + y_1^2 + z_1^2} + \sqrt{x_2^2 + y_2^2 + z_2^2} + \dots + \sqrt{x_n^2 + y_n^2 + z_n^2} \geq 10\sqrt{2}$$

ekanligi kelib chiqadi.

6-masala. Agar $a, b, c > 0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

Isbot. Har bir ildiz ostidagi ifodalarni quyidagicha yozib olamiz:

$$\sqrt{a^2 - ab + b^2} = \sqrt{b^2 - 2 \cdot b \cdot \frac{a}{2} + \frac{a^2}{4} + \frac{3a^2}{4}} = \sqrt{\left(b - \frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}a}{2}\right)^2}$$

$$\sqrt{b^2 - bc + c^2} = \sqrt{b^2 - 2 \cdot b \cdot \frac{c}{2} + \frac{c^2}{4} + \frac{3c^2}{4}} = \sqrt{\left(b - \frac{c}{2}\right)^2 + \left(\frac{\sqrt{3}c}{2}\right)^2}$$

Endi vektorlarni quyidagicha tanlab olamiz:

$\vec{x}\left(b - \frac{a}{2}; \frac{\sqrt{3}a}{2}\right)$ va $\vec{y}\left(\frac{c}{2} - b; \frac{\sqrt{3}c}{2}\right)$. Bu vektorlarni yig'indisi

$$\vec{z} = \vec{x} + \vec{y} \Rightarrow \vec{z}\left(\frac{c - a}{2}; \frac{\sqrt{3}}{2}(a + c)\right) \Rightarrow$$

$$\Rightarrow |\vec{z}| = \sqrt{\frac{1}{4}(c^2 - 2ac + a^2) + \frac{3}{4}(a^2 + 2ac + c^2)} = \sqrt{a^2 + ac + c^2}$$

Bizga ma'lumki, $|\vec{x}| + |\vec{y}| \geq |\vec{x} + \vec{y}| = |\vec{z}|$. Bu tengsizlikka topilgan qiymatlarini qo'ysak

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

tengsizlikning isboti kelib chiqadi.

7-masala. Agar α, β, γ ixtiyoriy uchburchakning ichki burchaklari bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$cos\alpha + cos\beta + cos\gamma \leq \frac{3}{2}$$

Isbot. Tekislikda bitta nuqtadan chiquvchi $\vec{r}_1, \vec{r}_2, \vec{r}_3$ birlik vektorlarni olamiz. Ular oralaridagi burchaklar $180° - \alpha, 180° - \beta, 180° - \gamma$ bo'lsin. Bizga ma'lumki ushbu

$$(\vec{r}_1 + \vec{r}_2 + \vec{r}_3)^2 \geq 0$$

tengsizlik o'rinli. Bunga ko'ra

$$\vec{r}_1^{\,2} + \vec{r}_2^{\,2} + \vec{r}_3^{\,2} + 2\vec{r}_1\vec{r}_2 + 2\vec{r}_1\vec{r}_3 + 2\vec{r}_2\vec{r}_3 \geq 0 \Rightarrow$$

$$\Rightarrow 3 + 2\cos(180° - \alpha) + 2\cos(\ 180° - \beta) + 2\cos(\ 180° - \gamma) \geq 0$$

$$\Rightarrow 3 - 2[\cos\alpha + \cos\beta + \cos\gamma] \geq 0 \Rightarrow cos\alpha + cos\beta + cos\gamma \leq \frac{3}{2}$$

tengsizlikning isboti kelib chiqadi.

8-masala. Agar α, β, γ ixtiyoriy uchburchakning ichki burchaklari bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$sin^2\alpha + sin^2\beta + sin^2\gamma \leq \frac{9}{4}$$

Isbot. Tekislikda bitta nuqtadan chiquvchi $\vec{r}_1, \vec{r}_2, \vec{r}_3$ birlik vektorlarni olamiz. Ular oralaridagi burchaklar $2\alpha, 2\beta, 2\gamma$ bo'lsin. Bizga ma'lumki ushbu

$$(\vec{r}_1 + \vec{r}_2 + \vec{r}_3)^2 \geq 0$$

tengsizlik o'rinli. Bunga ko'ra

$$\vec{r}_1^{\,2} + \vec{r}_2^{\,2} + \vec{r}_3^{\,2} + 2\vec{r}_1\vec{r}_2 + 2\vec{r}_1\vec{r}_3 + 2\vec{r}_2\vec{r}_3 \geq 0 \Rightarrow$$

$$\Rightarrow 3 + 2\cos 2\alpha + 2\cos 2\beta + 2\cos 2\gamma \geq 0 \Rightarrow$$

$$\Rightarrow 3 + 2[3 - 2sin^2\alpha - 2sin^2\beta - 2sin^2\gamma] \geq 0 \Rightarrow sin^2\alpha + sin^2\beta + sin^2\gamma \leq \frac{9}{4}$$

tengsizlikning isboti kelib chiqadi.

9-masala. Agar α, β, γ ixtiyoriy uchburchakning ichki burchaklari bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$sin\frac{\alpha}{2}+sin\frac{\beta}{2}+sin\frac{\gamma}{2}\leq\frac{3}{2}$$

Isbot. Tekislikda bitta nuqtadan chiquvchi $\vec{r}_1, \vec{r}_2, \vec{r}_3$ birlik vektorlarni olamiz. Ular oralaridagi burchaklar $\frac{\pi}{2}+\frac{\alpha}{2},\ \frac{\pi}{2}+\frac{\beta}{2},\ \frac{\pi}{2}+\frac{\gamma}{2}$ bo'lsin. Bizga ma'lumki ushbu

$$(\vec{r}_1+\vec{r}_2+\vec{r}_3)^2\geq 0$$

tengsizlik o'rinli. Bunga ko'ra

$$\vec{r}_1^{\,2}+\vec{r}_2^{\,2}+\vec{r}_3^{\,2}+2\vec{r}_1\vec{r}_2+2\vec{r}_1\vec{r}_3+2\vec{r}_2\vec{r}_3\geq 0\Rightarrow$$

$$\Rightarrow 3+2\cos\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)+2\cos\left(\frac{\pi}{2}+\frac{\beta}{2}\right)+2\cos\left(\frac{\pi}{2}+\frac{\gamma}{2}\right)\geq 0\Rightarrow$$

$$\Rightarrow 3-2\left[sin\frac{\alpha}{2}+sin\frac{\beta}{2}+sin\frac{\gamma}{2}\right]\geq 0\Rightarrow sin\frac{\alpha}{2}+sin\frac{\beta}{2}+sin\frac{\gamma}{2}\leq\frac{3}{2}$$

ekanligi kelib chiqadi.

10-masala. Agar α,β,γ ixtiyoriy uchburchakning ichki burchaklari bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$sin\alpha+sin\beta+sin\gamma\leq\frac{3\sqrt{3}}{2}$$

Isbot. Yuqoridagi 8-masalada isbotlangan tengsizlikdan foydalanamiz:

$$sin^2\alpha+sin^2\beta+sin^2\gamma\leq\frac{9}{4}$$

Bu tensizlikning har ikkala tomonini 3 ga ko'paytiramiz.

$$3sin^2\alpha+3sin^2\beta+3sin^2\gamma\leq\frac{27}{4}$$

bu tengsizlikning chap qismini baholaymiz.

$$3(sin^2\alpha+sin^2\beta+sin^2\gamma)=sin^2\alpha+sin^2\beta+sin^2\gamma+$$

$$+2(sin^2\alpha+sin^2\beta+sin^2\gamma)\geq sin^2\alpha+sin^2\beta+sin^2\gamma+$$

$$+2sin\alpha sin\beta+2sin\beta sin\gamma+2sin\gamma sin\alpha=(sin\alpha+sin\beta+sin\gamma)^2\Rightarrow$$

$$\Rightarrow 3(sin^2\alpha+sin^2\beta+sin^2\gamma)\geq(sin\alpha+sin\beta+sin\gamma)^2$$

Bundan esa

$$(sin\alpha + sin\beta + sin\gamma)^2 \le 3sin^2\alpha + 3sin^2\beta + 3sin^2\gamma \le \frac{27}{4} \Rightarrow$$

$$\Rightarrow (sin\alpha + sin\beta + sin\gamma)^2 \le \frac{27}{4} \Rightarrow sin\alpha + sin\beta + sin\gamma \le \frac{3\sqrt{3}}{2}$$

tengsizlikning isboti kelib chiqadi.

Faollashtiruvchi savollar.

1. *Vektor deb nimaga aytiladi?*
2. *Skalyar kattalik deb nimaga aytiladi?*
3. *Vektor kattalik deb nimaga aytiladi?*
4. *Vektorning moduli deb nimaga aytiladi?*
5. *Vektorning skalyar ko'paytmasi formulasini ayting?*
6. *Ikki vektor orasidagi burchak qanday topiladi?*
7. *Vektorning moduli qanday topiladi?*
8. *Uchburchak tengsizligini ayting?*

Mustaqil yechish uchun masalalar

1. Agar $x_1 + x_2 + \cdots + x_n = p$ va $y_1 + y_2 + \cdots + y_n = q$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.
$$\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} + \cdots + \sqrt{x_n^2 + y_n^2} \ge \sqrt{p^2 + q^2}$$
2. Agar $x_1 + x_2 + \cdots + x_n = 6$ va $y_1 + y_2 + \cdots + y_n = 8$ va $z_1 + z_2 + \cdots + z_n = 5$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.
$$\sqrt{x_1^2 + y_1^2 + z_1^2} + \sqrt{x_2^2 + y_2^2 + z_2^2} + \cdots + \sqrt{x_n^2 + y_n^2 + z_n^2} \ge 5\sqrt{5}$$
3. Tengsizlikni isbotlang: $\sqrt{sin^4x + 1} + \sqrt{cos^4x + 1} \ge \sqrt{5}$
4. Agar $a + b + c = 1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.
$$\sqrt{9a + 1} + \sqrt{9b + 1} + \sqrt{9c + 1} \le 6$$
5. Tengsizlikni isbotlang: $-\sqrt{a^2 + b^2} \le asinkx + bcoskx \le \sqrt{a^2 + b^2}$
6. $a, b, c \in R$ bo'lsa, $\sqrt{(a + c)^2 + b^2} + \sqrt{(a - c)^2 + b^2} \ge 2\sqrt{a^2 + b^2}$ tengsizlikni isbotlang.
7. Agar $x, y, z > 0$ bo'lsa, quyidagi tengsizlikni isbotlang.
$$\sqrt{x^2 + y^2 - xy} + \sqrt{x^2 + z^2} \ge \sqrt{y^2 + z^2 + \sqrt{3}yz}$$
8. Ushbu $\sqrt{x^2 + 2x + 4} + \sqrt{x^2 - x\sqrt{3} + 1}$ ifodaning eng kichik qiymatini toping.

9. Ushbu $\sqrt{x^2-6x+13}+\sqrt{x^2-14x+58}$ ifodaning eng kichik qiymatini toping.
10. $f(x)=\sqrt{x^2+4}+\sqrt{x^2-3\sqrt{3}x+9}$ funksiyaning eng kichik qiymatini toping
11. Agar $z^2+y^2=a^2$ va $u^2+v^2=b^2$ bo'lsa, $zu+yv$ ifodaning eng katta va eng kichik qiymatlarini toping. Bu yerda $ab\geq 0$.
12. Agar *x,y,z,t,k* haqiqiy sonlar uchun $x+y+z+t+k=40$ shart bajarilsa, quyidagi ifodaning eng kichik qiymatini toping.

$$\sqrt{x^2+1}+\sqrt{y^2+4}+\sqrt{z^2+9}+\sqrt{t^2+16}+\sqrt{k^2+400}$$

Agar α,β,γ ixtiyoriy uchburchakning ichki burchaklari bo'lsa, u holda quyidagi tengsizliklarni isbotlang.

13. $cos\frac{\alpha}{2}+cos\frac{\beta}{2}+cos\frac{\gamma}{2}\leq\frac{3\sqrt{3}}{2}$
14. $cos\alpha\cdot cos\beta\cdot cos\gamma\leq\frac{1}{8}$
15. $sin\frac{\alpha}{2}\cdot sin\frac{\beta}{2}\cdot sin\frac{\gamma}{2}\leq\frac{1}{8}$
16. $sin\alpha\cdot sin\beta\cdot sin\gamma\leq\frac{3\sqrt{3}}{8}$
17. $cos\frac{\alpha}{2}\cdot cos\frac{\beta}{2}\cdot cos\frac{\gamma}{2}\leq\frac{3\sqrt{3}}{8}$
18. $ctg\frac{\alpha}{2}+ctg\frac{\beta}{2}+ctg\frac{\gamma}{2}\geq 3\sqrt{3}$
19. $sin^2\frac{\alpha}{2}+sin^2\frac{\beta}{2}+sin^2\frac{\gamma}{2}\geq\frac{3}{4}$
20. $cos^2\alpha+cos^2\beta+cos^2\gamma\geq\frac{3}{4}$
21. $cos^2\frac{\alpha}{2}+cos^2\frac{\beta}{2}+cos^2\frac{\gamma}{2}\leq\frac{9}{4}$
22. $tg\frac{\alpha}{2}+tg\frac{\beta}{2}+tg\frac{\gamma}{2}\geq\sqrt{3}$
23. $ctg\alpha+ctg\beta+ctg\gamma\geq\sqrt{3}$

5-§. Trigonometriyaga oid olimpiada masalalar

Tayanch so'zlar: *Trigonometriya, sinus, kosinus, tangens, kotangens, Koshi-Bunyakovski-Shvars tengsizligi, uchburchak, perimetr, ayniyat, yig'indi, burchak, gradus.*

Bu bo'lim trigonometriyaga bag'ishlangan bo'lib, unda asosan olimpiadada uchraydigan trigonometrik ayniyatlarni, tengsizliklarni isbotlash va murakkabroq bo'lgan trigonometrik formulalarning isbotlari ko'rsatilgan.

1-masala. Agar α, β, γ uchburchakning burchaklari bo'lsa, u holda quyidagi tenglikni isbotlang

$$tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma}{2}+tg\frac{\gamma}{2}tg\frac{\alpha}{2}=1$$

Isbot. Bizga ma'lumki, $\alpha+\beta+\gamma=\pi\Rightarrow\frac{\alpha}{2}+\frac{\beta}{2}=\frac{\pi}{2}-\frac{\gamma}{2}$. Bu tenglikdan

$$tg\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=tg\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)\Rightarrow\frac{tg\frac{\alpha}{2}+tg\frac{\beta}{2}}{1-tg\frac{\alpha}{2}tg\frac{\beta}{2}}=\frac{1}{tg\frac{\gamma}{2}}\Rightarrow$$

$$\Rightarrow tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma}{2}+tg\frac{\gamma}{2}tg\frac{\alpha}{2}=1$$

ekanligi kelib chiqadi.

2-masala. Agar α, β, γ uchburchakning burchaklari bo'lsa, u holda quyidagi tengsizlikni isbotlang

$$tg^2\frac{\alpha}{2}+tg^2\frac{\beta}{2}+tg^2\frac{\gamma}{2}\geq 1$$

Isbot. Bizga ma'lumki $tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma}{2}+tg\frac{\gamma}{2}tg\frac{\alpha}{2}=1$ tenglik o'rinli. Bu tenglikka Koshi-Bunyakovski-Shvars tengsizligini qo'llaymiz:

$$1=tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma}{2}+tg\frac{\gamma}{2}tg\frac{\alpha}{2}\leq$$

$$\leq\sqrt{tg^2\frac{\alpha}{2}+tg^2\frac{\beta}{2}+tg^2\frac{\gamma}{2}}\cdot\sqrt{tg^2\frac{\beta}{2}+tg^2\frac{\gamma}{2}+tg^2\frac{\alpha}{2}}=tg^2\frac{\alpha}{2}+tg^2\frac{\beta}{2}+tg^2\frac{\gamma}{2}.$$

Bundan esa $tg^2\frac{\alpha}{2}+tg^2\frac{\beta}{2}+tg^2\frac{\gamma}{2}\geq 1$ ekanligi kelib chiqadi.

3-masala. Agar α, β, γ uchburchakning burchaklari bo'lsa, u holda quyidagi tenglikni isbotlang

$$cos^2\alpha+cos^2\beta+cos^2\gamma+2cos\alpha\cdot cos\beta\cdot cos\gamma=1$$

Isbot. $\alpha+\beta+\gamma=\pi\Rightarrow\alpha+\beta=\pi-\gamma\Rightarrow cos\gamma=-\cos(\alpha+\beta)$ tenglikdan foydalanamiz.

$$cos^2\alpha+cos^2\beta+cos^2(\alpha+\beta)-2cos\alpha\cdot cos\beta\cdot\cos(\alpha+\beta)=$$

$$= cos^2\alpha + cos^2\beta + (cos\alpha \cdot cos\beta - sin\alpha \cdot sin\beta)^2 -$$

$$-2cos\alpha \cdot cos\beta \cdot (cos\alpha \cdot cos\beta - sin\alpha \cdot sin\beta) = cos^2\alpha + cos^2\beta +$$

$$+cos^2\alpha \cdot cos^2\beta - 2sin\alpha \cdot cos\alpha \cdot sin\beta \cdot cos\beta + sin^2\alpha \cdot sin^2\beta -$$

$$-2cos^2\alpha \cdot cos^2\beta + 2sin\alpha \cdot cos\alpha \cdot sin\beta \cdot cos\beta = cos^2\alpha + cos^2\beta -$$

$$-(cos^2\alpha \cdot cos^2\beta - sin^2\alpha \cdot sin^2\beta) = \frac{1 + cos2\alpha}{2} + \frac{1 + cos2\beta}{2} -$$

$$-(cos\alpha \cdot cos\beta - sin\alpha \cdot sin\beta)(cos\alpha \cdot cos\beta + sin\alpha \cdot sin\beta) =$$

$$1 + \frac{cos2\alpha + cos2\beta}{2} - \cos(\alpha + \beta)\cos(\alpha - \beta) =$$

$$= 1 + \cos(\alpha + \beta)\cos(\alpha - \beta) - \cos(\alpha + \beta)\cos(\alpha - \beta) = 1$$

4-masala. Agar musbat *x*,*y*,*z* sonlar $xyz = x + y + z + 2$ tenglikni qanoatlantirsa, u holda quyidagi tengsizlikni isbotlang

$$5(x + y + z) + 18 \geq 8\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right)$$

Isbot. Yuqorida 3-masaladagi

$$cos^2\alpha + cos^2\beta + cos^2\gamma + 2cos\alpha \cdot cos\beta \cdot cos\gamma = 1$$

tenglikdan foydalanib,

$$\frac{cos\alpha}{cos\beta \cdot cos\gamma} \cdot \frac{cos\beta}{cos\alpha \cdot cos\gamma} \cdot \frac{cos\gamma}{cos\alpha \cdot cos\beta} = \frac{cos\alpha}{cos\beta \cdot cos\gamma} + \frac{cos\beta}{cos\alpha \cdot cos\gamma} + \frac{cos\gamma}{cos\alpha \cdot cos\beta} + 2$$

ifodani hosil qilamiz.

$$x = \frac{cos\alpha}{cos\beta \cdot cos\gamma}, \qquad y = \frac{cos\beta}{cos\alpha \cdot cos\gamma}, \qquad z = \frac{cos\gamma}{cos\alpha \cdot cos\beta}$$

deb belgilash kiritamiz va $cos\alpha + cos\beta + cos\gamma \leq \frac{3}{2}$ tengsizlikdan foydalansak,

$$\frac{1}{\sqrt{xy}} + \frac{1}{\sqrt{yz}} + \frac{1}{\sqrt{zx}} \leq \frac{3}{2} \Rightarrow 2\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right) \leq 3\sqrt{xyz} \Rightarrow$$

$$\Rightarrow 4\left(x + y + z + 2(\sqrt{xy} + \sqrt{yz} + \sqrt{zx})\right) \leq 9xyz \Rightarrow$$

$$\Rightarrow 8\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right) \leq 9(x+y+z+2)-4(x+y+z)=$$

$$=5(x+y+z)+18$$

5-masala. Agar $\alpha,\ \beta,\ \gamma$ uchburchakning burchaklari bo'lsa, u holda quyidagi tenglikni isbotlang

$$sin^2\frac{\alpha}{2}+sin^2\frac{\beta}{2}+sin^2\frac{\gamma}{2}+2sin\frac{\alpha}{2}\cdot sin\frac{\beta}{2}\cdot sin\frac{\gamma}{2}=1$$

Isbot. Bizga ma'lumki, $\alpha+\beta+\gamma=\pi \Rightarrow \frac{\alpha+\beta}{2}=\frac{\pi-\gamma}{2}$. Quyidagi ifodani soddalashtiramiz.

$$sin^2\frac{\gamma}{2}+2sin\frac{\alpha}{2}\cdot sin\frac{\beta}{2}\cdot sin\frac{\gamma}{2}=cos^2\frac{\alpha+\beta}{2}+2sin\frac{\alpha}{2}\cdot sin\frac{\beta}{2}\cdot cos\frac{\alpha+\beta}{2}=$$

$$=cos\frac{\alpha+\beta}{2}\left(cos\frac{\alpha+\beta}{2}+2sin\frac{\alpha}{2}\cdot sin\frac{\beta}{2}\right)=cos\frac{\alpha+\beta}{2}cos\frac{\alpha-\beta}{2}=$$

$$=\frac{cos\alpha+cos\beta}{2}=\frac{\left(1-2sin^2\frac{\alpha}{2}\right)+\left(1-2sin^2\frac{\beta}{2}\right)}{2}=1-sin^2\frac{\alpha}{2}-sin^2\frac{\beta}{2}.$$

Oxirgi tenglikdan esa

$$sin^2\frac{\alpha}{2}+sin^2\frac{\beta}{2}+sin^2\frac{\gamma}{2}+2sin\frac{\alpha}{2}\cdot sin\frac{\beta}{2}\cdot sin\frac{\gamma}{2}=1$$

tenglikning isboti kelib chiqadi.

6-masala. Yig'indini hisoblang:

$$tg1°tg2°+tg2°tg3°+tg3°tg4°+\cdots+tg44°tg45°$$

Yechish. Ifodani *S* orqali belgilaymiz.

$$S=tg1°tg2°+tg2°tg3°+tg3°tg4°+\cdots+tg44°tg45°=44+$$

$$+tg2°tg1°+tg3°tg2°+tg4°tg3°+\cdots+tg45°tg44°-44=$$

$$=(1+tg2°tg1°)+(1+tg3°tg2°)+(1+tg4°tg3°)+\cdots+$$

$$+(1+tg45°tg44°)-44.$$

Endi quyidagi ayniyatdan foydalanamiz:

$$tg(x-y)=\frac{tgx-tgy}{1+tgxtgy} \Rightarrow 1+tgxtgy=\frac{tgx-tgy}{tg(x-y)}$$

$$S=\frac{tg2^\circ-tg1^\circ}{tg(2^\circ-1^\circ)}+\frac{tg3^\circ-tg2^\circ}{tg(3^\circ-2^\circ)}+\frac{tg4^\circ-tg3^\circ}{tg(4^\circ-3^\circ)}+\dots+\frac{tg45^\circ-tg44^\circ}{tg(45^\circ-44^\circ)}-44=$$

$$S=\frac{1}{tg1^\circ}(tg2^\circ-tg1^\circ+tg3^\circ-tg2^\circ+tg4^\circ-tg3^\circ+\dots+tg45^\circ-tg44^\circ)-$$

$$-44=ctg1^\circ(tg45^\circ-tg1^\circ)-44=ctg1^\circ-45$$

7-masala. Yig'indini hisoblang:

$$\frac{1}{sin1^\circ sin2^\circ}+\frac{1}{sin2^\circ sin3^\circ}+\dots+\frac{1}{sin89^\circ sin90^\circ}$$

Yechish. Quyidagi yig'indining umumiy hadini baholaymiz:

$$\frac{1}{sinx\cdot\sin(x+1)}=\frac{1}{sin1^\circ}\cdot\frac{sin1^\circ}{sinx\cdot\sin(x+1)}=\frac{1}{sin1^\circ}\cdot\frac{\sin(x+1-x)}{sinx\cdot\sin(x+1)}=$$

$$=\frac{1}{sin1^\circ}\cdot\frac{\sin(x+1)\,cosx-\cos(x+1)\,sinx}{sinx\cdot\sin(x+1)}=\frac{1}{sin1^\circ}\cdot(ctgx-ctg(x+1))$$

$$\frac{1}{sin1^\circ sin2^\circ}+\frac{1}{sin2^\circ sin3^\circ}+\dots+\frac{1}{sin89^\circ sin90^\circ}=\frac{1}{sin1^\circ}\times$$

$$\times(ctg1^\circ-ctg2^\circ+ctg2^\circ-ctg3^\circ+ctg3^\circ-ctg4^\circ+\dots+ctg89^\circ-ctg90^\circ)=$$

$$=\frac{1}{sin1^\circ}\cdot ctg1^\circ=\frac{cos1^\circ}{sin^21^\circ}$$

8-masala. Birinchi chorak burchaklari uchun quyidagi tengsizlikni isbotlang

$$\cos(sinx)>\sin(cosx)$$

Isbot. $\cos(sinx)-\sin(cosx)>0$ ekanligini isbotlaymiz.

$$\cos(sinx)-\sin(cosx)=\cos(sinx)-\cos\left(\frac{\pi}{2}-cosx\right)=$$

$$=-2sin\left(\frac{sinx-cosx}{2}+\frac{\pi}{4}\right)sin\left(\frac{sinx+cosx}{2}-\frac{\pi}{4}\right)=$$

$$=2sin\left(\frac{sinx-cosx}{2}+\frac{\pi}{4}\right)sin\left(\frac{\pi}{4}-\frac{sinx+cosx}{2}\right)=2sin\alpha\cdot sin\beta$$

Endi α va β burchaklarning birinchi chorakda ekanligini ko'rsatamiz.

Bizga ma'lumki, $-\sqrt{2}\le sinx\pm cosx\le\sqrt{2}$ tengsizlik o'rinli. Bundan

$$-\frac{\sqrt{2}}{2} \le \frac{sinx \pm cosx}{2} \le \frac{\sqrt{2}}{2} \Rightarrow -\frac{\sqrt{2}}{2} \le \alpha - \frac{\pi}{4} \le \frac{\sqrt{2}}{2} \Rightarrow \frac{\pi}{4} - \frac{\sqrt{2}}{2} \le \alpha \le \frac{\pi}{4} + \frac{\sqrt{2}}{2}$$

Ekanligi kelib chiqadi. Xuddi shunday $\frac{\pi}{4} - \frac{\sqrt{2}}{2} \le \beta \le \frac{\pi}{4} + \frac{\sqrt{2}}{2}$. Hosil bo'lgan tengsizliklardagi α va β burchaklarning yuqori va quyi chegaralarini baholaymiz. Uning uchun ushbu $\frac{\sqrt{2}}{2} < \frac{\pi}{4}$ tengsizlikdan foydalanamiz:

$$\frac{\pi}{4} - \frac{\sqrt{2}}{2} \le \alpha \le \frac{\pi}{4} + \frac{\sqrt{2}}{2} \Rightarrow \frac{\pi}{4} - \frac{\pi}{4} < \alpha < \frac{\pi}{4} + \frac{\pi}{4} \Rightarrow 0 < \alpha < \frac{\pi}{2}$$

Shunga o'xshash $0 < \beta < \frac{\pi}{2}$. Bulardan

$$2sin\alpha \cdot sin\beta > 0 \Rightarrow \cos(sinx) > \sin(cosx)$$

ekanligi kelib chiqadi.

9-masala. Quyidagi yig'indini hisoblang:

$$sinx + 2sin2x + 3sin3x + \cdots + 2018sin2018x$$

Yechish. Quyidagicha belgilash kiritamiz:

$$f(x) = sinx + 2sin2x + 3sin3x + \cdots + 2018sin2018x$$

Bu funksiyaning har ikkala tomonini integrallaymiz:

$$\int f(x)dx = \int (sinx + 2sin2x + 3sin3x + \cdots + 2018sin2018x)dx =$$

$$= -cosx - cos2x - cos3x - \cdots - cos2018x = -S$$

$$S = cosx + cos2x + cos3x + \cdots + cos2016x + cos2017x + cos2018x \Rightarrow$$

$$\Rightarrow 2sin\frac{x}{2} \cdot S = 2sin\frac{x}{2} \cdot cosx + 2sin\frac{x}{2} \cdot cos2x + 2sin\frac{x}{2} \cdot cos3x + \cdots +$$

$$+2sin\frac{x}{2} \cdot cos2016x + 2sin\frac{x}{2} \cdot cos2017x + 2sin\frac{x}{2} \cdot cos2018x =$$

$$= sin\frac{3x}{2} - sin\frac{x}{2} + sin\frac{5x}{2} - sin\frac{3x}{2} + sin\frac{7x}{2} - sin\frac{5x}{2} + \cdots +$$

$$+sin\frac{4033x}{2} - sin\frac{4031x}{2} + sin\frac{4035x}{2} - sin\frac{4033x}{2} + sin\frac{4037x}{2} -$$

$$-sin\frac{4035x}{2} = sin\frac{4037x}{2} - sin\frac{x}{2} = 2sin1009x \cdot cos\frac{2019x}{2} \Rightarrow$$

$$\Rightarrow 2sin\frac{x}{2}\cdot S = 2sin1009x\cdot cos\frac{2019x}{2} \Rightarrow S = \frac{sin1009x\cdot cos\frac{2019x}{2}}{sin\frac{x}{2}}.$$

Topilgan yig'indini belgilashga olib borib qo'yamiz.

$$\int f(x)dx = -\frac{sin1009x\cdot cos\frac{2019x}{2}}{sin\frac{x}{2}}.$$

Endi bu tenglikning har ikkala tomonidan hosila olamiz:

$$f(x) = \left(-\frac{sin1009x\cdot cos\frac{2019x}{2}}{sin\frac{x}{2}}\right)'$$

Bundan esa

$$f(x) = \frac{sin2018x - 4096sin\frac{x}{2}\cdot cos\frac{4037x}{2}}{4sin^2\frac{x}{2}}$$

ekanligini topamiz. Demak,

$$sinx + 2sin2x + 3sin3x + \cdots + 2018sin2018x$$
$$= \frac{sin2018x - 4096sin\frac{x}{2}\cdot cos\frac{4037x}{2}}{4sin^2\frac{x}{2}}$$

bo'lar ekan.

10-masala. Ixtiyoriy 8 ta haqiqiy sonlardan har doim ikkita x, y sonlarni

$$0 \le \frac{x-y}{1+xy} \le \sqrt{3-2\sqrt{2}}$$

tengsizlikni qanoatlantiruvchi qilib olish mumkinligini isbotlang.

Isbot. Ixtiyoriy olingan 8 ta sonlarni o'sish tartibida joylashtiramiz. Aytaylik, ular $\alpha_1 < \alpha_2 < \cdots < \alpha_8$ bo'lsin va ularning arktangenslarini

$$\alpha_i = arctga_i\,, i = 1,2,\dots,8$$

deb belgilab olamiz. Tangens $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ oraliqda o'suvchi bo'lgani uchun

$$-\frac{\pi}{2} < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_8 < \frac{\pi}{2} < \alpha_1 + \pi$$

bo'ladi. $\alpha_1, \alpha_2, \dots, \alpha_8$ nuqtalar $[\alpha_1; \alpha_1 + \pi]$ oraliqda 8 ta $[\alpha_1; \alpha_2], [\alpha_2; \alpha_3], \dots$, $, [\alpha_8; \alpha_1 + \pi]$ kesmalarga ajratadi. Ma'lumki shu kesmalardan kamida bittasining uzunligi $\frac{\pi}{8}$ dan katta bo'lmaydi. Faraz qilaylik shu kesma $[\alpha_k; \alpha_{k+1}]$ bo'lsin. U holda $\alpha_{k+1} - \alpha_k \leq \frac{\pi}{8}$ va $\alpha_{k+1} \geq \alpha_k$ ekanligidan $\alpha_{k+1} - \alpha_k \geq 0$ bo'ladi, ya'ni $0 \leq \alpha_{k+1} - \alpha_k \leq \frac{\pi}{8}$ va tangens o'suvchi ekanligidan $0 \leq tg(\alpha_{k+1} - \alpha_k) \leq tg\frac{\pi}{8}$. Bundan $0 \leq tg(\alpha_{k+1} - \alpha_k) \leq \sqrt{3 - 2\sqrt{2}}$ ni hosil qilamiz. Oxirgi tengsizlikni quyidagicha yozib olamiz, $0 \leq \frac{tg\alpha_{k+1} - tg\alpha_k}{1 + tg\alpha_{k+1} \cdot tg\alpha_k} \leq \sqrt{3 - 2\sqrt{2}}$ va $tg\alpha_{k+1} = x$, $tg\alpha_k = y$ deb belgilasak $0 \leq \frac{x-y}{1+xy} \leq \sqrt{3 - 2\sqrt{2}}$ kelib chiqadi.

Agar $[\alpha_1; \alpha_1 + \pi]$ kesmaning uzunligi $\alpha_1 + \pi - \alpha_8 \leq \frac{\pi}{8}$ bo'lib qolsa $tg(\alpha_1 + \pi - \alpha_8) = tg(\alpha_1 - \alpha_8)$ bo'ladi va x, y qiymatlar mos ravishda $tg\alpha_1$ va $tg\alpha_8$ larga teng bo'ladi.

11-masala. Agar A natural sonni 7 ga bo'lganda qoldiq 4+$tg^2\alpha + ctg^2\alpha$ bo'lsa, u holda $\frac{22sin^2\alpha + 7}{7cos^2\alpha + 22}$ ni hisoblang.

Yechish: $A = 7k + r,\ r \leq 6$ bo'lishi kerak.

$$r = 4 + tg^2\alpha + ctg^2\alpha \geq 4 + 2tg\alpha \cdot ctg\alpha = 6 \Rightarrow r \geq 6$$

$$\begin{cases} r \leq 6 \\ r \geq 6 \end{cases} \Rightarrow r = 6 \Rightarrow tg\alpha = ctg\alpha \Rightarrow \alpha = \frac{\pi}{4} + \pi n, n \in Z$$

ekanligi kelib chiqadi.

$$\frac{22sin^2\alpha + 7}{7cos^2\alpha + 22} = \frac{22 \cdot \left(\frac{\sqrt{2}}{2}\right)^2 + 7}{7 \cdot \left(\frac{\sqrt{2}}{2}\right)^2 + 22} = \frac{11 + 7}{\frac{7}{2} + 22} = \frac{12}{17}$$

12-masala. Agar $sinx + cos2x$ va $cosx + sin2x$ sonlari noldan farqli ratsional sonlar bo'lsa, u holda $sinx$ va $cosx$ sonlari ham ratsional bo'lishini isbotlang.

Isbot: Quyidagicha shartni yozib olamiz:

$$\begin{cases} sinx + cos2x = \dfrac{m}{n}, & m,k \in Z\backslash\{0\} \\ cosx + sin2x = \dfrac{k}{l}, & n,l \in N \end{cases}$$

$$\begin{cases} sinx + 1 - 2sin^2x = \dfrac{m}{n} \\ cosx(2sinx + 1) = \dfrac{k}{l} \end{cases} \Rightarrow \begin{cases} (2sinx + 1)(1 - sinx) = \dfrac{m}{n} \\ (2sinx + 1)cosx = \dfrac{k}{l} \end{cases} \Rightarrow$$

$$\Rightarrow \frac{1 - sinx}{cosx} = \frac{lm}{kn} = \frac{b}{a} \Rightarrow \frac{cos^2x}{(1 - sinx)^2} = \frac{a^2}{b^2} \Rightarrow \frac{(1 - sinx)(1 + sinx)}{(1 - sinx)^2} = \frac{a^2}{b^2} \Rightarrow$$

$$\Rightarrow \frac{1 + sinx}{1 - sinx} = \frac{a^2}{b^2} \Rightarrow sinx = \frac{a^2 - b^2}{a^2 + b^2}$$

$$cosx = \sqrt{1 - \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2} = \frac{2ab}{a^2 + b^2}$$

Bulardan $sinx = \frac{a^2-b^2}{a^2+b^2}$, $cosx = \frac{2ab}{a^2+b^2}$ ekanligi kelib chiqadi. $\frac{a^2-b^2}{a^2+b^2}$ va $\frac{2ab}{a^2+b^2}$ ratsionalligidan $sinx$ va $cosx$ ning ham ratsionalligi kelib chiqadi.

13-masala. Agar $x \in \left(0; \frac{\pi}{2}\right)$ bo'lsa, $\left(1 + \frac{7}{sinx}\right)\left(1 + \frac{19}{cosx}\right) > 293$ tengsizlikni isbotlang.

Isbot: $x \in \left(0; \frac{\pi}{2}\right) \Rightarrow \begin{cases} sinx < 1 & | \cdot 7cosx \\ cosx < 1 & | \cdot 19sinx \end{cases} \Rightarrow \begin{cases} 7sinxcosx < 7cosx \\ 19sinxcosx < 19sinx \end{cases}$

Bu tengsizlikni hadma-had qo'shamiz: $13sin2x < 7cosx + 19sinx$. Hosil bo'lgan tengsizlikning har ikkala tomoniga $133sin2x$ ni qo'shib

$146sin2x < 7cosx + 19sinx + 133sin2x$ ni hosil qilamiz. Bizga ma'lumki, $sin2x \leq 1$ tengsizlik o'rinli.

$$146sin2x < 7cosx + 19sinx + 133sin2x \leq 7cosx + 19sinx + 133 \Rightarrow$$

$$\Rightarrow 293sinxcosx < 7cosx + 19sinx + sinxcosx + 133 \Rightarrow$$

$$\Rightarrow \frac{7cosx + sinxcosx + 19sinx + 133}{sinxcosx} > 293 \Rightarrow$$

$$\Rightarrow \frac{(sinx + 7)(cosx + 19)}{sinxcosx} > 293 \Rightarrow \left(1 + \frac{7}{sinx}\right)\left(1 + \frac{19}{cosx}\right) > 293.$$

14-masala. Agar x, y haqiqiy sonlar bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$|cosx| + |cosy| + |\cos(x+y)| \geq 1$$

Isbot: Dastlab quyidagi tengsizlikni isbotlaymiz:

$$|cosx| + |cosy| \geq |\sin(x+y)|$$

$$|\sin(x+y)| = |sinxcosy + cosxsiny| \leq \underbrace{|sinx|}_{\leq 1}|cosy| + |cosx|\underbrace{|siny|}_{\leq 1} \leq$$
$$\leq 1 \cdot |cosy| + |cosx| \cdot 1 = |cosx| + |cosy|$$

Endi isbotlash talab etilgan tengsizlikka qo'llaymiz.

$$|cosx| + |cosy| + |\cos(x+y)| \geq |\sin(x+y)| + |\cos(x+y)| =$$
$$= \left|\cos\left(\frac{\pi}{2} - (x+y)\right)\right| + |\cos(x+y)| \geq \left|\sin\frac{\pi}{2}\right| = 1.$$

15-masala. Agar x_1, x_2, x_3, x_4, x_5 haqiqiy sonlar uchun

$x_1 + x_2 + x_3 + x_4 + x_5 = 0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$|cosx_1| + |cosx_2| + |cosx_3| + |cosx_4| + |cosx_5| \geq 1$$

Isbot: Yuqoridagi misolda qo'llanilgan tengsizlikdan foydalanamiz.

$$|cosx_1| + |cosx_2| + |cosx_3| + |cosx_4| + |cosx_5| \geq$$

$$\geq |\sin(x_1 + x_2)| + |\sin(x_3 + x_4)| + |cosx_5| =$$

$$= \left|\cos\left(\frac{\pi}{2} - (x_1 + x_2)\right)\right| + \left|\cos\left(\frac{\pi}{2} - (x_3 + x_4)\right)\right| + |cosx_5| \geq$$

$$\geq \left|\sin\left(\pi - (x_1 + x_2 + x_3 + x_4)\right)\right| + |cosx_5| =$$

$$= \left|\cos\left(\frac{\pi}{2} - (x_1 + x_2 + x_3 + x_4)\right)\right| + |\cos(-x_5)| \geq$$

$$\geq \left|\sin\left(\frac{\pi}{2} - (x_1 + x_2 + x_3 + x_4 + x_5)\right)\right| = \left|\sin\frac{\pi}{2}\right| = 1.$$

16-masala. $0 < \alpha < \beta < \gamma < 2\pi$ munosabatlarni qanoatlantiradigan α, β, γ sonlar berilgan bo'lsin. Agar ixtiyoriy haqiqiy x son uchun

$\cos(x+\alpha) + \cos(x+\beta) + cos\gamma = 0$ tenglik o'rinli bo'lsa, $\beta - \alpha$ ni toping.

Yechish: Tenglikni quyidagicha yozib olamiz:

$$(cos\alpha + cos\beta)cosx - (sin\alpha + sin\beta)sinx = -cos\gamma$$

Bu tenglik barcha x larda o'zgarmas bo'lishi uchun $cosx$ va $sinx$ oldidagi koeffitsiyentlar nolga teng bo'lishi kerak.

$$\begin{cases} sin\alpha + sin\beta = 0 \\ cos\alpha + cos\beta = 0 \end{cases} \Rightarrow \begin{cases} 2sin\dfrac{\alpha+\beta}{2}cos\dfrac{\alpha-\beta}{2} = 0 \\ 2cos\dfrac{\alpha+\beta}{2}cos\dfrac{\alpha-\beta}{2} = 0 \end{cases} \Rightarrow cos\frac{\alpha-\beta}{2} = 0 \Rightarrow$$

$$\Rightarrow \frac{\beta-\alpha}{2} = \frac{\pi}{2} + \pi k, k \in Z \Rightarrow \beta - \alpha = \pi + 2\pi k, k \in Z$$

$0 < \alpha < \beta < \gamma < 2\pi$ shartni hisobga olib $\beta - \alpha = \pi$ ekanligini topamiz.

Faollashtiruvchi savollar.

1. *Trigonometriyaning asosiy ayniyatlarini ayting?*
2. *Uchburchakning ichki burchaklari yig'indisi nechchiga teng?*
3. *Sinx, cosx, tgx, ctgx funksiyalarning ishoralarini ayting?*
4. *Trigonometrik funksiyalarning qiymatlar sohasini ayting?*
5. *Yig'indini ko'paytmaga keltirish formulasini ayting?*
6. *Ko'paytmani yig'indiga keltirish formulasini ayting?*
7. *Ikkilangan burchak formulasini ayting?*
8. *Darajani pasaytirish formulasini ayting?*

Mustaqil yechish uchun masalalar

Agar α, β, γ uchburchakning burchaklari bo'lsa, u holda quyidagi ayniyatlarni isbotlang

1. $\frac{sin\gamma}{cos\alpha \cdot cos\beta} = tg\alpha + tg\beta$
2. $cos\alpha + cos\beta + cos\gamma = 1 + 4sin\frac{\alpha}{2} \cdot sin\frac{\beta}{2} \cdot sin\frac{\gamma}{2}$
3. $sin\alpha + sin\beta + sin\gamma = 4cos\frac{\alpha}{2} \cdot cos\frac{\beta}{2} \cdot cos\frac{\gamma}{2}$
4. $sin2\alpha + sin2\beta + sin2\gamma = 4sin\alpha \cdot sin\beta \cdot sin\gamma$
5. $sin^2\alpha + sin^2\beta + sin^2\beta = 2 + 2cos\alpha \cdot cos\beta \cdot cos\gamma$
6. Agar $0 < x < \frac{\pi}{2}, m > 0, n > 0$ sonlar uchun
$$\frac{\sin(x-\alpha)}{\sin(x-\beta)} = m, \frac{\cos(x-\alpha)}{\cos(x-\beta)} = n$$
ekanligi ma'lum bo'lsa, $\cos(\alpha - \beta)$ ni toping.
7. $tg20° + tg40° + \sqrt{3}tg20° \cdot tg40°$ ni hisoblang.
8. $\frac{1}{cos\alpha \cdot cos2\alpha} + \frac{1}{cos2\alpha \cdot cos3\alpha} + \cdots + \frac{1}{cos2018\alpha \cdot cos2019\alpha}$ ni soddalashtiring.
9. Quyidagi yig'indini hisoblang: $sin\alpha + sin2\alpha + sin3\alpha + \cdots + sin2019\alpha$

10. Quyidagi yig'indini hisoblang: $cos\alpha + cos2\alpha + cos3\alpha + \cdots + cos2019\alpha$

11. Quyidagi tenglamani yeching: $\frac{cosx}{|cosx|} + \frac{|sinx|}{sinx} = -2$

12. Tenglikni isbotlang: $sin18° = \frac{\sqrt{5}-1}{4}$

13. Quyidagi tenglamani yeching: $\sqrt{1 - sinx} = cosx$

14. Har bir *a* haqiqiy son uchun quyidagi tenglamani yeching.

$$(a-1)\left(\frac{1}{sinx} + \frac{1}{cosx} + \frac{1}{sinxcosx}\right) = 2$$

15. Agar $5\sin\beta = \sin(2\alpha + \beta)$ bo'lsa, $\frac{tg(\alpha+\beta)}{tg\alpha}$ ifodaning qiymatini toping.

16. Agar

$$2ctg^2\alpha \cdot ctg^2\beta \cdot ctg^2\gamma + ctg^2\alpha \cdot ctg^2\beta + ctg^2\alpha \cdot ctg^2\gamma + ctg^2\beta \cdot ctg^2\gamma = 1$$

bo'lsa, $cos^2\alpha + cos^2\beta + cos^2\beta$ yig'indining qiymatini toping.

17. Ayniayni isbotlang: $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\, sin\,\beta$

18. Agar *n* soni birdan katta bo'lgan natural son bo'lsa,

$$cos\frac{2\pi}{n} + cos\frac{4\pi}{n} + cos\frac{6\pi}{n} + \cdots + cos\frac{2n\pi}{n}$$

yig'indi nimaga teng.

19. O'tkir burchakli uchburchakning yuzi $\sqrt{3}$ va *x*, *y*, *z* burchaklari uchun

$$cos^2x + cos^2y + cos^2z = cosx \cdot cosy + cosy \cdot cosz + cosz \cdot cosx$$

shart bajarilsa, shu uchburchakning perimetrini toping.

20. Agar α, β, γ biror uchburchakning burchaklari bo'lsa, u holda

$$5cos\alpha + 10cos\beta + 4cos\gamma$$

ning eng katta qiymatini toping.

21. Agar $\alpha + \beta + \gamma = 0$ bo'lsa, quyidagi ayniyatni isbotlang.

$$sin\alpha + sin\beta + sin\gamma = -4sin\frac{\alpha}{2} \cdot sin\frac{\beta}{2} \cdot sin\frac{\gamma}{2}$$

22. Yig'indini hisoblang: $sin\frac{2\pi}{n} + sin\frac{4\pi}{n} + sin\frac{6\pi}{n} + \cdots + sin\frac{2(n-1)\pi}{n}$

23.Yig'indini hisoblang: $cos\frac{2\pi}{n} + cos\frac{4\pi}{n} + cos\frac{6\pi}{n} + \cdots + cos\frac{2(n-1)\pi}{n}$

24. Yig'indini hisoblang:

$$sin\alpha + \sin(\alpha + \frac{2\pi}{n}) + \sin(\alpha + \frac{4\pi}{n}) + \cdots + \sin(\alpha + \frac{2(n-1)\pi}{n})$$

25. Hisoblang: $cos36° - cos72°$

26. Hisoblang: $cos\frac{\pi}{7} - cos\frac{2\pi}{7} + cos\frac{3\pi}{7}$

27. Hisoblang: $cos\frac{2\pi}{7} \cdot cos\frac{4\pi}{7} \cdot cos\frac{8\pi}{7}$

28. Hisoblang: $cos\frac{2\pi}{9} \cdot cos\frac{4\pi}{9} \cdot cos\frac{8\pi}{9}$

29. Yig'indini hisoblang: $cos\frac{2\pi}{2m+1}+cos\frac{4\pi}{2m+1}+cos\frac{6\pi}{2m+1}+\cdots+cos\frac{2m\pi}{2m+1}$

30. Ko'paytmani hisoblang: $cos\alpha\cdot cos2\alpha\cdot cos4\alpha\cdot\ldots\cdot cos2^n\alpha$

31. Isbotlang: $ctg\alpha-tg\alpha=2ctg2\alpha$

32. Isbotlang: $tg\alpha+2tg2\alpha+4tg4\alpha+\cdots+2^n tg2^n\alpha=ctg\alpha-2^{n+1}ctg2^{n+1}\alpha$

33. Isbotlang: $cos\frac{\pi}{2n+1}cos\frac{2\pi}{2n+1}cos\frac{3\pi}{2n+1}+\cdots+cos\frac{n\pi}{2n+1}=\frac{1}{2^n}$

34. Isbotlang: $sin\alpha+sin2\alpha+sin3\alpha+\cdots+sinn\alpha=\frac{sin\frac{n\alpha}{2}sin\frac{(n+1)\alpha}{2}}{sin\frac{\alpha}{2}}$

35. Isbotlang: $cos\alpha+cos2\alpha+cos3\alpha+\cdots+cosn\alpha=\frac{sin\frac{(2n+1)\alpha}{2}}{2sin\frac{\alpha}{2}}-\frac{1}{2}$

36. Isbotlang:

$$cos\alpha+\cos(\alpha+x)+\cos(\alpha+2x)+\cdots+\cos(\alpha+nx)=\frac{sin\left(\alpha+\left(n+\frac{1}{2}\right)x\right)-sin\left(\alpha-\frac{1}{2}x\right)}{2sin\frac{x}{2}}$$

37. Istalgan α,β,γ burchaklar (uchburchak burchaklari bo'lishi shart emas) uchun quyidagi ayniyatlarni isbotlang:

$$sin\alpha+sin\beta+sin\gamma-\sin(\alpha+\beta+\gamma)=4sin\frac{\alpha+\beta}{2}sin\frac{\beta+\gamma}{2}sin\frac{\gamma+\alpha}{2}$$

38. Istalgan α,β,γ burchaklar (uchburchak burchaklari bo'lishi shart emas) uchun quyidagi ayniyatlarni isbotlang:

$$cos\alpha+cos\beta+cos\gamma+\cos(\alpha+\beta+\gamma)=4cos\frac{\alpha+\beta}{2}cos\frac{\beta+\gamma}{2}cos\frac{\gamma+\alpha}{2}$$

39. $cos\alpha+cos\beta=m$ va $sin\alpha+sin\beta=n$ bo'lsa, $\cos(\alpha+\beta)$ ni toping.

40. Hisoblang: $arctg\frac{1}{3}+arctg\frac{1}{5}+arctg\frac{1}{7}+arctg\frac{1}{8}$

41. Tenglamani yeching: $arctg\frac{1}{3}+arctg\frac{1}{4}+arctg\frac{1}{5}+arctg\frac{1}{x}=\frac{\pi}{4}$

42. $x+\frac{1}{x}=2cos\alpha$ bo'lsa, $x^n+\frac{1}{x^n}$ ni toping.

43. *ABC* uchburchakda $\cos(2\alpha-\beta)+\sin(\alpha+\beta)=2$ va *AB*=4 bo'lsa, *BC*=?

44. Ixtiyoriy 8 ta haqiqiy sonlardan har doim ikkita x,y sonlarni

$$0\le\frac{x-y}{1+xy}\le\sqrt{3-2\sqrt{2}}$$

tengsizlikni qanoatlantiruvchi qilib olish mumkinligini isbotlang.

III BOB. GEOMETRIYA

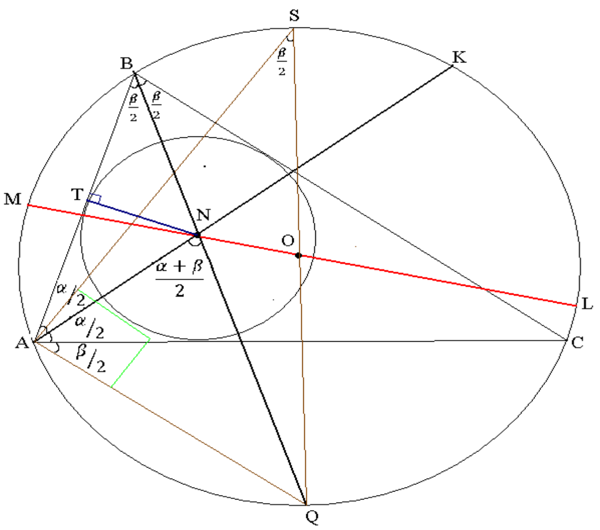
1-§. Bir nechta teoremalarning turli xil isbotlari

Tayanch so'zlar: *Teorema, Eyler teoremasi, Menelay teoremasi, Cheva teoremasi, Varinyon teoremasi, Ptolomey teoremasi, Geron formulasi, Simson teoremasi, uchburchak, kvadrat, to'rtburchak, parallelogramm, aylana, vatar, diametr, vektor, to'g'ri chiziq, diagonal.*

Yuzta masalani bir xil usulda yechgandan ko'ra bitta masalani yuz xil usulda yechgan yaxshi deganidek, bu paragrafda Geometriyada uchraydigan ajoyib formulalarning turli xil isbotlarini keltiramiz.

1-teorema(Eyler). Uchburchakka tashqi chizilgan va ichki chizilgan aylanalar markazlari orasidagi masofa $d^2 = R^2 - 2Rr$ formula orqali topiladi.

1-isboti. Quyidagicha belgilashlar kiritamiz.



$$ON = d, OL = OM = R, NT = r.$$

BQ va AK – bissektrissalar. $\angle CBQ = \angle CAQ$ chunki $\breve{CQ}$ yoyga tiralgan. Bundan ΔANQ teng yonli $AQ = NQ$. Sababi ΔABN da bitta $\angle ANQ$ tashqi burchak $\frac{\alpha+\beta}{2}$ ga teng.

$$\begin{cases} LN = OL + ON = R + d \\ MN = OM - ON = R - d \end{cases}$$

Bu tengliklarni hadma-had ko'paytirib

$$LN \cdot MN = R^2 - d^2$$

tenglikni olamiz. Vatarlar haqidagi teoremaga ko'ra

$$BN \cdot NQ = LN \cdot MN = R^2 - d^2 \Rightarrow BN \cdot NQ = R^2 - d^2.$$

Yuqoridagi $AQ = NQ$ tenglikni e'tiborga olib

$$BN \cdot NQ = BN \cdot AQ = R^2 - d^2 \Rightarrow AQ \cdot BN = R^2 - d^2 \qquad (1)$$

tenglikni hosil qilamiz. $\Delta TBN \sim \Delta ASQ$ chunki $\angle ABQ = \angle ASQ$ bitta yoyga tiralgan ichki burchak. $\angle CAQ = \angle BTN$, sababi $\angle SAQ$ diametrga tiralgan ichki burchak. Bu tengliklardan quyidagilarni yozamiz:

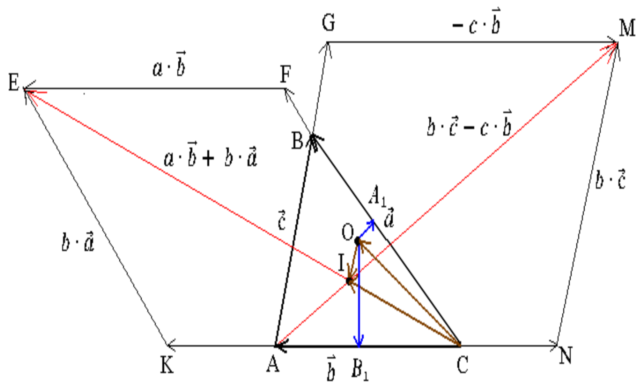
$$\frac{r}{AQ} = \frac{BN}{2R} \Rightarrow AQ \cdot BN = 2Rr \qquad (2)$$

Hosil bo'lgan (2) va (1) tengliklardan

$$\begin{cases} AQ \cdot BN = R^2 - d^2 \\ AQ \cdot BN = 2Rr \end{cases} \Rightarrow d^2 = R^2 - 2Rr$$

ekanligi kelib chiqadi.

2-isboti. Vektorlar yordamida isbotlaymiz.



Bizga ma'lumki, uchburchakka ichki chizilgan aylananing markazi burchak bissektrissalari kesishgan nuqtasi bo'ladi va *I* (imarkaz) harfi bilan belgilanadi.

Ushbu $\overrightarrow{CB} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $|CB| = a$, $|CA| = b$, $|AB| = c$ belgilashlarni kiritib olamiz. $\overrightarrow{CI}$ vektor ham, $b \cdot \vec{a} + a \cdot \vec{b}$ vektor ham ACB burchak bissektrissasi bo'yicha yo'nalganligi uchun ular o'zaro kollinear. Chunki $a \cdot \vec{b} = b \cdot \vec{a}$, bundan $CKEF$ romb ekanligi kelib chiqadi. $\overrightarrow{CE}$ rombning diagonali bo'lgani uchun burchak bissektrissasi bo'ladi. Bundan

$$\overrightarrow{CI} = \lambda(b\cdot\vec{a} + a\cdot\vec{b})$$

bo'ladi. Bunga ko'ra $\overrightarrow{AI} = \overrightarrow{AC} + \overrightarrow{CI} = \lambda b\cdot\vec{a} + (\lambda\vec{a} - 1)\cdot\vec{b}$. Xuddi shunday $\overrightarrow{AI}$ vektor ham, $c\cdot\overrightarrow{AC} + b\cdot\overrightarrow{AB}$ vektor ham BAC burchak bissektrissasi bo'yicha yo'nalgan. Chunki $c\cdot\vec{b} = b\cdot\vec{c}$, bundan $AGMN$ romb ekanligi kelib chiqadi. Shunga ko'ra

$$\overrightarrow{AI} = \mu(c\cdot\overrightarrow{AC} + b\cdot\overrightarrow{AB}) = \mu b\vec{a} - (\mu b + \mu c)\cdot\vec{b}$$

tenglikni hosil qilamiz. $\overrightarrow{AI}$ uchun olingan ikkita tenglikni bir-biridan ayirsak, ushbu

$$(\lambda - \mu)b\vec{a} - (\lambda a + \mu b + \mu c - 1)\vec{b} = 0$$

tenglik hosil bo'ladi. Bu yerda $\vec{a}$ va $\vec{b}$ vektorlar kollinear emasligini hisobga olsak, $\lambda = \mu$ va $\lambda a + \mu b + \mu c = 1$ tengliklarni olamiz. Bundan ushbu

$$\overrightarrow{CI} = \frac{1}{a+b+c}\cdot(b\cdot\vec{a} + a\cdot\vec{b})$$

tenglik hosil bo'ladi.

Agar A_1 orqali BC tomonning o'rtasini va B_1 orqali AC tomonning o'rtasini belgilasak, u holda $\overrightarrow{CO} + \overrightarrow{OA_1} = \frac{1}{2}\vec{a}$ va $\overrightarrow{CO} + \overrightarrow{OB_1} = \frac{1}{2}\vec{b}$ tengliklarni mos ravishda $\vec{a}$ va $\vec{b}$ vaktorlarga ko'paytirib, ushbu

$$\overrightarrow{CO}\cdot\vec{a} = \frac{1}{2}a^2 \text{ va } \overrightarrow{CO}\cdot\vec{b} = \frac{1}{2}b^2 \qquad (1)$$

tengliklarni olamiz. Chunki $\overrightarrow{OA_1} \perp \vec{a}$ va $\overrightarrow{OB_1} \perp \vec{b}$. Bundan $\overrightarrow{OA_1}\cdot\vec{a} = 0$ va $\overrightarrow{OB_1}\cdot\vec{b} = 0$.

Endi ushbu

$$\overrightarrow{OI} = \overrightarrow{CI} - \overrightarrow{CO} = \frac{1}{a+b+c}\cdot(b\cdot\vec{a} + a\cdot\vec{b}) - \overrightarrow{CO}$$

tenglikning kvadratini (1) formulalardan foydalanib hisoblaymiz:

$$\overrightarrow{OI}^2 = \frac{1}{(a+b+c)^2}(b\cdot\vec{a} + a\cdot\vec{b})^2 - \frac{2}{a+b+c}(b\cdot\vec{a} + a\cdot\vec{b}) + R^2 = \frac{2a^2b^2(1+\cos\gamma)}{(a+b+c)^2} - \frac{ab(a+b)}{a+b+c} + R^2$$

Bu yerda ushbu

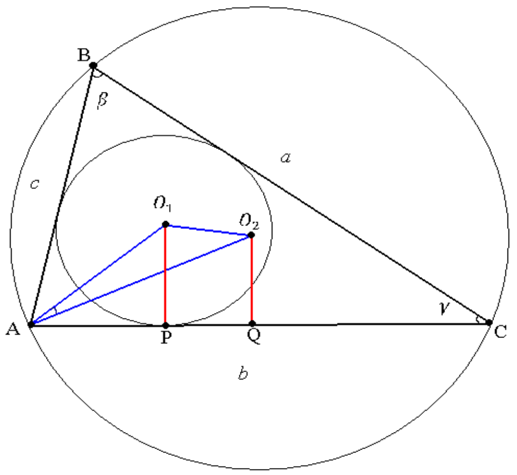
$$1 + \cos\gamma = 1 + \frac{a^2 + b^2 - c^2}{2ab} = \frac{(a+b+c)(a+b-c)}{2ab}$$

tenglikdan foydalansak,

$$\overrightarrow{OI}^2 = R^2 - \frac{abc}{a+b+c} = R^2 - 2\cdot\frac{abc}{4S}\cdot\frac{2S}{a+b+c} = R^2 - 2Rr$$

Eyler formulasi kelib chiqadi.

3-isboti. Trigonometriyadan foydalanib isbotlaymiz.



Belgilash kiritamiz: $O_1O_2 = d$, $AO_2 = R$, $O_1P = r$, $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$, $\alpha + \beta + \gamma = 180°$, $\frac{\alpha+\beta}{2} = 90° - \frac{\gamma}{2}$, $p = \frac{a+b+c}{2}$. Bizga ma'lumki, uchburchakka ichki chizilgan aylananing markazi bissektrissalar kesishish nuqtasida, unga tashqi chizilgan aylana markazi esa tomonlariga o'tkazilgan o'rta perpendikulyarlar kesishish nuqtasida yotadi. Shularga asosan:

$$AO_1 = \frac{r}{\sin\frac{\alpha}{2}},\ \angle PAO_1 = \frac{\alpha}{2},\ \angle AO_2Q = \beta,\ \angle QAO_2 = 90° - \beta \Rightarrow \angle O_1AO_2 = \beta + \frac{\alpha}{2} - 90°$$

tengliklarni olamiz. Quyidagi formulalarni ko'rib o'taylik:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R \qquad (1)$$

$$r = \frac{S}{p} \qquad (2)$$

(1) formulani (2) formulaga asosan quyidagicha yozamiz:

$$r = \frac{S}{p} = \frac{ab\sin\gamma}{a+b+c} = \frac{4R^2\sin\alpha\sin\beta\sin\gamma}{2R(\sin\alpha + \sin\beta + \sin\gamma)} = \frac{2R\sin\alpha\sin\beta\sin\gamma}{\sin\alpha + \sin\beta + \sin\gamma} \qquad (3)$$

Endi AO_1O_2 uchburchakda kosinuslar teoremasini qo'llab, quyidagini hosil qilamiz:

$$d^2=\frac{r^2}{\sin^2\frac{\alpha}{2}}+R^2-\frac{2Rr}{\sin\frac{\alpha}{2}}\cdot\cos\left(\beta+\frac{\alpha}{2}-90^\circ\right)=R^2+\frac{r\cdot r}{\sin^2\frac{\alpha}{2}}-\frac{2Rr}{\sin\frac{\alpha}{2}}\cdot\sin\left(\beta+\frac{\alpha}{2}\right)$$

(3) formulaga asosan

$$d^2=R^2+\frac{r}{\sin^2\frac{\alpha}{2}}\cdot\frac{2R\sin\alpha\sin\beta\sin\gamma}{\sin\alpha+\sin\beta+\sin\gamma}-\frac{2Rr}{\sin\frac{\alpha}{2}}\cdot\sin\left(\beta+\frac{\alpha}{2}\right)=$$

$$=R^2-2Rr\left(\frac{\sin\left(\beta+\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}-\frac{\sin\alpha\sin\beta\sin\gamma}{\sin^2\frac{\alpha}{2}(\sin\alpha+\sin\beta+\sin\gamma)}\right).$$

Bu tenglikdagi qavs ichidagi ifodani 1 ga tengligini ko'rsatish kifoya.

$$\frac{\sin\left(\beta+\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}-\frac{\sin\alpha\sin\beta\sin\gamma}{\sin^2\frac{\alpha}{2}(\sin\alpha+\sin\beta+\sin\gamma)}=\frac{\sin\left(\beta+\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}-\frac{2\cos\frac{\alpha}{2}\sin\beta\sin\gamma}{\sin\frac{\alpha}{2}(\sin\alpha+\sin\beta+\sin\gamma)}=$$

$$=\frac{\sin\left(\beta+\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}-\frac{2\cos\frac{\alpha}{2}\sin\beta\sin\gamma}{\sin\frac{\alpha}{2}\left(2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\sin\gamma\right)}=\frac{\sin\left(\beta+\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}-$$

$$-\frac{4\cos\frac{\alpha}{2}\sin\beta\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}\left(2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\right)}=\frac{\sin\left(\beta+\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}-\frac{2\cos\frac{\alpha}{2}\sin\beta\sin\frac{\gamma}{2}}{\sin\frac{\alpha}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\right)}=$$

$$=\frac{\sin\left(\beta+\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}-\frac{4\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}=\frac{\sin\left(\beta+\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}-\frac{2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}=$$

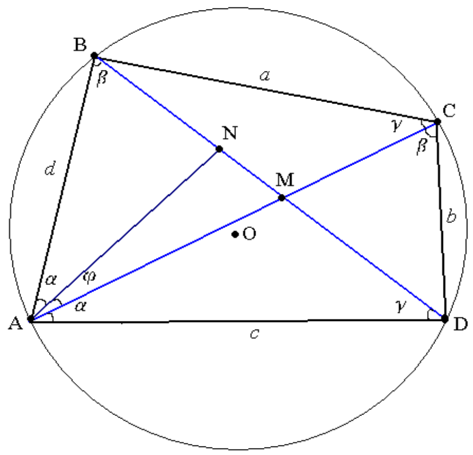
$$=\frac{\sin\left(90^\circ-\frac{\gamma}{2}+\frac{\beta}{2}\right)-2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}=\frac{\cos\left(\frac{\gamma}{2}-\frac{\beta}{2}\right)-2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}=$$

$$= \frac{\cos\frac{\gamma}{2}\cos\frac{\beta}{2} + \sin\frac{\gamma}{2}\sin\frac{\beta}{2} - 2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\frac{\alpha}{2}} = \frac{\cos\frac{\beta+\gamma}{2}}{\sin\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} = 1.$$

Teorema (Ptolomey). Ixtiyoriy qavariq to'rtburchakka tashqi aylana chizilgan bo'lsa, u holda uning diagonallarining ko'paytmasi qarama-qarshi tomonlari ko'paytmasi yig'indisiga tengligini isbotlang, ya'ni

$$d_1 d_2 = bd + ac$$

1-isboti. Burchak yasab isbotlaymiz.



$\angle CAD$ ga teng bo'lgan $\angle BAC$ ichidan $\angle BAN$ yasaymiz. Chizmaga qarab quyidagilarni yozamiz:

$$\Delta ANB \sim \Delta ADC \Rightarrow \frac{BN}{DC} = \frac{AB}{AC}$$

$$\Delta ABC \sim \Delta ADN \Rightarrow \frac{ND}{BC} = \frac{AD}{AC}$$

Bu tengliklardan:

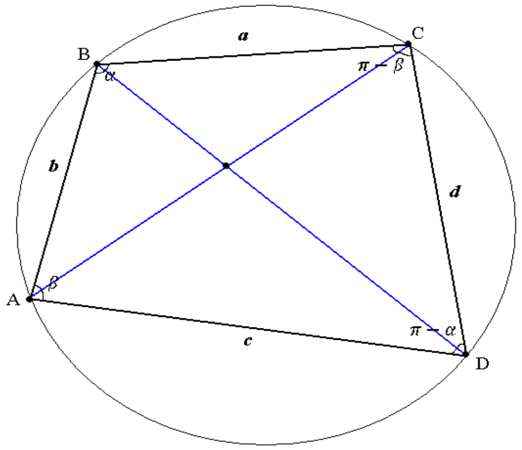
$$\begin{cases} BN = \frac{AB}{AC} \cdot DC = \frac{bd}{d_1} \\ ND = \frac{AD}{AC} \cdot BC = \frac{ac}{d_1} \end{cases}$$

Bu tengliklarni hadma-had qo'shib, $BN + ND = d_2$ ekanligidan

$$BN + ND = \frac{1}{d_1}(bd + ac) \Rightarrow d_1 d_2 = bd + ac$$

tenglikni hosil qilamiz.

2-isboti. Kosinuslar teoremasidan foydalanib isbotlaymiz.



Bizga ma'lumki, qavariq to'rtburchakka tashqi aylana chizilgan bo'lsa, uning qarama-qarshi burchaklarining yig'indisi $180°$ ga teng bo'ladi, ya'ni

$$\begin{cases}\angle ABC=\alpha\Rightarrow\angle ADC=\pi-\alpha\\ \angle BAD=\beta\Rightarrow\angle BCD=\pi-\beta\end{cases}$$

tengliklarni yozishimiz mumkin. Chizmadagi belgilashlarga asosan ΔABC va ΔADC larda kosinuslar teoremasini qo'llaymiz:

$$\begin{cases}d_1^2=d^2+c^2-2dc\cos\alpha\\ d_1^2=a^2+b^2-2ab\cos(\pi-\alpha)\end{cases}\Rightarrow\begin{cases}\cos\alpha=\dfrac{d^2+c^2-d_1^2}{2dc}\\ \cos\alpha=\dfrac{d_1^2-a^2-b^2}{2ab}\end{cases}\Rightarrow$$

$$\Rightarrow\frac{d^2+c^2-d_1^2}{2dc}=\frac{d_1^2-a^2-b^2}{2ab}\Rightarrow ab(d^2+c^2-d_1^2)=dc(d_1^2-a^2-b^2)\Rightarrow$$

$$\Rightarrow ab(d^2+c^2)-abd_1^2=dcd_1^2-dc(a^2+b^2)\Rightarrow$$

$$d_1^2=\frac{abd^2+abc^2+dca^2+dcb^2}{dc+ab}\Rightarrow d_1^2=\frac{(ad+bc)(ac+bd)}{dc+ab}$$

Xuddi shunga o'xshash ΔABD va ΔBDC larda kosinuslar teoremasini qo'llab

$$d_2^2=\frac{(bd+ac)(ab+dc)}{bc+ad}$$

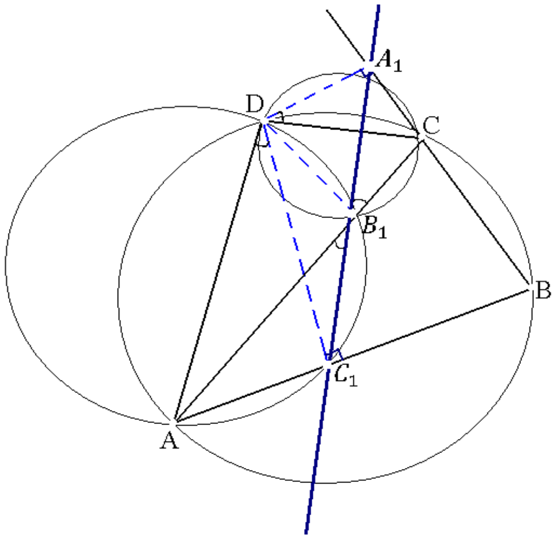
ekanligini topamiz. Topilgan d_1^2 va d_2^2 qiymatlarini hadma-had ko'paytirib

$$d_1^2 d_2^2 = (ac+bd)^2 \Rightarrow d_1 d_2 = bd + ac$$

tenglikni hosil qilamiz.

3-isboti. Simson teoremasidan foydalanib isbotlaymiz:

Teorema (Simson). Aylanada A, B, C nuqtalar berilgan bo'lsin. U holda ixtiyoriy P nuqtadan $AB,\ BC,\ CA$ tomonlarga tushirilgan perpendikulyarlarning asoslari bitta to'g'ri chiziqda yotishi uchun P nuqtaning ham shu aylanada yotishi zarur va yetarli.



D dan $AB,\ BC,\ CA$ tomonlarga tushirilgan perpendikulyarlarning asoslari mos ravishda $C_1,\ A_1,\ B_1$ bo'lsin. $DA_1 \perp CA_1$ va $DB_1 \perp CB_1$ ekanligidan $D,\ A_1,\ B_1,\ C$ nuqtalar CD markazli aylanada yotadi. Bundan CD diametrga sinuslar teoremasini qo'llasak:

$$A_1B_1 = CD\sin\angle A_1CB_1 = CD\sin\angle BCA \qquad (1)$$

ABC uchburchakka tashqi chizilgan aylana radiusi R bo'lsin, u holda sinuslar teoremasiga ko'ra:

$$AB = 2R\sin\angle BCA \qquad (2)$$

(1) va (2) tengliklardan:

$$A_1B_1 = \frac{AB\cdot CD}{2R} \qquad (3)$$

Yuqoridagi kabi ish yuritib quyidagi tenglikni ham olish mumkin:

$$B_1C_1 = \frac{BC \cdot AD}{2R}, \quad C_1A_1 = \frac{AC \cdot DB}{2R} \qquad (4)$$

Simson teoremasiga ko'ra A_1, B_1, C_1 nuqtalar bitta to'g'ri chiziqda yotadi, bundan:

$$A_1B_1 + B_1C_1 = A_1C_1 \qquad (5)$$

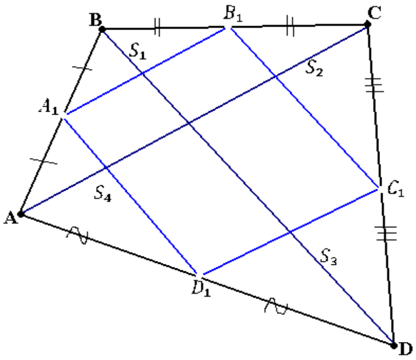
(3), (4) larni (5) ga olib borib qo'ysak:

$$\frac{AB \cdot CD}{2R} + \frac{BC \cdot AD}{2R} = \frac{AC \cdot DB}{2R}$$

tenglikka ega bo'lamiz. Bundan $AD \cdot BC + AB \cdot CD = AC \cdot BD$ ekanligi kelib chiqadi.

Teorema (Varinyon). To'rtburchakning tomonlari o'rtalarini tutashtiruvchi kesmalardan hosil bo'lgan to'rtburchak parallelogramm bo'ladi va uning yuzi to'rtburchak yuzasining yarmiga teng.

1-isboti. Belgilash kiritamiz: $S_{A_1BB_1} = S_1$, $S_{B_1CC_1} = S_2$, $S_{C_1DD_1} = S_3$, $S_{D_1AA_1} = S_4$.



Dastlab, $A_1B_1C_1D_1$ parallelogramm ekanligini isbotlaymiz. Bizga ma'lumki, uchburchakning o'rta chizig'i asosiga parallel va uning uzunligining yarmiga teng. $A_1B_1C_1D_1$ to'rtburchakning qarama-qarshi tomonlari o'zaro parallel va teng ekanligini ko'rsatamiz.

ΔABC va ΔADC da A_1B_1 va C_1D_1 mos ravishda o'rta chiziqlari bo'lganligi uchun $A_1B_1 \parallel AC$, $A_1B_1 = \frac{1}{2}AC$ va $C_1D_1 \parallel AC$, $C_1D_1 = \frac{1}{2}AC$. Bundan esa $A_1B_1 \parallel C_1D_1$, $A_1B_1 = C_1D_1$ kelib chiqadi.

Xuddi shunday ΔABD va ΔBCD da A_1D_1 va B_1C_1 o'rta chiziqlari. $A_1D_1 \parallel BD$, $A_1D_1 = \frac{1}{2}BD$ va $B_1C_1 \parallel BD$, $B_1C_1 = \frac{1}{2}BD$. Bundan $A_1D_1 \parallel B_1C_1$, $A_1D_1 = B_1C_1$ ekanligi kelib chiqadi.

Endi $A_1B_1C_1D_1$ parallelogrammning yuzi $ABCD$ to'rtburchakning yuzining yarmiga tengligini isbotlaymiz. ΔABD va ΔBCD mos ravishda ΔAA_1D_1 va ΔCC_1B_1 ga o'xshashligidan

$$+\begin{cases} S_{ABD} = 4S_{AA_1D_1} \\ S_{BCD} = 4S_{CC_1B_1} \end{cases} \Rightarrow S_{ABCD} = 4(S_2 + S_4) \Rightarrow S_2 + S_4 = \frac{1}{4}S_{ABCD}.$$

Xuddi shunga o'xshash ΔABC va ΔADC mos ravishda ΔBB_1D_1 va ΔDD_1C_1 ga o'xshashligidan

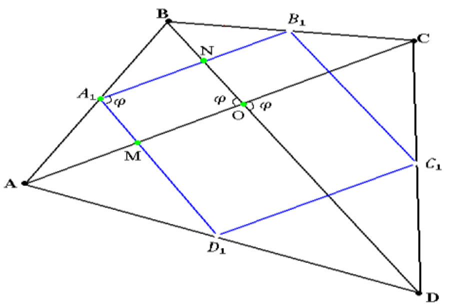
$$+\begin{cases} S_{ABC} = 4S_{BB_1A_1} \\ S_{ADC} = 4S_{DD_1C_1} \end{cases} \Rightarrow S_{ABCD} = 4(S_1 + S_3) \Rightarrow S_1 + S_3 = \frac{1}{4}S_{ABCD}$$

$S_{A_1B_1C_1D_1} = S_{ABCD} - (S_1 + S_2 + S_3 + S_4)$ ekanligidan

$$S_{A_1B_1C_1D_1} = S_{ABCD} - \frac{1}{4}(S_{ABCD} + S_{ABCD}) = S_{ABCD} - \frac{1}{2}S_{ABCD} = \frac{1}{2}S_{ABCD}$$

ekanligi kelib chiqadi.

2-isboti. Ixtiyoriy qavariq to'rtburchakning yuzi $S = \frac{1}{2}d_1d_2\sin\varphi$ formula



orqali topiladi. Shu formulaga asosan $S_{ABCD} = \frac{1}{2}\cdot AC\cdot BD\sin\varphi$ o'rinli bo'ladi. $A_1B_1 \parallel AC,\ A_1D_1 \parallel BD$ bo'lganligi sababli A_1NOM parallelogrammligi kelib chiqadi. Parallelogrammning qarama-qarshi burchaklari tengligidan $\angle NA_1M = \varphi$ bo'ladi. Ikkinchi tomondan $A_1B_1C_1D_1$ parallelogrammning yuzi

$$S_{A_1B_1C_1D_1} = A_1B_1 \cdot A_1D_1 \sin\varphi$$

bo'ladi. A_1B_1 va A_1D_1 mos ravishda AC va BD diagonallarining yarmiga teng ekanligidan $S_{A_1B_1C_1D_1} = \frac{1}{2}AC\cdot BD\sin\varphi$. Bundan esa

$$S_{A_1B_1C_1D_1} = \frac{1}{4}AC\cdot BD\sin\varphi = \frac{1}{2}\cdot\frac{1}{2}AC\cdot BD\sin\varphi \Rightarrow S_{A_1B_1C_1D_1} = \frac{1}{2}S_{ABCD}$$

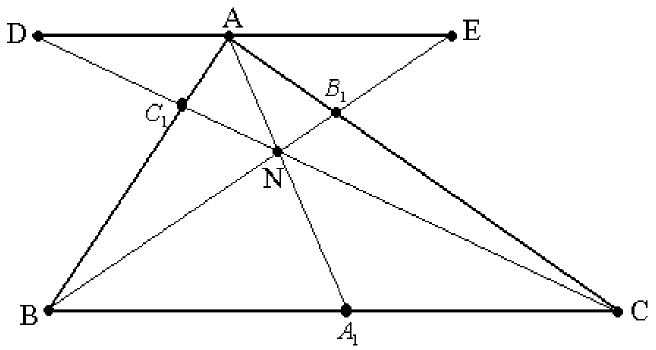
ekanligi kelib chiqadi.

Teorema (Cheva). Agar AA_1, BB_1, CC_1 to'g'ri chiziqlar uchburchak ABC ning uchlaridan chiqib, bir nuqtada kesishsa yoki parallel bo'lib AB, BC, CA tomonlarni yoki ularning davomini nuqtalarda kesib o'tsa, u holda

$$\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} = 1$$

tenglik o'rinli bo'ladi.

1-isboti. Uchburchak ABC ning A uchidan BC tomonga parallel bo'lgan DE ni o'tkazamiz, natijada ADC_1 va BCC_1 uchburchaklar hosil bo'ladi.



$\angle BC_1C = \angle DC_1A$ va $\angle C_1BC = \angle C_1AD$ larga asosan ΔADC_1 o'xshash ΔBCC_1 ekanidan

$$\frac{DA_1}{BC} = \frac{AC_1}{C_1B} \qquad (1)$$

bo'ladi. Xuddi shunday ΔAEB_1 o'xshash ΔCBB_1 ga asosan

$$\frac{BC}{AE} = \frac{B_1C}{AB_1} \qquad (2)$$

ekanini yoza olamiz. Hosil qilingan o'xshash uchburchaklardan bevosita

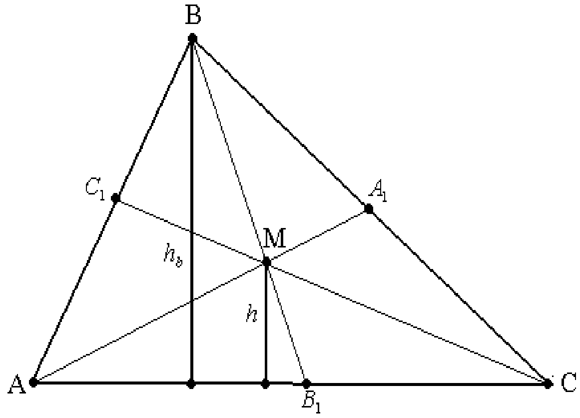
$$\frac{AE}{BA_1} = \frac{AN}{NA_1} = \frac{DA}{A_1C} \text{ yoki } \frac{BA_1}{A_1C} = \frac{AE}{DA} \qquad (3)$$

kelib chiqadi. Natijada (1), (2) va (3) larni hadlab ko'paytirsak,

$$\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} = 1$$

ekani kelib chiqadi.

2-isboti. ABC uchburchakning B burchagidan balandlik tushiramiz va uni



h_b deb belgilaymiz.

M nuqtadan AC tomonga perpendikulyar qilib h kesmani tushiramiz. h kesma $\triangle AMB_1$ va $\triangle CMB_1$ uchburchaklarning M nuqtadan tushirilgan balandligi bo'ladi. $\frac{S_{ABB_1}}{S_{CBB_1}} = \frac{\frac{h_b \cdot AB_1}{2}}{\frac{h_b \cdot CB_1}{2}} = \frac{AB_1}{CB_1}$ va $\frac{S_{AMB_1}}{S_{CMB_1}} = \frac{\frac{h \cdot AB_1}{2}}{\frac{h \cdot CB_1}{2}} = \frac{AB_1}{CB_1}$ tengliklardan quyidagi munosabatni keltirib chiqaramiz:

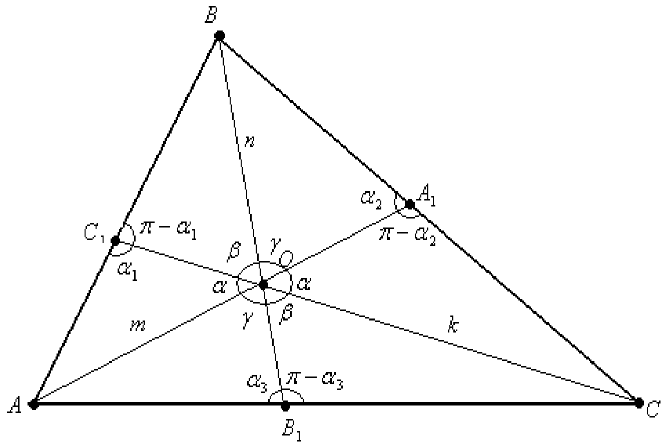
$$S_{AMB} = S_{ABB_1} - S_{AMB_1} = S_{CBB_1} \cdot \frac{AB_1}{CB_1} - S_{CMB_1} \cdot \frac{AB_1}{CB_1} = \frac{AB_1}{CB_1}\left(S_{CBB_1} - S_{CMB_1}\right) = S_{CMB} \cdot \frac{AB_1}{CB_1}$$

Bundan esa $\frac{S_{AMC}}{S_{BMA}} = \frac{A_1C}{BA_1}$ va $\frac{S_{BMC}}{S_{AMA}} = \frac{BC_1}{AC_1}$ tengliklarni ham keltirib chiqarish mumkin. Hosil bo'lgan munosabatlarni isbot talab qilinayotgan tenglikka olib borib qo'ysak:

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = \frac{S_{AMC}}{S_{BMC}} \cdot \frac{S_{BMA}}{S_{AMC}} \cdot \frac{S_{BMC}}{S_{BMA}} = 1$$

ekanligi kelib chiqadi.

3-isboti. Chizmada belgilashlar kiritamiz:



$$\angle AOC_1 = \angle COA_1 = \alpha,\ \angle BOC_1 = \angle COB_1 = \beta,\ \angle BOA_1 = \angle AOB_1 = \gamma$$

$$\angle AC_1O = \alpha_1,\ \angle BA_1O = \alpha_2,\ \angle AB_1O = \alpha_3,\ AO = m,\ BO = n,\ CO = k.$$

ΔAOC_1 va ΔBOC_1 larda sinuslar teoremasini qo'llaymiz:

$$\begin{cases} \dfrac{AC_1}{\sin\alpha} = \dfrac{m}{\sin\alpha_1} \\ \dfrac{C_1B}{\sin\beta} = \dfrac{n}{\sin(\pi-\alpha_1)} \end{cases} \Rightarrow \frac{AC_1}{C_1B} = \frac{\sin\alpha}{\sin\beta}\cdot\frac{m}{n} \qquad (1)$$

Xuddi shunday ΔBOC_1 va ΔCOA_1 , ΔCOB_1 va ΔAOB_1 larda ham sinuslar teoremasini qo'llab:

$$\frac{BA_1}{A_1C} = \frac{\sin\gamma}{\sin\alpha}\cdot\frac{n}{k} \qquad (2)$$

$$\frac{CB_1}{B_1A} = \frac{\sin\beta}{\sin\gamma}\cdot\frac{k}{m} \qquad (3)$$

tengliklarni hosil qilamiz. Bu topilgan (1), (2) va (3) tengliklarni hadma-had ko'paytirib :

$$\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A} = \frac{\sin\alpha}{\sin\beta}\cdot\frac{m}{n}\cdot\frac{\sin\gamma}{\sin\alpha}\cdot\frac{n}{k}\cdot\frac{\sin\beta}{\sin\gamma}\cdot\frac{k}{m} = 1$$

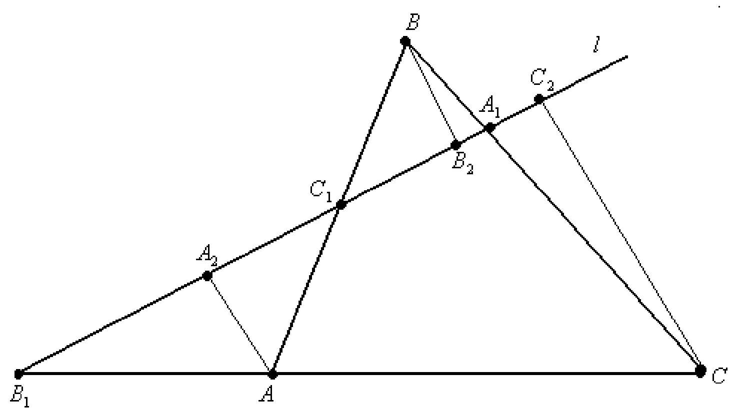
isbotlash talab etilgan tenglikni hosil qilamiz.

Teorema (Menelay). Agar ixtiyoriy ABC uchburchakning AB va BC tomonlarini mos ravishda C_1 va A_1, AC tomonning davomini B_1 nuqtada biror l to'g'ri chiziq kesib o'tsa, u holda

$$\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}=1$$

tenlik bajariladi.

Isbot. A, B, C uchlariga l to'g'ri chiziqqa AA_2, BB_2, CC_2 perpendikulyarlar tushiramiz. U holda $\Delta AA_2C_1 \sim \Delta BB_2C_1$ dan $\frac{AC_1}{C_1B}=\frac{AA_2}{BB_2}$ ni, $\Delta BB_2A_1 \sim \Delta CC_2A_1$ dan



$\frac{BA_1}{A_1C}=\frac{BB_2}{CC_2}$ ni va $\Delta AB_1A_2 \sim \Delta CB_1C_2$ dan $\frac{CB_1}{B_1A}=\frac{CC_2}{AA_2}$ larni topib, mos tomonlarini ko'paytirib yuborsak,

$$\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}=1$$

tenglik kelib chiqadi.

Lemma (Menelay). Yuqoridagi shartlar bajarilsa, u holda

$$\frac{AC_1}{AB}\cdot\frac{BC}{CA_1}\cdot\frac{A_1B_1}{B_1C_1}=1$$

tenglik o'rinli bo'ladi.

Isbot. ΔCB_1A_1 da Menelay teoremasini qo'llaymiz. AB to'g'ri chiziq uchun:

$$\frac{A_1C_1}{C_1B_1}\cdot\frac{B_1A}{AC}\cdot\frac{BC}{BA_1}=1\Rightarrow\frac{A_1C_1\cdot BC}{C_1B_1}=\frac{AC\cdot BA_1}{B_1A} \qquad (1)$$

Endi isbotlash talab etilgan tenglikni soddalashtiramiz:

$$\frac{AC_1}{AB}\cdot\frac{BC}{CA_1}\cdot\frac{A_1B_1}{B_1C_1}=1\Rightarrow\frac{AC_1}{AC_1+C_1B}\cdot\frac{BA_1+A_1C}{CA_1}\cdot\frac{A_1C_1+C_1B_1}{B_1C_1}=1\Rightarrow$$

$$\Rightarrow\left(1+\frac{BA_1}{CA_1}\right)\left(1+\frac{A_1C_1}{B_1C_1}\right)=1+\frac{C_1B}{AC_1}\Rightarrow\frac{BA_1}{CA_1}+\frac{A_1C_1}{B_1C_1}+\frac{BA_1}{CA_1}\cdot\frac{A_1C_1}{B_1C_1}=\frac{C_1B}{AC_1}\Rightarrow$$

$$\Rightarrow\left(1+\frac{BA_1}{CA_1}\right)\left(1+\frac{A_1C_1}{B_1C_1}\right)=1+\frac{C_1B}{AC_1}\Rightarrow\frac{BA_1}{CA_1}+\frac{A_1C_1}{B_1C_1}+\frac{BA_1}{CA_1}\cdot\frac{A_1C_1}{B_1C_1}=\frac{C_1B}{AC_1}\Rightarrow$$

$$\Rightarrow\frac{BA_1}{CA_1}+\frac{A_1C_1(CA_1+A_1B)}{B_1C_1\cdot CA_1}=\frac{C_1B}{AC_1}\Rightarrow\frac{BA_1}{CA_1}+\frac{A_1C_1\cdot BC}{B_1C_1\cdot CA_1}=\frac{C_1B}{AC_1}\Rightarrow$$

$$\Rightarrow\frac{BA_1}{CA_1}+\frac{A_1C_1\cdot BC}{B_1C_1}\cdot\frac{1}{CA_1}=\frac{C_1B}{AC_1}$$

(1) tenglikdan foydalanamiz:

$$\frac{BA_1}{CA_1}+\frac{A_1C_1\cdot BC}{B_1C_1}\cdot\frac{1}{CA_1}=\frac{C_1B}{AC_1}\Rightarrow\frac{BA_1}{CA_1}+\frac{AC\cdot BA_1}{B_1A\cdot CA_1}=\frac{C_1B}{AC_1}\Rightarrow\frac{BA_1}{CA_1}\left(1+\frac{AC}{B_1A}\right)=\frac{C_1B}{AC_1}\Rightarrow$$

$$\Rightarrow\frac{BA_1}{CA_1}\cdot\frac{B_1A+AC}{B_1A}\cdot\frac{AC_1}{C_1B}=1\Rightarrow\frac{BA_1}{CA_1}\cdot\frac{CB_1}{B_1A}\cdot\frac{AC_1}{C_1B}=1$$

Bu oxirgi tenglik esa Menelay teoremasining o’zginasi.

Faollashtiruvchi savollar.

1. *Uchburchak ta’rifini ayting?*
2. *To’rtburchak ta’rifini ayting?*
3. *Aylana ta’rifini ayting?*
4. *Radius ta’rifini ayting?*
5. *Vatar ta’rifini ayting?*
6. *Eyler formulasini ayting?*
7. *Geron formulasini ayting?*
8. *Menelay teoremasini ayting?*
9. *Cheva teoremasini ayting?*
10. *Varinyon teoremasini ayting?*

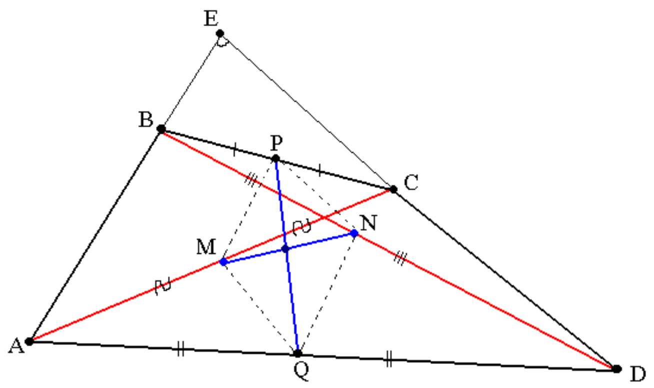
2-§. Olimpiada masalalari

Tayanch so'zlar: *Menelay teoremasi, Cheva teoremasi, uchburchak, kvadrat, to'rtburchak, parallelogramm, aylana, vatar, diametr, vektor, to'g'ri chiziq, diagonal.*

Mazkur paragraf geometriyada uchraydigan olimpiada masalalariga bag'ishlangan bo'lib, unda masalalarning to'liq yechimi va isbotlari keltirilgan.

1-masala. ABC qavariq to'rtburchakda diagonallarining o'rtasini tutashtiruvchi kesmaning uzunligi AD va BC tomonlarining o'rtalarini tutashtiruvchi kesmaning uzunligiga teng. AB va CD tomonlarini davom ettirishdan hosil bo'lgan burchak kattaligini toping.

Yechish. Masala shartiga mos chizmani chizamiz.



Berilgan shartlar:

$MN = PQ,\ BP = PC,\ AQ = QD,\ AM = MC,\ BN = ND$. Topish kerak $\angle AED = ?$.

ΔABD da NQ o'rta chiziq bo'lgani uchun $NQ = \frac{1}{2}AB$ va $\angle ABD = \angle NQD$.

ΔABC da MP o'rta chiziq bo'lgani uchun $MP = \frac{1}{2}AB \Rightarrow MP = NQ$.

ΔACD da MQ o'rta chiziq bo'lgani uchun $MQ = \frac{1}{2}CD$ va $\angle CDA = \angle MQA$.

ΔBCD da PN o'rta chiziq bo'lgani uchun $PN = \frac{1}{2}CD \Rightarrow PN = MQ$.

Bu ma'lumotlardan $MP = NQ, PN = MQ,\ MN = PQ,\ MP \parallel NQ,\ MQ \parallel PN \quad \Rightarrow \quad MPNQ$ to'g'ri to'rtburchak ekanligi kelib chiqadi. $\angle BAD = \alpha \Rightarrow \angle NQD = \alpha$, $\angle CDA = \beta \Rightarrow \angle MQA = \beta$. $\angle MNQ = 90^\circ$ ekanligidan

$$\angle MQA + \angle NQD + \angle MNQ = 180° \Rightarrow \alpha + \beta + 90° = 180° \Rightarrow \alpha + \beta = 90°$$

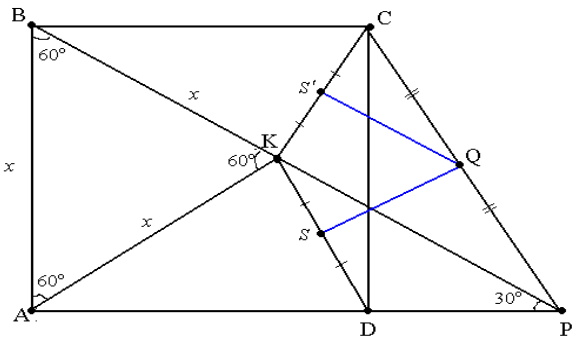
ni hosil qilamiz. ΔAED da

$$\angle AED = 180° - (\angle BAD + \angle CDA) \Rightarrow \angle AED = 180° - (\alpha + \beta) \Rightarrow \angle AED = 90°$$

ekanligi kelib chiqadi.

2-masala. Berilgan $ABCD$ kvadrat ichida ABK teng tomonli uchburchak chizilgan. BK va AD to'g'ri chiziqlar P nuqtada kesishadi. KD va CP kesmalar o'rtalarini tutashtiruvchi kesma kvadrat tomonining yarmiga teng bo'lishini ko'rsating.

Isbot. Chizmadagi belgilashlarga ko'ra S nuqtaning CK to'g'ri chiziqdagi simmetrik nuqtasini S' orqali belgilaymiz va $CK = DK$.

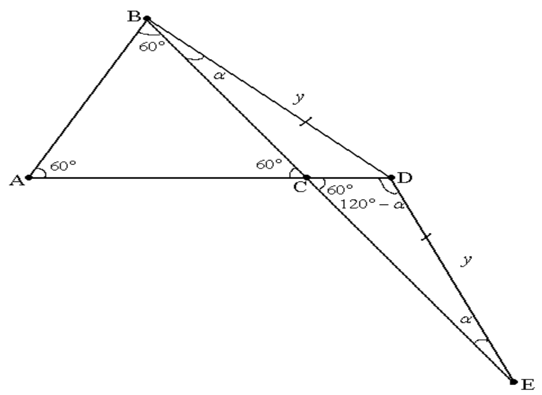


Muntazam uchburchakning tomoni uzunligini x orqali belgilasak, uchburchak ABP to'g'ri burchakli uchburchak ekanligidan $BP = 2x \Rightarrow BK = KP = x$. ΔCKP da QS' o'rta chiziq ekanligidan

$$QS' = \frac{1}{2}KP \Rightarrow QS' = \frac{1}{2}x,\ QS' = QS \Rightarrow QS = \frac{1}{2}x$$

ekanligi kelib chiqadi.

3-masala. Muntazam ABC uchburchakning AC tomoni C yo'nalishda davom ettirilib unda D nuqta, BC tomoni ham C nuqtada davom ettirilib unda E



nuqta shunday olinganki, $BD = DE$. U holda $AD = CE$ ekanligini isbotlang.

Isbot. Berilgan shartlarga asoslanib chizmani chizamiz.

$\angle EBD = \angle DEB = \alpha,\ BD = DE = y$ belgilashlar kiritamiz. Uchburchak CDE va ABD larda sinuslar teoremasini qo'llaymiz:

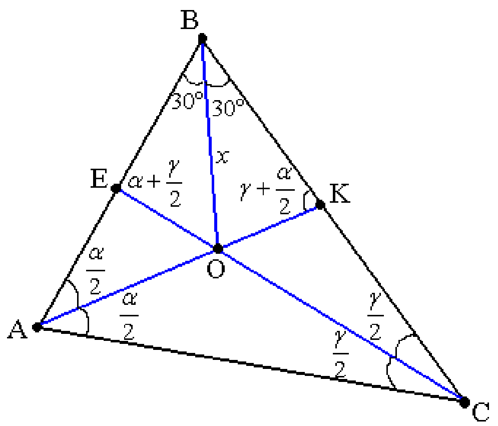
$$\frac{CE}{\sin(120^\circ - \alpha)} = \frac{y}{\sin 60^\circ} \Rightarrow CE = \frac{y\sin(60^\circ + \alpha)}{\sin 60^\circ}$$

$$\frac{AD}{\sin(60^\circ + \alpha)} = \frac{y}{\sin 60^\circ} \Rightarrow AD = \frac{y\sin(60^\circ + \alpha)}{\sin 60^\circ}.$$

Bu tengliklardan esa $AD = CE$ ekanligi kelib chiqadi.

4-masala. ABC uchburchakda $\angle B = 60^\circ$. AK va CE bissektrisalar O nuqtada kesishadi. U holda $OK = OE$ tenglikni isbotlang.

Isbot. Berilgan shartlarga asoslanib chizmani chizamiz.



AK va CE bissektrisalar bo'lgani uchun BO bissektrisa. Chizmadan quyidagi tengliklarni hosil qilamiz:

$$BO = x,\ \alpha + \gamma = 120^\circ \Rightarrow \gamma = 120^\circ - \alpha \Rightarrow \frac{\gamma}{2} = 60^\circ - \frac{\alpha}{2}.$$

Endi uchburchak BOE va BOK larda sinuslar teoremasini qo'llaymiz:

$$\frac{x}{\sin\left(\alpha + \frac{\gamma}{2}\right)} = \frac{OE}{\sin 30^\circ} \Rightarrow OE = \frac{x}{2\sin\left(\alpha + 60^\circ - \frac{\alpha}{2}\right)} \Rightarrow OE = \frac{x}{2\sin\left(60^\circ + \frac{\alpha}{2}\right)}$$

$$\frac{x}{\sin\left(\gamma+\frac{\alpha}{2}\right)}=\frac{OK}{\sin 30^\circ}\Rightarrow OK=\frac{x}{2\sin\left(120^\circ-\alpha+\frac{\alpha}{2}\right)}\Rightarrow OK=\frac{x}{2\sin\left(120^\circ-\frac{\alpha}{2}\right)}.$$

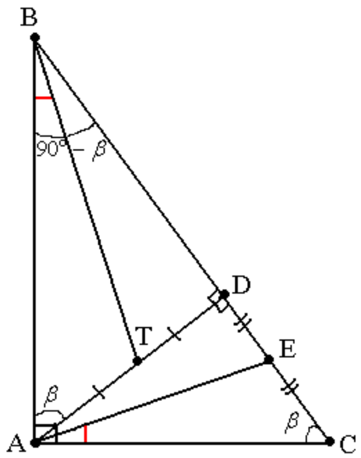
Hosil bo'lgan bu tengliklardan hadma-had bo'lib,

$$\frac{OE}{OK}=\frac{\sin\left(120^\circ-\frac{\alpha}{2}\right)}{\sin\left(60^\circ+\frac{\alpha}{2}\right)}=\frac{\sin\left(180^\circ-\left(60^\circ+\frac{\alpha}{2}\right)\right)}{\sin\left(60^\circ+\frac{\alpha}{2}\right)}=\frac{\sin\left(60^\circ+\frac{\alpha}{2}\right)}{\sin\left(60^\circ+\frac{\alpha}{2}\right)}=1\Rightarrow OK=OE$$

tenglikni hosil qilamiz.

5-masala. ABC uchburchakda $\angle A=60^\circ$. AD gipotenuzaga tushirilgan balandlik. T va E nuqtalar AD va DC tomonlarning o'rtalari. U holda $\angle ABT=\angle CAE$ ekanligini isbotlang.

Isbot. Chizmada belgilashlar kiritamiz:



$\Delta ABD\sim\Delta ADC$, BT va AE to'g'ri chiziqlar mos ravishda ΔABD va ΔACD larning $90^\circ-\beta$ ga teng burchaklardan chiqqan medianalar. Demak, ular ajratgan burchaklar mos ravishda teng, ya'ni $\angle EAD=\angle TBD,\ \angle ABT=\angle CAE$.

6-masala. PAQ burchakka ichki chizilgan aylana burchak tomonlariga P va Q nuqtalarda urinadi. $BC\ (B\in AP, C\in AQ)$ to'g'ri chiziq aylanaga T nuqtada urinadi. BQ va CP to'g'ri chiziqlar M nuqtada kesishadi. A, T, M nuqtalar bir to'g'ri chiziqda yotishini isbotlang.

Isbot. A, T, M nuqtalar bir to'g'ri chiziqda yotishi uchun $\frac{CT}{TB}\cdot\frac{BM}{MQ}\cdot\frac{AQ}{AC}=1$ ekanligini ko'rsatish yetarli. Quyidagi uchburchaklarda Menelay teoremasini qo'llaymiz:

$$\Delta CPQ \text{ da } \frac{CM}{MP}\cdot\frac{PM}{MQ}\cdot\frac{AQ}{AC}=1\Rightarrow\frac{PN}{NQ}=\frac{AC}{AQ}\cdot\frac{PM}{MC} \qquad (1)$$

$$\Delta BPQ \text{ da } \frac{BM}{MQ}\cdot\frac{MQ}{PM}\cdot\frac{AP}{AB}=1\Rightarrow\frac{BM}{MQ}=\frac{AB}{AP}\cdot\frac{PN}{NQ} \qquad (2)$$

(1) va (2) tengliklardan

$$\frac{BM}{MQ}=\frac{AB}{AP}\cdot\frac{AC}{AQ}\cdot\frac{MP}{MC} \qquad (3)$$

tenglikni hosil qilamiz.

$$\Delta BCP \text{ da } \frac{BT}{CT}\cdot\frac{CM}{MP}\cdot\frac{AP}{AB}=1\Rightarrow\frac{AB}{AP}=\frac{BT}{CT}\cdot\frac{MC}{MP} \qquad (4)$$

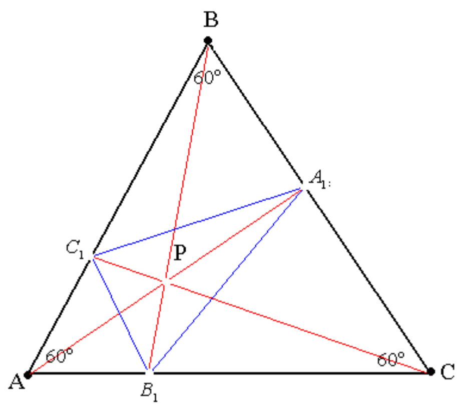
(3) va (4) dan

$$\frac{BM}{MQ}=\frac{AB}{AP}\cdot\frac{AC}{AQ}\cdot\frac{MP}{MC}=\frac{BT}{CT}\cdot\frac{MC}{MP}\cdot\frac{MP}{MC}\cdot\frac{AC}{AQ}\Rightarrow\frac{BM}{MQ}\cdot\frac{AQ}{AC}\cdot\frac{CT}{BT}=1$$

ekanligi kelib chiqadi.

7-masala. ABC muntazam uchburchakning ichida ixtiyoriy P nuqta olingan. AP, BP, CP to'g'ri chiziqlar BC, CA, AB kesmalarni mos ravishda A_1, B_1, C_1 nuqtalarda kesadi. U holda $A_1B_1\cdot B_1C_1\cdot C_1A_1\geq A_1B\cdot B_1C\cdot C_1A$ tengsizlikni isbotlang.

Isbot. $\Delta A_1CB_1,\ \Delta AC_1B_1,\ \Delta A_1BC_1$ larda kosinuslar teoremasini qo'llaymiz:



$$A_1B_1^{\ 2} = A_1C^2 + B_1C^2 - CA_1 \cdot CB_1 \geq 2 \cdot A_1C_1 \cdot B_1C - CA_1 \cdot CB_1 = A_1C_1 \cdot B_1C \Rightarrow A_1B_1^{\ 2} \geq A_1C_1 \cdot B_1C$$

$$B_1C_1^{\ 2} \geq AC_1 \cdot AB_1,\ A_1C_1^{\ 2} \geq A_1B \cdot BA_1.$$

Hosil bo'lgan tengsizliklarni hadma-had ko'paytirib

$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq \sqrt{AB_1 \cdot B_1C \cdot BC_1 \cdot C_1A \cdot CA_1 \cdot A_1B} \qquad (1)$$

tengsizlikni hosil qilamiz. ΔABC da Cheva teoremasini qo'llaymiz:

$$\frac{AB_1}{B_1C} \cdot \frac{BC_1}{C_1A} \cdot \frac{CA_1}{A_1B} = 1 \Rightarrow A_1B \cdot B_1C \cdot C_1A = AB_1 \cdot BC_1 \cdot CA_1 \qquad (2)$$

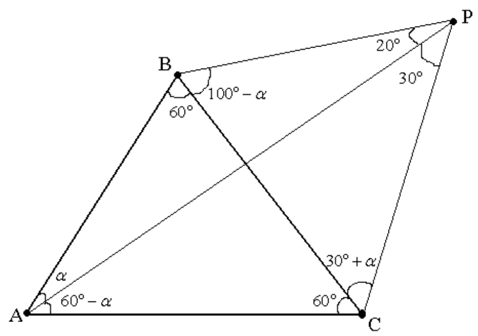
Yuqoridagi (1) tengsizlikka (2) tenglikni qo'llab,

$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq \sqrt{AB_1 \cdot B_1C \cdot BC_1 \cdot C_1A \cdot CA_1 \cdot A_1B} \Rightarrow A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A$$

isbotlash talab etilgan tengsizlikni hosil qilamiz.

8-masala. ABC muntazam uchburchakning BC tomonini kesib o'tuvchi AP nurda P nuqta shunday olinganki, $\angle APB = 20°,\ \angle APC = 30°$ bo'lsa, $\angle BAP$ ni toping.

Yechish. Masalaning shartiga mos chizma chizamiz.



Uchburchak tashqarisida olingan nuqta uchun Chevaning sinuslar teoremasini qo'llaymiz:

$$\frac{\sin(30° + \alpha)}{\sin(90° + \alpha)} \cdot \frac{\sin(60° - \alpha)}{\sin\alpha} \cdot \frac{\sin(160° - \alpha)}{\sin(100° - \alpha)} = 1 \Rightarrow \frac{\sin(30° + \alpha)}{\cos\alpha} \cdot \frac{\sin(60° - \alpha)}{\sin\alpha} \cdot \frac{\sin(20° + \alpha)}{\cos(10° - \alpha)} = 1$$

Hosil bo'lgan tenglikni soddalashtiramiz.

$$\sin(30° + \alpha) \cdot \sin(60° - \alpha) \cdot \sin(20° + \alpha) = \sin\alpha \cdot \cos\alpha \cdot \cos(10° - \alpha) \Rightarrow$$

$$\Rightarrow \frac{1}{2}[\cos(30° - 2\alpha) - \cos 90°]\cdot \sin(20° + \alpha) = \sin\alpha \cdot \cos\alpha \cdot \cos(10° - \alpha) \Rightarrow$$

$$\Rightarrow \cos(30° - 2\alpha)\cdot \sin(20° + \alpha) = \sin 2\alpha \cdot \cos(10° - \alpha) \Rightarrow$$

$$\Rightarrow \cos(30° - 2\alpha)\cdot \cos(70° - \alpha) = \sin 2\alpha \cdot \cos(10° - \alpha) \Rightarrow$$

$$\Rightarrow \frac{1}{2}[\cos(40° + \alpha) + \cos(100° - 3\alpha)] = \sin 2\alpha \cdot \sin(80° + \alpha) \Rightarrow$$

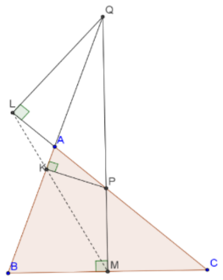
$$\Rightarrow \frac{1}{2}[\cos(40° + \alpha) + \cos(100° - 3\alpha)] = \frac{1}{2}[\cos(80° - \alpha) - \cos(80° + 3\alpha)] \Rightarrow$$

$$\Rightarrow \cos(40° + \alpha) + \cos(90° + 10° - 3\alpha) = \cos(80° - \alpha) - \cos(90° - (10° - 3\alpha)) \Rightarrow$$

$$\Rightarrow \cos(40° + \alpha) - \sin(10° - 3\alpha) = \cos(80° - \alpha) - \sin(10° - 3\alpha) \Rightarrow$$

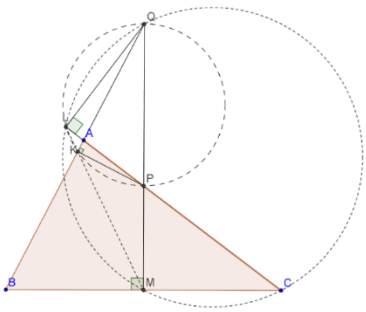
$$\Rightarrow \cos(40° + \alpha) = \cos(80° - \alpha) \Rightarrow 40° + \alpha = 80° - \alpha \Rightarrow \alpha = 20°$$

9-masala. O’tkir burchakli *ABC* uchburchakda *AC* > *AB*, *M*-*BC* tomon o’rtasi. *BC* tomon o’rta perpendikulari *AC* va *AB* to’g’ri chziqlarni mos ravishda *P* va *Q* nuqtalarda kesib o’tadi. *P* va *Q* nuqtalarning *AB* va AC to’g’ri chiziqlardagi proyeksiyasi mos ravishda *K* va *L* nuqtalar. *K*,*L* va *M* bir to’g’ri chiziqda yotishini



isbotlang.

Yechish. $\angle QMC = 90° = \angle QLC$ bundan *L*,*Q*,*C*,*M* nuqtalarning barchasi bir aylanada (*CQ* diametrli aylanada) yotadi. $\angle QKP = 90° = \angle QLP$ bundan *K*,*L*,*P*,*Q* nuqtalarning barchasi bir aylanada (*PQ* diametrli aylanada) yotadi. *BQC* uchburchak teng yonli uchburchak, chunki *QM* ham mediana, ham balandlik. Demak, $\angle BQM = \angle CQM = \alpha$. $\angle MLC = \angle MQC = \alpha = \angle KQP = \angle KLP = \angle KLC$. Bundan *K*,*L* va *M* bir to’g’ri chiziqda yotadi. ■



Faollashtiruvchi savollar.

1. *Uchburchakning o'rta chizig'i deb nimaga aytiladi?*
2. *Uchburchak medianasining xossalarini ayting?*
3. *Uchburchak bissektrisasining xossalarini ayting?*
4. *Chevaning sinuslar teoremasini ayting?*
5. *Qanday holda qavariq to'rtburchakka tashqi aylana chizish mumkin?*
6. *Qanday holda qavariq to'rtburchakka ichki aylana chizish mumkin?*
7. *Rombga tashqi aylana chizish mumkinmi?*
8. *Parallelogrammga ichki aylana chizish mumkinmi?*

IV BOB. OLIMPIADA MASALALARI VA TESTLARI

1-§. Yozma ish materiallari

Tayanch so'zlar: *Tengsizlik, parametr, kvadrat ildiz, tenglama, katet, gipotenuza, modul, proporsiya, Eyler formulasi, parallelogramm, aylana, vatar, diametr, vektor, to'g'ri chiziq, diagonal.*

1- variant

1. Yig'indini hisoblang: $\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{2018}+\sqrt{2019}}$
2. Tengsizlikni isbotlang: $\sqrt{30+\sqrt{30+\cdots+\sqrt{30}}}+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}<8$
3. Tenglamani yeching: $(x^2-2207^2)^2=8828x+1$
4. $|x^2-3x+2|=a$ tenglama a ning qanday qiymatida 3 ta yechimga ega bo'ladi?
5. To'g'ri burchakli uchburchak katetlari a va b ma'lum. To'g'ri burchakli uchburchak gipotenuzasiga tushirilgan bissektrissasini toping.

2- variant

1. Tengsizlikni isbotlang: $\sqrt{1\cdot 2}+\sqrt{2\cdot 3}+\sqrt{3\cdot 4}+\cdots+\sqrt{226\cdot 227}<25764$
2. Tenglamani butun sonlarda yeching. $x^3+9x^2+16x-xy-2y-3=0$
3. $2^n+4^n, n\in N$ ko'rinishhdagi sonlar to'plamida nechta to'la kvadrat bo'ladigan son bor?
4. $(x+1)^{2n}-x^{2n}-2x-1$ ko'phadni $2x^3+3x^2+x$ ko'phadga bo'lgandagi qoldiqni toping. $(n\in N)$
5. $ABCD$ $(BC\parallel AD)$ trapetsiyada $2AB=CD, BC<AD, \angle BAD+\angle CDA=120°$ bo'lsa, $ABCD$ trapetsiyaning burchaklarini toping.

3- variant

1. Noldan farqli $a,b,c\in R$ sonlar uchun
$$\frac{a^2}{b^2+c^2}+\frac{b^2}{a^2+c^2}+\frac{c^2}{b^2+a^2}\geq\frac{3}{2}$$
tengsizlikni isbotlang.
2. Ayniyatni isbotlang: $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$
3. Tenglamani yeching.
$$\frac{a+b-x}{c}+\frac{a+c-x}{b}+\frac{b+c-x}{a}=1-\frac{4x}{a+b+c}$$
4. $\forall n\in N$ da $\frac{10^n+8}{9}$ ifodaning butun son ekanligini isbotlang.

5. ABC uchburchakning AK va CE bissektrissalari O nuqtada kesishadi. Agar $\angle B = 60°$ bo'lsa, u holda $OE = OK$ ekanligini isbotlang.

4- variant

1. a, b, c musbat sonlar uchun
$$\frac{b+c}{\sqrt{a^2+bc}} + \frac{a+c}{\sqrt{b^2+ac}} + \frac{b+a}{\sqrt{c^2+ab}} \geq 4$$
tengsizlikni isbotlang.
2. ABC uchburchakda A_1, B_1, C_1 nuqtalar mos ravishda BC, AC, AB tomonlarining o'rtalari, BH esa uning balandligi. Agar AHC_1 va CHA_1 uchburchaklarga tashqi chizilgan aylanalar M nuqtada kesishsa ($M \neq H$), u holda $\angle ABM = \angle CBB_1$ tenglikni isbotlang.
3. a va b butun sonlar shundayki, $\forall m, n \in N$ da $an^2 + bm^2$ ifoda to'la kvadrat bo'ladi. $ab = 0$ bo'lishini isbotlang.
4. Tenglamani yeching: $\frac{x^3}{\sqrt{4-x^2}} + x^2 - 4 = 0$
5. x ning qanday qiymatlarida $\frac{sinx+tgx}{cosx+ctgx}$ ifoda musbat bo'ladi.

5- variant

1. Tenglamani butun sonlarda yeching: $x^2 = y^2 + 2y + 13$
2. $y = \frac{x}{ax^2+b}$ funksiyaning eng katta qiymatini toping.
3. ABC muntazam uchburchak ichidan ixtiyoriy P nuqta olinib, undan BC, AC, AB tomonlariga mos ravishda PD, PE, PF perpendikulyarlar tushirilgan
$$\frac{PD+PE+PF}{BD+CE+AF} = ?$$
4. Hisoblang: $cos\frac{2\pi}{15} + cos\frac{4\pi}{15} - cos\frac{7\pi}{15} - cos\frac{\pi}{15}$
5. $10^{900} - 2^{1000}$ ifodani 1994 ga bo'inishini isbotlang.

6- variant

1. Parametrga bog'liq tenglamani yeching:$\sqrt{a - \sqrt{a-x}} = x$
2. a, b, c musbat haqiqiy sonlar uchun quyidagi tengsizlikni isbotlang.
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{2}{a+b} + \frac{2}{c+b} + \frac{2}{a+c} \geq \frac{9}{a+b+c}$$
3. Tenglamaning haqiqiy yechimlarini toping: $x \cdot [x] = 2018$
4. PAQ burchakka ichki chizilgan aylana burchak tomonlariga P va Q nuqtalarda urinadi. BC ($B \in AP, C \in AQ$) to'g'ri chiziq aylanaga T nuqtada

urinadi. BQ va CP to'g'ri chiziqlar M nuqtada kesishadi. A, T, M nuqtalar bir to'g'ri chiziqda yotishini isbotlang.

5. $ABCD$ rombga ichki aylana chizilgan. Aylananing P nuqtasida o'tkazilgan urinma AB, BC, AD tomonning davomlarini N, Q, M nuqtalarda kesadi. Agar $MN{:}NP{:}PQ = 7{:}1{:}2$ bo'lsa, romb burchaklarini toping.

7- variant

1. Barcha $n \geq 2$ natural sonlar uchun
$$\frac{n^{1003} + n^{1002} + n^{1001} + 1}{n+1}$$
murakkab son ekanligini isbotlang.
2. $a, b, c, d > 0$ va $a^2 + b^2 + c^2 + d^2 = 1$ o'rinli bo'lsa,
$$(1-a)(1-b)(1-c)(1-d) \geq abcd$$
tengsizlikni isbotlang.
3. ABC – o'tkir burchakli teng yonli uchburchak ($AB = AC$). CD – balandlik. C nuqtani markaz qilib CD radiusli ω_1 aylana chizilgan. AC tomonning davomi ω_1 aylanani P nuqtada kesadi. B nuqtani markaz qilib BD radiusli ω_2 aylana ω_1 aylanani ikkinchi marta E nuqtada kesadi. $\angle PDE$ ni toping.
4. $sin2\pi x = sin\pi x$ tenglamaning $[-1; 1)$ kesmada nechta ildizi bor?
5. $\frac{1+x^2}{1+x}$ ifodaning eng kichik qiymatini toping.

8- variant

1. Hisoblang: $sin47° + sin61° - sin11° - sin25°$
2. Agar teng yonli trapetsiyaning balandligi h, yon tomoni esa unga tashqi chizilgan aylana markazidan α burchak ostida ko'rinsa, trapetsiyaning yuzini toping.
3. ABC uchburchakka ichki chizilga aylana AB, BC, AC tomonlariga mos ravishda N, P, K nuqtalarda urinadi. BK kesma shu aylanani L nuqtada kesadi. $T = AL \cap NK, Q = CL \cap KP$ bo'lsin. BK, NQ, PT to'g'ri chiziqlar bitta nuqtada kesishishini isbotlang.
4. Tenglamani butun sonlarda yeching: $3 \cdot 2^x + 1 = y^2$
5. Hisoblang: $\frac{2}{\sqrt{4 - 3\sqrt[4]{5} + 2\sqrt{5} - \sqrt[4]{125}}}$

9- variant

1. Tenglamni butun sonlarda yeching: $(x+1)(y^2 - x^2 - 4) = x^2$

2. $a, b, c > 0 \quad (a+b-c)(a+c-b)(b+c-a) = abc$ bo'lsa, $a = b = c$ ekanligini isbotlang.
3. Ixtiyoriy $ABCD$ qavariq to'rtburchakka ichki aylana chizish mumkin. AC va BD diagonallari O nuqtada kesishadi. $P_{AOB} = P_{COD} = P_{BOC}$ shart bajarilsa, $ABCD$ romb ekanligini isbotlang.
4. $y = 2sin^2x + 4cos^2x + 6sinxcosx$ funksiyaning eng katta va eng kichik qiymatlarini toping.
5. Agar $(a, b) = 1$ bo'lib, $A \vdots a$ va $A \vdots b$ bo'lsa, u holda $A \vdots ab$ bo'ladi. Shuni isbotlang.

10- variant

1. Tenglamaning haqiqiy yechimlarini toping.
$$(x^2 - 3x - 2)^2 - 3(x^2 - 3x - 2) - 2 - x = 0$$
2. $ABCD$ to'rtburchakka aylana ichki chizilgan. $\angle BAD = \alpha, AC = d,$ $AB = BC = AD + CD$ bo'lsa, ABC uchburchakning yuzini toping.
3. $x, y, z > 0,\ x + y + z = 3$ bo'lsa,
$$\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} \geq \frac{1}{9} + \frac{2}{27}(xy + yz + zx)$$
4. Agar A, B, C uchburchakning burchaklari bo'lsa, $sin\frac{A}{2} \cdot sin\frac{B}{2} \cdot sin\frac{C}{2} \leq a$ shartni qanoatlantiruvchi a ning eng kichik qiymatini toping.
5. Ferma teoremasini isbotlang: p tub son, $a \in Z$ bo'lsa, u holda $a^p - a \vdots p$ o'rinli.

11- variant

1. $a, b, c > 0$ bo'lsa,
$$2(a^2 + 1)(b^2 + 1)(c^2 + 1) \geq (a + 1)(b + 1)(c + 1)(abc + 1)$$
tengsizlikni isbotlang.
2. $\frac{x^2-1}{y+1} + \frac{y^2-1}{x+1}$ ifoda butun son bo'lsa, $\frac{x^2-1}{y+1}$ va $\frac{y^2-1}{x+1}$ ifodalar ham butun ekanligini isbotlang.
3. Agar x, y, z, t lar butun sonlar $x + y + z + t = 0$ shartni qanoatlantirsa, u holda $\frac{x^4+y^4+z^4+t^4}{2} + 2xyzt$ ifoda to'la kvadrat bo'lishini isbotlang.
4. Agar $(a, b) = 1,\ a, b \in Z$ bo'lsa, u holda $a^2 + b^2$ ko'rinishdagi sonning $4k + 3$ ko'rinishdagi bo'luvchisi mavjud emas. Shuni isbotlang.
5. Tomonlari a, b, c bo'lgan uchburchakka tashqi chizilgan aylana markazidan og'irlik markazigacha bo'lgan masofani toping.

12- variant

1. $\begin{cases} x^2+y^2=4 \\ z^2+t^2=9 \\ xt+zy=6 \end{cases}$ bo'lsa, $\max\{x+z\}$ toping.
2. Munatazam uchburchakning ichidan shunday nuqta olinganki, bu nuqtadan uchburchak uchlarigacha bo'lgan masofalar 3, 4, 5 ga teng bo'lsa, uning yuzini toping.
3. $ABCD$ kvadrat ichida ixtiyoriy M nuqta olingan.
$$\angle MAB+\angle MBC+\angle MCD+\angle MDA>135°$$
ekanligini isbotlang.
4. Eyler formulasini isbotlang: Uchburchakka tashqi chizilgan va ichki chizilgan aylanalar markazlari orasidagi masofa quyidagi formula orqali topiladi
$$d^2=R^2-2Rr$$
5. Ixtiyoriy k natural son uchun k^5-5k^3+4k ifodani 120 ga bo'linishini isbotlang.

13- variant

1. Tenglamani yeching: $\sqrt{x^2-5x+1}+\sqrt{8x-x^2-12}=\sqrt{3x-11}$
2. Tengsizlikni yeching:
$$\sqrt{7x^3-28x^2+5x-20}+\sqrt{-3x^3+12x^2-4x+16}\le 7x^2+6x-136$$
3. Teng yonli trapetsiyaning diagonallari uni 2 ta teng yonli uchburchakka bo'ladi. Trapetsiyaning burchaklarini toping.
4. Uchburchakning yuzi 5 sm^2. Uning ikki tomonlari uzunliklari mos ravishda 3 sm va 4 sm. Berilgan tomonlar hosil qilgan burchak bissektrissalari bilan bo'lingan uchburchaklar yuzalarini toping.
5. $a>0$ bo'lsa, $a^{10}+\frac{3}{a^2}+\frac{4}{a}\ge 8$ tengsizlikni isbotlang.

14- variant

1. Tenglamani yeching: $(x^2-16)(x-3)+9x^2=0$
2. Agar $\alpha,\ \beta,\ \gamma$ uchburchakning burchaklari bo'lsa, u holda quyidagi tenglikni isbotlang.
$$cos^2\alpha+cos^2\beta+cos^2\gamma+2cos\alpha\cdot cos\beta\cdot cos\gamma=1$$
3. Koordinatalar tekisligida (x,y) koordinatalari $x^2+y^2=x^2y^2+1$ tenglamani qanoatlantiruvchi nuqtalar to'plamini ko'rsating.
4. $ABCD$ qavariq to'rtburchakda diagonallarining o'rtasini tutashtiruvchi kesmaning uzunligi AD va BC tomonlarining o'rtalarini tutashtiruvchi

kesmaning uzunligiga teng. AB va CD tomonlarini davom ettirishdan hosil bo'lgan burchak kattaligini toping.

5. Hisoblang: $\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\cdots+\frac{1}{100^2-1}$

15- variant

1. Tenglamani yeching: $(x+3)^4+(x+5)^4=4$
2. Birinchi chorak burchaklari uchun quyidagi tengsizlikni isbotlang.
$$\cos(sinx) > \sin(cosx)$$
3. Koordinatalar tekisligida (*x*,*y*) koordinatalari $x^2-\left(y+\frac{1}{y}\right)x+1=0$ tenglamani qanoatlantiruvchi nuqtalar to'plamini ko'rsating.
4. *ABC* uchburchakning *A* uchidan o'tkazilgan medianasini davom ettirganda shu uchburchakka tashqi chizilgan aylanani *D* nuqtada kesib o'tadi. *AC* va *DC* vatarlarning xar biri 1 ga teng bo'lsa, *BC* kemaning uzunligini toping.
5. Odam tinch turgan eskalatorda 75 sekundda ko'tariladi. Eskalator tinch turgan odamni 50 sekundda olib chiqadi. Tepaga harakat qilayotgan odam tepaga chiqayotgan eskalator bo'yicha qancha vaqtda chiqadi?

16- variant

1. Tenglamani qanoatlantiruvchi x va y ning barcha butun qiymatlarini toping.
$$9x^2y^2+6xy^2-9x^2y+2x^2+y^2-18xy+7x-5y+8=0$$
2. Parametr a ning har bir qiymatiga berilgan tengsizlikning barcha yechimlarini toping.
$$x+2a-2\sqrt{3ax+a^2}>0$$
3. Tenglamani yeching.
$$25x^2-20x+6=(\sqrt{2}-cos\frac{\pi x}{4})(\sqrt{2}+cos\frac{\pi x}{4}$$
4. Ayalana *ABC* uchburchakning *A* va *C* uchlaridan o'tib, uning *AB* tomonini *D* nuqtada, *BC* tomonini *E* nuqtada kesib o'tadi. Agar $AD=5, AC=2\sqrt{7}$, $BE=4$ va $BD:CE=3:2$ bo'lsa, *CDB* burchakni toping.
5. *KLMN* trapetsiyaning *KN* va *LM* tomonlari o'zaro parallel bo'lib, $KN=3$. *M* burchagi 120^0 ga teng. *LM* va *MN* to'g'ri chiziqlar *KLM* uchburchakka tashqi chizilgan aylanaga urinma bo'lsalar, *KLM* uchburchak yuzini toping.

17- variant

1. $(1;2)$ oraliqda a parametrning istalgan qiymatida $(2x-1)x^2<(a+1)x+3a$ tengsizlikni qanoatlantiruvchi barcha *x* ning qiymatlarini toping.

2. $x \neq y \neq z \neq 1, xyz = 1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.
$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$
3. Tenglamalar sistemasini yeching:
$$\begin{cases} 4\sin y - 6\sqrt{2}\cos x = 5 + 4cos^2 y \\ \cos 2x = 0 \end{cases}$$
4. Tomonlarining uzunliklari butun sonlar bilan o'lchanadigan uchburchakka radiusi 1 ga teng bo'lgan aylana ichki chizilgan. Uchburchak tomonlarini toping.
5. Asoslaridan birining burchaklari 40^0 va 50^0. O'rta chizig'ining uzunligi 4 ga teng bo'lgan trapetsiya berilgan. agar trapetsiyaning asoslari o'rtasini birlashtiruvchi kesmaning uzunligi 1 ga teng bo'lsa, trapetsiyaning kichik asosining uzunligini toping.

18- variant

1. $1 < a < b < c$ shartni qanoatlantiruvchi va $(a-1)(b-1)(c-1)$ ifoda $abc - 1$ ning bo'luvchisi bo'ladigan barcha natural a, b, c larni toping.
2. Tengsizlikni yeching:
$$log_{\ 0,5}\left(log_4 \frac{x-11}{x-8}\right) \geq 0$$
3. $f: R \to f$ bo'ladigan va quyidagi tenglikni qanoatlantiradigan barcha $f(x)$ funksiyalarni toping.
$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$$
4. To'g'ri chiziq aylanaga *A* nuqtada urinadi. *BC* kesma aylananing diametri. Agar diametrning oxirgi nuqtalari urinma to'g'ri chiziqdan *a* va *b* masofaga teng uzoqlikda joylashgan bo'lsa, *A* nuqtadan diametrgacha bo'lgan masofani toping.
5. *ABC* uchburchak ichida *M* nuqta shunday olinganki, $\angle AMC = 90°$, $\angle AMB = 150°$ va $\angle BMC = 120°$ shartni qanoatlantiradi. *AMC*, *AMB*, *BMC* uchburchaklarga tashqi chizilgan aylana markazlari mos ravishda *P*, *Q*, *R* nuqtalarda bo'lsa, u holda $S_{PQR} > S_{ABC}$ bo'lishini isbotlang.

19- variant

1. $0 < \alpha < \beta < \gamma < 2\pi$ munosabatlarni qanoatlantiradigan α, β, γ sonlar berilgan bo'lsin. Agar ixtiyoriy haqiqiy x son uchun
$$\cos(x+\alpha) + \cos(x+\beta) + \cos\gamma = 0$$
tenglik o'rinli bo'lsa, $\gamma - \alpha$ ni toping.

2. Quyidagi tenglamaning $(-\pi;\pi)$ oraliqda juft bo'lmagan yechimlarga ega bo'ladigan *a* ning barcha qiymatlarini toping.

$$(|a|-2)\cos 3x+(2-|4-a|)\cos 2x+(2-|a|)\sin 2x=0$$

3. Tenglamani yeching:

$$\frac{\sqrt{2-x^2+2x}+x-2}{log_3\left(\frac{5}{2}-x\right)+log_3 2}\le 0$$

4. *ABC* uchburchakning *CD* medianasida *E* nuqta yotadi. *E* nuqtadan o'tuvchi va *AB* tomonga *A* nuqtada urinuvchi ω_1 aylana va *B* nuqtada urinuvchi ω_2 aylanalar *AC* va *BC* tomonlarni mos ravishda *M* va *N* nuqtada kesadi. *CMN* uchburchakka tashqi chizilgan aylana ω_1 va ω_2 aylanalarga urinishini isbotlang.
5. Piramidaning asos tomoni a ga teng bo'lgan muntazam uchburchakdan iborat. Ikkita yon yoq asos tekisligiga perpendikulyar, teng yon qirralar esa o'zaro α burchak hosil qiladi. Berilgan piramidaga tengdosh va u bilan umumiy asosga ega bo'lgan uchburchakli to'g'ri prizmaning balandligini toping.

20- variant

1. Bir biriga teng bo'lmagan musbat a,b,c sonlar uchun quyidagi tengsizlikni isbotlang.

$$\left|\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{a+c}{c-a}\right|>1$$

2. $f(x,y)=|y|+3|x|-3$ va $g(x,y,a)=y^2+(x-a)(x+a)$ funksiyalar berilgan. Parametr *a* ning qanday musbat eng kichik qiymatida quyidagi tenglamalar sistemasi to'rttaga teng turli yechimga ega bo'ladi.

$$\begin{cases} f(x,y)=0 \\ g(x,y,a)=0 \end{cases}$$

3. Tengsizlikni yeching:

$$2\sqrt{7\cdot 10^x-2\cdot 4^x-5\cdot 25^x}<4\cdot 5^x$$

4. Radiusi $\sqrt{3}-1$ va *BAC* burchagi 60^0 bo'lgan aylanaga *ABC* uchburchak ichki chizilgan. Agar *BC* tomonga hamda *AB* va *AC* tomonlarning davomiga urunuvchi aylananing radiusi 1 ga teng bo'lsa, uchburchakning *ABC* va *ACB* burchaklarini toping.
5. Og'ma prizmaning asosi teng yonli trapetsiyadan iborat bo'lib, uning a ga teng bo'lgan yon tomoni kichik asosiga teng. O'tkir burchagi esa β ga teng. Prizmaning ustki asosining uchlaridan biri ostki asosining hamma uchlaridan

baravar uzoqlikda joylashgan. Agar prizmaning yon qirrasi asos tekisligi bilan α ga teng bo'lgan burchak tashkil qilsa, uning hajmini toping.

21- variant

1. Musbat sonlardan iborat $x_1, x_2, x_3, \dots, x_n$ ketma-ketlikda $x_1 = \frac{1}{2}$ va $n > 1$ uchun $x_{n+1} = 1 - x_1 x_2 x_3 \dots x_n$ bo'lsa, $0{,}99 < x_{10} < 1$ ni isbotlang.
2. Quyidagi tenglamadagi y, z ni qanoatlantiruvchi x ning barcha qiymatlari ichidan eng kichik qiymatini toping.
$$x^2 + 2y^2 + z^2 + xy - xz - yz = 1$$
3. Musbat a, b, c sonlari $a + b + c = 3$ shartni bajarsa, quyidagi tengsizlikni isbotlang.
$$\frac{a}{1+(b+c)^2} + \frac{b}{1+(c+a)^2} + \frac{c}{1+(a+b)^2} \le \frac{3(a^2+b^2+c^2)}{a^2+b^2+c^2+12abc}$$
4. PQ kesma tomonlari KL=1, PQ=3 ga teng bo'lgan KLMN to'g'ri to'rtburchak yotgan tekislikka parallel. KLMN to'g'ri to'rtburchakning barcha tomonlari va PQ kesma qandaydir sharga urunadi. Shu sharning hajmini toping.
5. *ABCD* trapetsiya berilgan (bunda $BC \parallel AD$). *P*, *M*, *Q*, *N* nuqtalar mos ravishda *AB*, *BC*, *CD* va *DA* tomonlarning o'rtasi. *AQ*, *PD* va *MN* kesmalarning bitta nuqtada kesishishini isbotlang.

22- variant

1. Ixtiyoriy olingan $a_1 < a_2 < \dots < a_{1984}$ sonlar ketma-ketligi uchun d bilan $a_2 - a_1,\ a_3 - a_2, \dots, a_{1984} - a_{1983}$ sonlarning eng kichigini belgilaylik. Isbotlang:
$$\frac{a_{1984} - a_1}{d} \ge 1983$$
2. Uchburchakning uchta balandligi bo'yicha bu uchburchakka ichki chizilgan aylananing radiusini hisoblang.
3. $1^{2003} + 2^{2003} + \dots + 2000^{2003} + 2001^{2003}$ yig'indini 13 ga bo'lgandagi qoldiqni toping.
4. $abc = 1\ va\ a, b, c \in R_+$ bo'lsa, quyidagi tengsizlikni isbotlang:
$$\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \ge \sqrt{a} + \sqrt{b} + \sqrt{c} + 3$$
5. Tenglamani yeching. $3 \cdot \{x\} + 2 \cdot [x] = 5$. Bu yerda $\{a\} - a$ ning kasr qismi, $[a] - a$ ning butun qismi.

23- variant

1. Tenglamani natural sonlarda yeching: $x^2 - xy + y - x = 2$
2. x va y ning butun qiymatlarida $2x+3y$ ifoda 17 ga qoldiqsiz bo'linsa, $9x+5y$ ifoda ham 17 ga qoldiqsiz bo'linishini isbotlang.
3. O nuqta ABC uchburchakning medianalar kesishish nuqtasi, BC tomonidan shunday D nuqta olinganki, bunda $OD \parallel AC$ shart bajariladi. $AODB$ to'rtburchak yuzining $AODC$ to'rtburchak yuzasiga nisbatini toping.
4. Ixtiyoriy natural $n > 2$ lar uchun $n^n - n^2 + n - 1$ sonining $(n-1)^2$ ga bo'linishini isbotlang.
5. ABC to'g'ri burchakli uchburchakda $(\angle C = 90°)$ CK - bissektrisa, CH - balandlik va $\frac{2}{CK^2} - \frac{1}{CH^2} = \frac{1}{10}$ bo'lsa, ABC uchburchakning yuzini toping.

24- variant

1. Musbat a, b, c sonlar $a + b + c = 1$ shartni qanoatlantirsa, u holda
$$\sqrt{\frac{ab}{ab+c}} + \sqrt{\frac{bc}{bc+a}} + \sqrt{\frac{ac}{ac+b}} \leq \frac{3}{2}$$
tengsizlikni isbotlang.
2. Uchburchak ABC ning burchaklari mos ravishda α, β, γ bo'lsa,
$$tg^2\frac{\alpha}{2} + tg^2\frac{\beta}{2} + tg^2\frac{\gamma}{2} \geq 1$$
tengsizlikni isbotlang.
3. ABCD - trapetsiyada $AD \parallel BC,\ CD = 1,\ \angle DBA = 23°\ va\ \angle BDC = 46°$, $BC{:}\,AD = 9{:}\,5$ bo'lsa, CD tomon uzunligini toping.
4. Integralni hisoblang: $\int_{-4}^{4} x^3|x|dx$
5. Tenglamani yeching.
$$\sqrt{x+7} \cdot \sqrt{2x+2} + \sqrt{x+7} \cdot \sqrt{5x-13} + \sqrt{2x+2} \cdot \sqrt{5x-13} = 8x - 4$$

25- variant

1. Tenglamani yeching: $[2x] + [3x] = 3$, bu yerda $[a] -$ a sonning butun qismi.
2. Ixtiyoriy uchburchak uchun quyidagi tenglik o'rinli bo'lishini isbotlang: $\frac{h_a h_b h_c}{a h_b h_c + b h_a h_c + c h_a h_b} = \frac{2S}{a^2+b^2+c^2}$,
bu yerda a,b,c – uchburchak tomonlari, h_a, h_b, h_c lar mos ravishda a,b,c tomonlarga tushirilgan balandliklar va S -uchburchak yuzi
3. $x, y, z > 0\ va\ x + y + z = 1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang.

$$\frac{x^2+3xy}{x+y}+\frac{y^2+3yz}{y+z}+\frac{z^2+3xz}{x+z}\le 2$$

4. Aytaylik *ABC* uchburchak uchun *DE* kesma *AB* ga parallel, *AD=DC*, *F* nuqta *AB* tomonda joylashgan bo'lsin. Agar *FDE* uchburchakning yuzi 4 ga teng bo'lsa, *ABC* uchburchakning yuzi topilsin.
5. *ABC* uchburchakda $AB\neq BC,\ \angle A=20°,\ \angle C=45°$ ekanligi ma'lum. *BM* mediana davom ettirib, unda *K* nuqta qo'yilgan, bunda *BM=MK*, *BH* balandlik davom ettirilib, unda *N* nuqta qo'yilgan, bunda *BH=HN* tengliklar o'rinli. *KAN* burchakni toping.

26- variant

1. Agar α,β,γ uchburchak burchaklari bo'lsa, u holda quyidagi tenglikni isbotlang.
$$sin^2\frac{\alpha}{2}+sin^2\frac{\beta}{2}+sin^2\frac{\gamma}{2}+2sin\frac{\alpha}{2}sin\frac{\beta}{2}sin\frac{\gamma}{2}=1$$
2. Ramanujon ayniyatini isbotlang.
$$\sqrt[3]{\sqrt[3]{2}-1}=\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}$$
3. Haqiqiy *x*,*y*,*z* sonlari uchun quyidagi tengsizlikni isbotlang.
$$\frac{x+y}{2}\cdot\frac{x^2+y^2}{2}\cdot\frac{x^3+y^3}{2}\le\frac{x^6+y^6}{2}$$
4. *ABC* uchburchakka *O* markazli ichki aylana chizilgan. *T* nuqtada u *BC* tomoniga urinadi. Bunda $\angle BOT:\angle COT=3:4$. Agar ichki chizilgan aylana radiusi 3 va $\angle A=30°$ bo'lsa, *BC* tomonini toping.
5. $\forall n\in N$ da n^3+5n ifodani 6 ga bo'linishini isbotlang.

27- variant

1. Agar α,β,γ uchburchak burchaklari bo'lsa, u holda quyidagi tenglikni isbotlang.
$$sin^2\alpha+sin^2\beta+sin^2\gamma=2+2cos\alpha cos\beta cos\gamma$$
2. $x_1\ va\ x_2$ $x^2-(a+d)x+ad-bc=0$ tenglamaning ildizlari bo'lsa, u holda $x_1^3\ va\ x_2^3$ ushbu $y^2-(a^3+b^3+3abc+3bcd)y+(ad-bc)^3=0$ tenglamaning ildizlari bo'lishini isbotlang.
3. k^5+3 ning k^2+1 ga bo'linadigan *k* ning barcha butun qiymatlarini toping.
4. Agar $xyz>0$ bo'lsa, $\frac{xy}{z}+\frac{yz}{x}+\frac{xz}{y}\ge x+y+z$ tengsizlikni isbotlang.

5. Qavariq to'rtburchakning diagonallari uni to'rtta uchburchakka ajratadi. Agar *AOB*, *BOC*, *COD*, *AOD* uchburchaklarning yuzlari mos ravishda S_1, S_2, S_3, S_4 bo'lsa, $S_1 \cdot S_2 = S_3 \cdot S_4$ ekanligini isbotlang.

28- variant

1. Tenglamani yeching. $x^3 - [x] = 3$. $[x] - x$ ning butun qismi.
2. $ABCD$ trapetsiyada $AB \parallel CD$ va $\angle D = 90°$. E nuqta CD tomonda olingan bo'lib, $AE = BE$ va ΔAED va ΔCEB lar o'xshash, lekin teng emas. Agar $\frac{CD}{AB} = 2018$ berilgan bo'lsa, $\frac{BC}{AD}$ ni toping.
3. Yig'indini hisoblang:
$$\frac{1}{sin1° \cdot sin2°} + \frac{1}{sin2° \cdot sin3°} + \cdots + \frac{1}{sin89° \cdot sin90°}$$
4. Ma'lumki *a, b, c, d* butun sonlar barchasi $ab - cd$ ga bo'linadi. $ab - cd$ ning qiymatini toping.
5. $a, b, c > 0$ bo'lib $a + b + c \geq abc$ bo'lsa, $a^2 + b^2 + c^2 \geq \sqrt{3}abc$ tengsizlikni isbotlang.

29- variant

1. $, b, c > 0$ va $abc = 1$ shart bajarilsa, u holda quyidagi tengsizlikni isbotlang.
$$a + b + c \leq a^2 + b^2 + c^2$$
2. $x + \frac{1}{x} = 2cos\alpha$ bo'lsa, $x^n + \frac{1}{x^n}$ nimaga teng?
3. ABC uchburchakning *A* va *B* uchlaridan o'tuvchi aylana *BC* tomonni *D* nuqtada kesadi. *B* va *C* uchlaridan o'tuvchi aylana esa *AB* tomonni *E* nuqtada va birinchi aylanani boshqa *F* nuqtada kesib o'tadi. *A,E,D,C* nuqtalar markazi *O* nuqtada bo'lgan aylanaga tegishli ekanligi ma'lum bo'lsa, $\angle BFO$ ni toping.
4. Agar x_1, x_2, x_3 sonlari ushbu $x^3 - 3x + 1 = 0$ tenglamani ildizlari bo'lsa, $x_1^5 + x_2^5 + x_3^5$ ni toping.
5. Agar *(x,y)* juftliklar $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ tenglikni qanoatlantirsa,
$$\sqrt{x^2 + \sqrt[3]{x^4y^2}} + \sqrt{y^2 + \sqrt[3]{x^2y^4}} = a$$
bo'lishini isbotlang.

30- variant

1. Agar $a, b \epsilon Q$ va $n \in N$ sonlar uchun $a^{2n+1} + b^{2n+1} = 2a^nb^n$ bo'lsa, u holda $1 - ab$ son biror ratsional sonning kvadrati ekanligini isbotlang.

2. Muntazam *ABC* uchburchakning *BC* tomonini kesib o’tuvchi *AP* nurda *P* nuqta shunday olinganki, $\angle APB = 20°$,$\angle APC = 30°$ bo’lsa, $\angle BAP$ ni toping.
3. Ushbu $\sqrt{x^2+2x+4}+\sqrt{x^2-x\sqrt{3}+1}$ ifodaning eng kichik qiymatini toping.
4. $a, b, c > 0$ va $abc = 1$ shart bajarilsa, u holda quyidagi tengsizlikni isbotlang.
$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \le 1$$
5. Tenglamani yeching. $(6x+7)^2(3x+4)(x+1) = 1$

31- variant

1. a) $2^n - 1$ soni 7 ga qoldiqsiz bo’linadigan *n* ning barcha butun musbat qiymatlarini toping. b) *n* ning hech bir butun musbat qiymatida $2^n + 1$ soni 7 ga bo’linmasligini toping.
2. Tomonlarining uzunliklari *a*, *b*, *c* bo’lgan uchburchak yuzi *S* bo’lsin. U holda $a^2+b^2+c^2 \ge 4\sqrt{3}S$ tengsizliko’rinli bo’lishini isbotlang. Qaysi hollarda tengsizlik bajariladi.
3. Quyidagi funksiyalarning grafigini chizing: a) $y = \begin{cases} x, & agar\ x \le 0 \\ x^2, & agar\ x > 0 \end{cases}$,
b) $y = \frac{|x|}{x}$, c) $y = |x+1| + |x-2|$
4. Berilgan *ABCD* kvadratning *BC* va *CD* tomonlaridan mos ravishda *K* va *L* nuqtalar $\angle AKB = \angle AKL$ bo’ladigan qilib tanlangan. $\angle KAL =?$
5. 2001 yil 1 yanvar dushanba edi. 2001 yildan 2050 yilgacha nechta yilda yakshanbalar soni dushanbalar sonidan ko’p.

32- variant

1. Beshta do’st bugun qanday hafta kunligi to’g’risida suhbatlashmoqda. Abduqodir “kechadan oldingi kun juma edi” dedi. Xasan “ ertadan keyingi kun seshanba bo’ladi” dedi. Javlon “kecha shanba edi” dedi. Jasur “ertaga dushanba bo’ladi” dedi. Sanjar “bugun payshanb” dedi. Ulardan qaysi biri adashgan.
2. Teng yonli trapetsiyaga aylana ichki chizilgan. Aylana yon tomonini urinish nuqtasida *m* va *n* ga teng kesmalarga ajratadi. Trapetsiyaning yuzini toping.
3. Istalgan musbat *x* va *y* sonlar uchun $x^2(f(x)+f(y)) = (x+y)f(f(x)\cdot y)$ tenglikni qanoatlantiradigan barcha $f: R^+ \to R^+$ funksiyalarni toping.
4. $a_2, a_3, \dots, a_{10}$ shunday musbat haqiqiy musbat sonlarki, $a_2 \cdot a_3 \cdot \dots \cdot a_{10} = 1$ $(1+a_2)^2(1+a_3)^3 \dots (1+a_{10})^{10} > 10^{10}$ ekanini isbotlang.

5. $(3;-1)$ nuqtadan o'tib, $(2;-3)$ nuqtadan $\frac{9}{\sqrt{17}}$ birlik masofada joylashgan to'g'ri chiziq tenglamasini tuzing.

33- variant

1. $(x-1)(40-x)\sqrt{22x-x^2-40}=0$ tenglamaning eng katta ildizi *m*, eng kichik ildizi *n* bo'lsin.
$$\frac{\left(m^2-\frac{1}{n^2}\right)^m\cdot\left(n+\frac{1}{m}\right)^{n-m}}{\left(n^2-\frac{1}{m^2}\right)^n\cdot\left(m-\frac{1}{n}\right)^{m-n}}$$
2. $x^2+2(k-1)x+k+5=0$ tenglama *k* ning qanday qiymatlarida kamida bitta musbat ildizga ega bo'ladi.
3. $2x+3y=100$ tenglamaning natural sonlarda yechimi nechta.
4. Radiusi 4 ga teng bo'lgan doirani ikki o'zaro perpendikulyar to'g'ri chiziqlar bilan 4 ta teng bo'lakka bo'lishdi. Bo'laklardan biriga ichki aylana chizildi. Shu aylananing radiusini toping.
5. *ABC* uchburchakning *AC* tomonida *M* nuqta, *BC* tomonida *K* nuqta shunday olinganki, $AM=\frac{2}{5}AC$ va $BK=\frac{1}{3}BC$ tengliklar o'rinli. U holda *BM* kesma *AK* kesmani qanday nisbatda bo'ladi.

34- variant

1. O'zining raqamlari yig'indisidan 59 marta katta bo'lgan natural sonalrni toping.
2. Auditoriyada 42 nafar talaba bor. Birinchi talaba qiz bola 7 o'g'il bolani taniydi, ikkinchi qiz bola 8 ta o'g'il bolani taniydi, uchinchi qiz bola 9 ta o'g'il bolani taniydi va hokazo. Oxirgi talaba qiz bola auditoriyadagi barcha o'g'il bolalarni taniydi. Auditoriyadagi talaba qizlar sonini toping.
3. *ABC* teng yonli to'g'ri burchakli uchburchakning *BC* gipotenuzasida *M* va *N* nuqtalar olingan bunda, *BM*:*MN*:*NC*=3:4:5 (*M* – *B* va *N* orasida) bo'lsa, *MAN* burchakni toping.
4. n^2+2n $(n\neq 4)$ ifoda 4 raqami bilan tugasa, shu sonning o'nli yozuvda kamida 3 ta juft raqam ishtirok etishini isbotlang.
5. ABC uchburchak *AF* medianasining o'rtasi *D* nuqta. *CD* to'g'ri chiziq *AB* tomonni *E* nuqtada kesib o'tadi. Agar *BD*=*BF* bo'lsa, *AE*=*DE* ekanligini isbotlang.

35- variant

1. *ABCD* to'rtburchakda $AB^2 + CD^2 = BC^2 + AD^2$. $AC \perp BD$ ekanligini isbotlang.
2. a, b, c – geometrik progressiya va $a + b + c = 111$ bo'lgan barcha natural sonlar (a, b, c) uchliklarini toping.
3. O'tkir burchakli *ABC* uchburchakda asosi *BC* tomonda yotgan r_a radiusli yarimdoira yasalgan, bu yarimdoira qolgan ikkita tomonga urinadi. Xuddi shunday r_b, r_c lar aniqlangan. Agar r – *ABC* ga ichki chizilga aylana radiusi bo'lsa, $\frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$ tenglikni isbotlang.
4. $\begin{cases} 8a^2 - 3b^2 + 5c^2 + 16d^2 - 10ab + 42cd + 18a + 22b - 2c - 54d = 42 \\ 15a^2 - 3b^2 + 21c^2 - 5d^2 + 4ab + 32cd - 28a + 14b - 54c - 52d = -22 \end{cases}$ shartni qanoatlantiradigan barcha a, b, c, d ratsional sonlarni toping.
5. Qavariq *ABCD* to'rtburchak perimetri p bo'lsin. $\frac{p}{2} < AC + BD < p$ ni isbotlang.

36- variant

1. Berilgan $a, b \geq 0$ sonlari uchun quyidagi tengsizlikni isbotlang.
$$\left(a^2 + b + \frac{3}{4}\right)\left(b^2 + a + \frac{3}{4}\right) \geq \left(2a + \frac{1}{2}\right)\left(2b + \frac{1}{2}\right)$$
2. Uchburchak uchlaridan chiquvchi *AN*, *BM*, *CK* to'g'ri chiziqlar *O* nuqtada kesishadi. *BN*:*BC*=1:4, *AM*:*AC*=1:3 va $S_{ABC} = 60\ sm^2$ bo'lsa, *BOK* uchburchak yuzini toping.
3. Raqamlari yig'indisi kvadrati bilan kvadratining raqamlari yig'indisi teng bo'ladigan barcha ikki xonali sonlarni toping.
4. Ushbu $7^{9^{9^9}}$ sonning oxirgi ikkita raqamini toping.
5. ABC uchburchakda AN bissektrissasini BM mediana K nuqtada kesib o'tadi. Agar AB:AC=3:8 bo'lsa, $\frac{S_{MNK}}{S_{MNC}}$ toping.

37- variant

1. Hisoblang: $4\cos 20° - \sqrt{3}\,\text{ctg}\,20°$
2. ABC uchburchak ichidan K nuqta olingan, $\angle ABK = 30°, \angle KAB = 10°$, $\angle ACB = 80°$ va $AC = BC$. $\angle AKC = ?$
3. Tenglamalar sistemasini yeching:

$$\begin{cases} x+y+z=6 \\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{11}{6} \\ xy+yz+zx=11 \end{cases}$$

4. Agar $7\sin\beta=\sin(2\alpha+\beta)$ bo'lsa, $3tg(\alpha+\beta)=4tg\alpha$ ekanligini isbotlang.
5. Taqqoslang: 100^{100} va 101^{99}

38- variant

1. Tenglamani natural sonlarda yeching. $x!+y!=10z+13$
2. $x,y,z\in R$ bo'lsa, quyidagi tengsizlikni isbotlang.
$$\frac{x^2+y^2+1}{2x^2+1}+\frac{y^2+z^2+1}{2y^2+1}+\frac{z^2+x^2+1}{2z^2+1}\geq 3$$
3. *ABCD* teng yonli (*AB*=*CD*, *AD*>*BC*) trapetsiyaning *AC* diagonali *BH* balandligini *M* nuqtada kesib, *B* uchidan boshlab 3:2 nisbatda bo'ladi. *D* uchidan chiquvchi to'g'ri chiziq *M* nuqtadan o'tib *AB* yon tomonni *K* nuqtada kesadi. Agar trapetsiyaning yuzi $210\,sm^2$ bo'lsa, u holda *AKM* uchburchak yuzini toping.
4. *ABCD* parallelogrammda *AN* va *BM* to'g'ri chiziqlar *O* nuqtada kesishadi. *BN:BC*=5:7, *CM:CD*=1:5 va *BON* uchburchak yuzi $10\,sm^2$ bo'lsa, *ABCD* parallelogram yuzini toping.
5. Agar $\begin{cases} 2n+1=a^2 \\ 3n+1=b^2 \end{cases}$ bo'lsa, u holda $5n+3$ soni tub emasligini isbotlang.

39- variant

1. $(x^3+x^{-3})+(x^2+x^{-2})+(x+x^{-1})=6$ tenglamani yeching.
2. Tenglamaniyeching: $x=\{x\}\cdot[x]$, bunda [x]-butun qism, {x}-kasr qism.
3. Agar $x_1,x_2,\dots,x_n\in[a,b], 0<a<b$ bo'lsa,
$$n^2\leq(x_1+x_2+\dots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)\leq\frac{(a+b)^2}{4ab}n^2$$
tengsizlikniisbotlang.
4. To'g'ri burchakli uchburchakning gipotenuzasi *c* ga teng. Gipatenuzaga o'tkazilgan mediana uchburchakning katetlari orasida o'rta proporsional miqdor bo'lsa, katetlarining uzunliklarini toping.
5. *ABC* uchburchakning *BD* medianasini *AE* to'g'richiziq *F* nuqtada kesib o'tsa va *BF:FD*=3:2 bo'lsa, *AF:FE* ni toping.

40- variant

1. $(x^2-1)(x+2)(\text{x}+4)=2008$ tenglamani yeching.
2. $f(x)=ax+b$ funksiya uchun $f(1)\le f(2)$; $f(4)\le f(3)$ va $f(2018)=2018$ bo'lsa, $f(2017)-f(2016)$ ning qiymatini hisoblang.
3. Agar $x>0, y>0, z>0$ bo'lsa, $\sqrt{\frac{x}{y+z}}+\sqrt{\frac{y}{x+z}}+\sqrt{\frac{z}{x+y}}>2$ tengsizlikni isbotlang.
4. Aylananing N nuqtasidan o'tkazilgan to'g'ri chiziq ikkinchi konsentrik aylanani C va D nuqtalarda kesib o'tsa, u holda $NC\cdot ND=R_1^2-R_2^2$ $(R_1>R_2)$ bo'lishini isbotlang.
5. *ABC* uchburchakning *AB* va *AC* tomonlari mos ravishda 3 sm va 4 sm bo'lsa, *BD* medianani *AE* bissektrisa *F* nuqtada kesib o'tishi ma'lum bo'lsa, u holda *BF:FD* nisbatni toping.

41- variant

1. Agar A, B, C uchburchak burchaklari bo'lsa, quyidagi tengsizlikni isbotlang:

$$\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}\le\frac{3\sqrt{3}}{8}$$

2. $ABCD$ qavariq to'rtburchak berilgan bo'lib, uning diagonallari F nuqtada kesishadi. Agar $\angle ABC=40°,\ \angle DBA=50°,\ \angle DAC=20°$ va $\angle AFB=70°$ bo'lsa, $\angle CDB$ ni toping.
3. Tenglamani yeching: $tgx+|tgx|+2^{\cos 3x}+2^{-\cos 3x}=2$
4. Tenglamani yeching: $\left(x^2-2003^2\right)^2=8012x+1$
5. $x, y, z, p, q, r>0,\ p+q+r=1$ va $x^p\cdot y^q\cdot z^r=1$ bo'lsa, quyidagi tengsizlikni isbotlang:

$$\frac{p^2x^2}{qy+rz}+\frac{q^2y^2}{px+rz}+\frac{r^2z^2}{px+qy}\ge\frac{1}{2}$$

42- variant

1. Agar ko'phad $P(t)=t^2-4t$ ko'rinishda va $x\ge 1, y\ge 1$ bo'lsa, $P\left(x^2+y^2\right)\ge P(2xy)$ tengsizlikni isbotlang.
2. Tenglamani yeching: $19[x]-96\{x\}=0$
3. Funksional tenglamani yeching. $f(x-y)=f(x)+f(y)-2xy,\ x, y\in R$

4. Teng yonli ABC uchburchakning $(AB = BC)$ BC yon tomonidan N va M (N nuqta B ga, M nuqta C ga yaqin) nuqtalar $NM = AM$, $\angle MAC = \angle BAN$ shartni qanoatlantiradi. $\angle CAN$ ni toping.
5. Teng yonli ABC uchburchakning $(AB = AC)$ A uchidagi burchagi $30°$. AB va AC tomonlaridan mos ravishda Q va P nuqtalar olingan. Agar $\angle QPC = 45°$ va $PQ = BC$ bo'lsa, u holda $BC = CQ$ ekanligini isbotlang.

43- variant

1. Tenglamani yeching.

$$\sqrt[4]{x-2} + \sqrt[4]{3-x} = 1$$

2. Musbat x va y sonlari uchun quyidagi tengsizlikni isbotlang.

$$\frac{x^3+y^3}{2} \cdot \frac{x^4+y^4}{2} \le \frac{x^7+y^7}{2}$$

3. Hisoblang:

$$\sqrt{1 + 2018^2 + \frac{2018^2}{2019^2}} + \frac{2018}{2019}$$

4. Yig'indini hisoblang:

$$\cos x + \cos 2x + \cos 3x + \cdots + \cos nx$$

5. ABC uchburchakda $\angle BAC < 90°$. C va B uchlaridan tushirilgan balandliklar tashqi aylanani mos ravishda D va E nuqtalarda kesib o'tadi. Bunda $DE = BC$. U holda $\angle BAC = ?$

Faollashtiruvchi savollar.

1. *O'zaro qo'shma sonlar deb nimaga aytiladi?*
2. *Modul ta'rifini ayting?*
3. *Ko'phadning ildizi deb nimaga aytiladi?*
4. *Bezu teoremasini ayting?*
5. *Uchburchakning medianalari kesishish nuqtasi nima deb ataladi?*
6. *Uchburchakning bissektrisalari kesishish nuqtasi nima deb ataladi?*
7. *Uchburchakning balandliklari kesishish nuqtasi nima deb ataladi?*
8. *n ta elementdan m tadan gruppashlar soni nimaga teng?*
9. *n ta elementdan m tadan o'rinlashtirishlar soni nimaga teng?*
10. *Pifagor teoremasini ayting?*

2-§. Test materiallari

Tayanch so'zlar: *Tenglama, tengsizlik, ayniyat, o'nli kasr, funksiya, ko'phad, irratsional sonlar, trigonometriya, parametr, kvadrat ildiz, tenglama, katet, gipotenuza, modul, proporsiya, Eyler formulasi, parallelogramm, aylana, vatar, diametr, vektor, to'g'ri chiziq, diagonal.*

1-variant

1. $\frac{1}{13}$ ni o'nli kasrga yoyilganda, verguldan keyingi 100 – raqamini toping.
   A) 0 B) 3 C) 9 D) 2
2. $a = \frac{111110}{111111}$; $b = \frac{222221}{222223}$; $c = \frac{333331}{333334}$ kasrlarni o'sish tartibida yozing.
   A) $b < c < a$ B) $c < a < b$ C) $b < c < a$ D) $a < c < b$
3. $(x-1)(x-2)(x-3)(x-4)+10$ ning eng kichik qiymatini toping.
   A) 10 B) 9 C) 8 D) 6
4. Kema *A* dan *B* ga 5 kunda, *B* dan *A* ga 7 kunda yetadi. Sol *A* dan *B* ga necha kunda yetadi?
   A) 35 B) 12 C) 17,5 D) 24
5. $\frac{2x+1}{x} + \frac{4x}{2x+1} = 5$ tenglama ildizlari yig'indisini toping.
   A) -0,5 B) 0,5 C) 1,5 D) – 1,5
6. $y = \frac{1}{\sqrt{5+4x-x^2}}$ funksiya aniqlanish sohasiga tegishli eng kichik butun sonni toping.
   A) -1 B) 0 C) 1 D) 2
7. Agar $xyz = 1$ bo'lsa, $\frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx}$ ni soddalashtiring.
   A) 2 B) $xy + xz + yz$ C) 1 D) $1 + x + y + z$
8. $a = \sqrt[3]{2}$; $b = \sqrt[5]{4}$; $c = \sqrt[10]{8}$ sonlarni o'sish tartibida yozing.
   A) $c < a < b$ B) $b < a < c$ C) $a < b < c$ D) $c < a < b$
9. $f(x) = \frac{\sqrt{3x+2-2x^2}}{x}$ funksiya aniqlanish sohasida nechta butun son bor?
   A) 1 B) 2 C) 3 D) 4
10. Idishda 12 litr sulfat kislota bor. Unga qancha suv qo'shsak, 30 % li sulfat kislota eritmasi hosil bo'ladi?
   A) 26 B) 30 C) 32 D) 28
11. Hisoblang: $\left(1 + 4^{\frac{1}{3}} + 16^{\frac{1}{3}}\right)\left(1 - 2\sqrt[3]{4} + \sqrt[3]{16}\right)^{0,5}$
   A) 1 B) 2 C) 3 D) $\sqrt[3]{4}$
12. Agar $a + b + c = 0$ bo'lsa, $a^3 + b^3 + c^3$ ni soddalashtiring.
   A) abc B) $2abc$ C) $3abc$ D) $(a+b)(a+c)(b+c)$

13. a son b dan 25 % ortiq. b son a dan hecha foiz kam?

A) 25 B) 80 C) 75 D) 20

14. 711; 754; 883 sonlarni ayni bir songa bo‘lganda bir xil qoldiq chiqadi. Shu sonni toping.

A) 29 B) 13 C) 43 D) 31

15. x^3+5x^2+3x-9 ni ko‘paytuvchilarga ajrating.

A) $(x-1)(x-2)^2$ B) $(x+1)^2(x-3)$

C) $(x-1)(x+3)^2$ D) $(x-1)^2(x-3)$

16. x_1 va x_2 sonlar $x^2-x+m^2-2m=0$ tenglama ildizlari ${x_1}^3+{x_2}^3$ ning eng kichik qiymatini toping.

A) 4 B) 5 C) 8 D) 10

17. x^3+2x^2-x+5 ni $x+1$ ga bo‘lganda qoldiqni toping.

A) 5 B) 7 C) 8 D) 9

18. $\frac{2m^2-3mn+n^2}{3mn-m^2-2n^2}$ kasrni qisqartiring.

Javob ______________________________

19. Agar x_1 va x_2 sonlar $x^2+5x-3=0$ tenglama ildizlari bo‘lsa, ildizlari x_1+1 va x_2+1 bo‘lgan kvadrat tenglama tuzing.

Javob ______________________________

20. Hisoblang $\frac{\sqrt{2,5\cdot6,4}\cdot\left(\sqrt{6,4:2,5}-\sqrt{2,5:6,4}\right)}{\sqrt{(2,5+0,4)^2-4\cdot0,4\cdot2,5}}$

Javob ______________________________

21. $\begin{cases}3x+5>0\\2-5x\geq4\end{cases}$ tengsizliklar sistemasini yeching.

Javob ______________________________

22. $y=3-2x-x^2$ parabola uchinchi toping.

Javob ______________________________

23. Hisoblang $7,23\cdot4\frac{1}{7}-\frac{23}{100}:\frac{7}{29}$

Javob ______________________________

24. k ning qanday qiymatida $2x^3-3x^2+kx-6$ ko‘phadni $x-2$ ga bo‘lganda qoldiq 6 ga teng bo‘ladi?

Javob ______________________________

25. Radiusi 1 ga teng aylana markaziy burchaklarining kattaligi 1; 2; 6 sonlarida proporsional uch yoyga bo‘lingan. Katta yoyning uzunligini toping.

Javob ______________________________

2- variant

1. 25 % i o‘zining teskarisiga teng bo‘lgan natural sonni toping.
   A) 2   B) 4   C) 6   D) 8
2. Barcha ikki xonali sonlar ko‘paytmasi 3 ning qanday eng katta darajasiga qoldiqsiz bo‘linadi?
   A) 3^{40} B) 3^{41} C) 3^{44} D) 3^{42}
3. O‘zining kvadratidan eng ko‘p farq qiluvchi va kvadratidan katta bo‘lgan sonni toping.
   A) $\frac{2}{3}$ B) $\frac{3}{4}$ C) $\frac{1}{2}$ D) $\frac{1}{3}$
4. $\frac{\sqrt{6}}{3\sqrt{2}+2\sqrt{3}}$ kasrni qisqartiring.
   A) $\sqrt{3}+\sqrt{2}$ B) $\frac{1}{\sqrt{3}+\sqrt{2}}$ C) $\frac{5}{3}\sqrt{3}$ D) $5\sqrt{6}$
5. $\frac{4}{5}$; $\frac{30}{17}$ va $\frac{18}{85}$ sonlarga qoldiqsiz bo‘linuvchi eng kichik natural sonni toping.
   A) 45   B) 90   C) 85   D) 180
6. Soddalashtiring: $\left(\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}\right)^2$
   A) $2\sqrt{3}$ B) 2   C) 1   D) $\sqrt{2}-1$
7. $x^2+\frac{9}{x^2}=5\left(x-\frac{3}{x}\right)$ tenglamaning butun yechimlari yig‘indisini toping.
   A) 5   B) 3   C) 2   D) 9
8. Agar $a<0$, $b<0$, $c<0$ va $ab=\frac{2}{3}$, $bc=\frac{3}{4}$, $ca=\frac{4}{5}$ bo‘lsa, a, b, c sonlarni o‘sish tartibida yozing.
   A) $b<a<c$ B) $a<b<c$ C) $c<b<a$ D) $c<a<b$
9. $\sqrt{\frac{3x+8}{5-x}}>-1$ tengsizlikni butun yechimlari yig‘indisini toping.
   A) 10   B) 7   C) -2   D) 4
10. Tenglamani yeching: $\frac{x^3-125}{x-5}=8x+35$
    A) -2   B) 2   C) -3   D) 3
11. Agar x_1 va x_2 sonlar $x^2+2x-1=0$ tenglamaning ildizlari bo‘lsa, ${x_1}^4+{x_2}^4$ ni hisoblang.
    A) 34   B) 36   C) -2   D) 0
12. Yo‘lovchi tezligi 60 km/soat poyezd ichida ketayotib, qarshi kelayotgan poyezd deraza yonidan 4 sekundda o‘tganini aniqladi. Agar u poyezdnning uzunligi 120 m bo‘lsa, tezligini aniqlang.
    A) 55   B) 50   C) 40   D) 48

13. $(x-1)\sqrt{-x^2+x+2} \leq 0$ tengsizlikning butun yechimlari yig‘indisini toping.

A) 2 B) 1 C) -1 D) 0

14. $x^2-6x+|x-4|+8=0$ tenglama ildizlari yig‘indisini toping.

A) 8 B) 7 C) 4 D) -4

15. $\frac{2}{|x+2|} \geq 1$ tengsizlikni butun yechimlari nechta ?

A) 5 B) 4 C) 3 D) 2

16. $a(a+1)(a+2)(a+3)+1$ ifodani ko‘paytuvchilarga ajrating.

A) $(a^2+3a+1)^2$ B) $(a^2+2a+1)^2$

C) $(a^2+1)(a^2+3)$ D) $(a^2+a+1)^2$

17. Agar $a=\frac{1}{2}(\sqrt{3}+1)$ bo‘lsa, $4a^3+2a^2-8a+7$ ni qiymatini hisoblang.

A) $2\sqrt{3}+1$ B) 10 C) $\frac{\sqrt{3}}{2}$ D) 8

18. $\frac{4}{1+\frac{1}{a+\frac{1}{b}}} \cdot (ab+b+1)$ ni soddalashtiring

Javob ______________________________

19. 5 ta ruchka 3 ta daftar 1750 so‘m. 1 ta ruchka va 1 ta daftar 450 so‘m tursa, daftar ruchkadan qancha qimmat turadi?

Javob ______________________________

20. Agar $\sin\alpha=-\frac{\sqrt{5}}{5}$ va $\frac{3\pi}{2}<\alpha<2\pi$ bo‘lsa, $\sin(30^{\circ}-\alpha)$ ni hisoblang.

Javob ______________________________

21. Agar $\sin\alpha=0{,}7$ bo‘lsa, $\cos 2\alpha$ ni hisoblang.

Javob ______________________________

22. $y=x^2-4x+5$ funksiya grafigi qaysi grafiklardan o‘tadi?

Javob ______________________________

23. x_1 va x_2 sonlar $3x^2+4x-2=0$ tenglama ildizlari. U holda ildizlari $\frac{1}{x_1}$ va $\frac{1}{x_2}$ bo‘lgan kvadrat tenglama tuzing.

Javob ______________________________

24. Qanday sonning $\frac{3}{5}$ qismi 7,2 dan uch yarim martta katta?

Javob ______________________________

25. Nuqtadan aylanaga eng katta kesuvchi va urinma o‘tkazilgan. Urinma 20 sm, aylana radiusi 21 sm bo‘lsa, kesuvchi uzunliglini toping.

Javob ______________________________

3-variant

1. Ushbu $y = x^2 - 4x + 3$ parabolaning uchu koordinatalari tekisligining qayerida joylashgan?
   A) I B) II C) III D) IV
2. Tengsizlikni yeching. $(x-1)(x+2) > 0$
   A) $(-\infty; -2) \cup (1; +\infty)$ B) $(-\infty; +\infty)$ C) $(-\infty; -1)$ D) $(-2; 1)$
3. Funksiyaning aniqlanish sohasini toping. $y = \sqrt{\frac{(x-1)(3-x)}{x(4-x)}}$
   A) $[0; 1] \cup [3; 4]$ B) $[0; 1] \cup \{3; -4\}$ C) $(-\infty; 0) \cup [1; 3] \cup (4; \infty)$ D) $[1; 2]$
4. Hisoblang. $\frac{\left(\frac{1}{49}\right)^{-\frac{1}{2}} - \left(\frac{1}{8}\right)^{-\frac{1}{3}}}{64^{\frac{2}{3}}}$
   A) $\frac{3}{4}$ B) $\frac{5}{16}$ C) $\frac{2}{5}$ D) $\frac{4}{7}$
5. $f(x) = \frac{x^2-4x+8}{x^2-4x+5}$ funksiyaning qiymatlar sohasini toping.
   A) $[1{,}6; 5]$ B) $[1{,}6; 4]$ C) $(1; 4]$ D) $[1; 4]$
6. $y = \sqrt{9 - x^2}$ funksiyaning qiymatlar sohasini toping.
   A) $[0; 3]$ B) $[-\infty; \infty]$ C) $[-3; 3]$ D) $[0; \infty)$
7. $240°$ radian o'lchovini toping.
   A) $\frac{3\pi}{4}$ B) $\frac{2\pi}{3}$ C) $\frac{4\pi}{3}$ D) $\frac{5\pi}{4}$
8. Arifmetik progressiyada $a_3 + a_5 = 12$ bo'lsa, $S_7 =$?
   A) 42 B) 18 C) 36 D) 48
9. $\frac{5\pi}{4}$ radian necha gradus?
   A) $220°$ B) $225°$ C) $230°$ D) $235°$
10. 64; 32; 16 geometrik progressiyaning 9 – hadi 6 – hadidan nechta kam?
    A) 1,025 B) 1,5 C)1,25 D) 1,75
11. Ikkita o'xshash uchburchaklarning yuzlari 6 va 24, ulardan birining perimetri ikkinchisinikidan 6 ga ortiq. Katta uchburchakning perimetrini toping.
    A) 18 B) 12 C) 20 D) 8
12. $AB \parallel CD$, *CA* va *DB* to'g'ri chiziqlar *O* nuqtada kesishadi. *OA*=5sm, *OB*=4sm, *OD*=9sm, *OC*=?
    A) 10,8 B) 10,5 C) 11,25 D) 11,3
13. *ABC* uchburchakda *AD* mediana *AB* va *AC* tomonlar bilan mos ravishda $30°$ va $60°$ li burchak qiladi. Agar $AB = \sqrt{3}$ ga teng bo'lsa, *AC* ni toping?
    A) $\frac{\sqrt{3}}{2}$ B) 1 C) $\frac{\sqrt{3}}{3}$ D) $1\frac{1}{2}$

14. To'g'ri burchakli uchburchakning burchaklaridan biri $60°$ ga, gipotenuzaga tushirilgan mediana 15 ga teng. Kichik katetining uzunligini toping.

A) 7,5 B) 10,5 C) 15 D) 12

15. Radiusi 8 ga teng aylananing $\frac{\pi}{8}$ radianga teng bo'lgan yoyning uzunligini aniqlang?

A) $\frac{\pi}{64}$ B) π C) 2π D) 4π

16. To'g'ri burchakli uchburchakka ichki chizilgan aylananing radiusi qaysi javobda to'g'ri ko'rsatilgan?

A) $r = \frac{a+b-c}{2}$ B) $r = \frac{a+b+c}{2}$ C) $r = \frac{c+b-a}{2}$ D) $r = \frac{c+a-b}{2}$

17. Aylananing uzunligi $18\sqrt{2}\pi$ ga teng. Aylanadagi AB vatar 90° yoyni tortib turadi. Vatarning uzunligini toping.

A) 18 B) 8 C) 16 D) 8,5

18. Muntazam uchburchakning balandligi 9 sm. Uchburchakka ichki chizilgan aylananing radiusini toping.

A) 3 B) 4 C) 5 D) 10

19. $\frac{-3x^2-4x-5}{3x-1} > 0$ tengsizlikni yeching.

Javob: ________________________________

20. *AB* to'g'ri chiziq markazi *O* va radiusi *r* bo'lgan aylanaga B nuqtada urinadi. Agar *OA*=2 sm, *r*=1,5 sm bo'lsa, *AB* kesma uzunligini toping.

Javob: ________________________________

21. Hisoblang: $\sqrt{51{,}5^3 + 51{,}5^2 \cdot 26{,}5 - 51{,}5 \cdot 26{,}5^2 - 26{,}5^3}$

Javob: ________________________________

22. Agar $\sqrt{38-x} - \sqrt{11-x} = 3$ bo'lsa, $\sqrt{38-x} + \sqrt{11-x}$ ni hisoblang.

Javob: ________________________________

23. $x^2 - 17x + q = 0$ tenglama ildizlarining kvadratlari ayirmasi 85 ga teng. q ni toping.

Javob: ________________________________

24. $\begin{cases} x+5 > 8-2x \\ 2+x < 6-x \end{cases}$ tengsizliklar sistemasini yeching.

Javob:________________________________

25. $\begin{cases} \frac{3x+y}{x-1} - \frac{x-y}{2y} = 2 \\ x - y = 4 \end{cases}$ tenglamalar sistemasini yeching.

Javob:________________________________

4- variant

1. ABC uchburchak berilgan. AB tomonida M nuqta, BC tomonida N nuqta shunday belgilanganki, bunda MN||AC va MN uchburchakni ikkita tengdosh shakllarga ajratadi. MB kesmada P nuqta, NB kesmada Q nuqta shunday belgilanganki, bunda PQ||MN va PQ kesma MBN uchburchakni ikkita tengdosh shakllarga ajratadi. Agar ABC uchburchakning perimetri 120 sm va AC tomon uzunligi perimetrining 60% ini tashkil qilsa, PQ ning uzunligini toping.

   A) 60 B)72 C)36 D)$12\sqrt{3}$

2. ABC uchburchakning AB tomonida M nuqta, BC tomonida N nuqta shunday olinganki, bunda AM:MB=2:7 , CN:NB=2:3 . ABC uchburchakning yuzi MBN uchburchakning yuziga nisbatini toping.

   A) 49:4 B)7:2 C)5:7 D)81:25

3. *ACB* to'g'ri burchakli uchburchakda *C* burchak-to'g'ri burchak va *AC*>*BC*. *C* uchidan *CK* bissektrissa va *CM* mediana o'tkazilgan. *CKM* uchburchakning yuzi *ABC* uchburchak yuzining 12,5% ini tashkil qiladi. *ABC* burchakning tangensini toping.

   A) $\frac{8}{3}$ B) $\frac{\sqrt{5}}{\sqrt{3}}$ C) $\frac{25}{9}$ D) $\frac{5}{3}$

4. *ABC* uchburchakda *AB*=6sm, *BC*=9sm, *AC*=14sm. *A* nuqtadan *AD* bissektrissa va *B* nuqtadan *BF* mediana o'tkazilgan va ular *E* nuqtada kesishadi. *BE:EF* nisbatini toping.

   A) 6:7 B) 2:1 C) 2:3 D) 3:2

5. (-2;-7) nuqta *Ox* o'qiga nisbatan simmetrik bo'lgan B nuqtadan C(6;22) nuqtagacha masofani toping.

   A)17 B) $\sqrt{241}$ C) 19 D) 21

6. ABC to'g'ri burchakli uchburchakda C to'g'ri burchak. Gipatenuzasining uzunligi $\sqrt{26}$ sm, yuzi esa 6 sm^2 . tgA+tgB ning qiymatini toping.

   A) $2\frac{8}{13}$ B) $\frac{3\sqrt{2}}{2}$ C) $2\frac{1}{6}$ D) $4\sqrt{6}$

7. *ABC* uchburchakning *AC* tomonida *M* va *N* nuqtalar shunday olinganki, bunda *ABM*, *MBN*, *NBC* uchburchaklar yuzlari mos ravishda $12sm^2$, $40\ sm^2$va $32\ sm^2$ga teng. *AC*=42 sm bo'lsa, *MN* ni toping.

   A) 26 B) topib bo'lmaydi C)20 D)21

8. Markazi *O* nuqtada bo'lgan doirada ikkita *AB* va *CD* diametrlar o'tkazilgan, natijada to'rtta burchak hosil bo'lgan. Burchaklarning biri ikkinchisidan 40° ga katta. Aylanada *M* va *N* nuqtalar shunday olinganki *MOD* burchak *AOD*

burchakdan 10 marta kichik, *NOD* burchak *BOD* burchakdan 10 marta kichik. *MCN* burchak kattaligini toping.

A) 18° B) 9° C) 36° D) 17°

9. To'g'ri burchakli uchburchakning to'g'ri burchagi uchidan o'tkazilgan balandlik gipotenuzani 4 sm va 36 sm li kesmalarga ajratadi. Uchburchakning yuzini toping.

A) $200\sqrt{2}$ B) $180\sqrt{3}$ C) 240 D) 480

10. To'g'ri burchakli uchburchakning katetlari yig'indisi 73 sm ga, gipotenuzasi 53 sm ga teng bo'lsa, unga ichki chizilgan aylana radiusini toping.

A) 10 B)15 C) 20 D) 25

11. Tenglamaning ildizlari kvadratlarining yig'indisini toping:

$$x^2-\frac{2^{10}\cdot 4^{10}\cdot 8^5\cdot 16^7}{32^{14}}\cdot x+15=0$$

A) 41 B)$28+4\sqrt{2}$ C) 34 D) 42

12. Toshkentlik matematiklarning uchdan biri faylasuflar, toshkentlik faylasuflarning yettidan biri matematiklardir. X-matematiklar soni, y-faylasuflar soni bo'lsa, ifodani soddalashtiring:

$$|x-y|+|2x-y|+|3x-y|+x$$

A) x+y B) 7x-3 C) 3x-y D) 3-7x

13. Tenglamani yeching: $|x^2+7x+21|+|x^2+3x+11|=|2x^2+4x+5|$

A)$2\pm\sqrt{1+\sqrt{3}}$ B) -4.5 va 7.2 C) -4.5 D) ildizi yo'q

14. Soddalashtiring: $\sqrt[5]{\sqrt[3]{x}\cdot\sqrt{x}}\cdot\sqrt[3]{x^2}\cdot\sqrt[6]{x}$

A) x^2 B) x C) $\sqrt[12]{x^5}$ D) $\sqrt{x}$

15. Tenglamaning eng kichik manfiy ildizini toping:

$$\frac{1}{x(x+1)}+\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x+3)}+\frac{1}{(x+3)(x+4)}=\frac{1}{3}$$

A) manfi yildizi yo'q B) $-5\sqrt{2}$ C) -6 D)-2

16. To'g'ri chiziq A(7;-1) va B(2;5) nuqtalar orqali o'tadi. Bu to'g'ri chiziqning tenglamasi qanday?

A) 6x+5y-37=0 B) 5x+6y-37=0 C) 6x-5y+37=0 D) 5x+5y+37=0

17. $\begin{cases} 4x+2y=11 \\ 12x+(a^3-a^2+a+5)x=42 \end{cases}$ tenglamalar sistemasi yechimga ega bo'lmaydigan a ning nechta qiymatlari bor?

A) bunday *a* mavjud emas B)2 C)3 D)1

18. Tengsizlikni yeching: $\frac{(x-1)^2\cdot(x+5)^6\cdot(-x-4)^9}{(x+6)^7\cdot(x+9)^8}>0$

Javob:____

19. Maxsulotning narxi uch marta o'zgardi, birinchi marta 10% ga pasaytirildi, ikkinchi marta 30% ga pasaytirildi. Uch marta o'zgarishdan so'ng maxsulot narxi 7,1 % ga ortdi. Uchinchi marta maxsulot narxi necha foizga oshirildi?
    Javob:____
20. 480 kg mevaning 73 % i suvdan iborat . Bir qancha vaqtdan keyin tarkibidagi suv 10 % gacha kamaydi. Endi mevalarning og'irligi necha kilogramm?
    Javob:____
21. Argumentning qanday eng kichik qiymatida $y = \frac{(9x-8)^{24}-1}{(81x^2-144x+64)^6}$ funksiya eng kichik qiymatni qabul qiladi?
    Javob:____
22. Perimetri 16 ga teng to'g'ri to'rtburchak qanday eng katta yuzaga ega?
    Javob:____
23. Xovuzdan bir tekis suv chiqarila boshlagandan 6 soat o'tgach yana 574 m^3 suv qoldi, yana 4 soatdan keyin 406 m^3 suv qoldi. Xovuzda qancha suv bo'lgan edi?
    Javob:____
24. Nasos uch soatda xovuzdagi suvning $\frac{3}{4}$ qismini tortib chiqaradi. 2 soat 40 minut ishlab, nasos to'xtatildi. To'xtatilgandan so'ng xovuzda 28 m^3 suv qolgan bo'lsa, xovuzning xajmini toping.
    Javob:____
25. ABC to'g'ri burchakli uchburchakda $\angle C = 90°$, $ctgA = \frac{15}{8}$. ABC uchburchakning perimetrini toping.
    Javob:____

5- variant

1. Ishni ikkita ishchi orasida shunday bo'lish kerakki, ikkinchi ishchiga birinchisidan 25% kam tegsin. Xar bir ishchi butun ishning necha foizidan oladi?
   A) 37,5% va 62,5%   B)100% va 75%
   C) 125% va 100%   D) $57\frac{1}{7}$% va $42\frac{6}{7}$%
2. Ifodaning qiymatini toping: $\sqrt{7+3\sqrt{2}}$
   A) $\sqrt{\frac{7-\sqrt{67}}{2}} + \sqrt{\frac{7-\sqrt{67}}{2}}$ B) $\sqrt{\frac{18-\sqrt{98}}{2}} + \sqrt{\frac{18+\sqrt{98}}{2}}$

C) $\sqrt{\frac{7-\sqrt{31}}{2}}+\sqrt{\frac{7+\sqrt{31}}{2}}$ D) $\sqrt{\frac{7-\sqrt{18}}{2}}+\sqrt{\frac{7+\sqrt{18}}{2}}$

3. Velosapedchi yo'lning uchdan bir qismini 1m/s tezlik bilan, keyingi uchdan bir qismini 2m/s tezlik bilan, oxirgi qismini 3m/s tezlik bilan yurdi. Velosapedchining butun yo;ldagi o'rtacha tezligini toping.
   A) 2 B) $1\frac{7}{11}$ C) $\sqrt{6}$ D) $2\frac{1}{3}$
4. $x\cdot y=x+y$ ekani ma'lum, y butun son bo'ladigan x ning nechta butun qiymati mavjud.
   A) 2 B)1 C) bitta ham yo'q D) cheksiz ko'p
5. Beshta yashikda bir xil miqdorda olma bor. Agar har bir yashikdan 60 tadan olma olinsa, barcha yashikdagi olmalar soni avval ikkita yashikda nechta olma bo'lsa, shuncha bo'lib qoladi. Dastlab har bir yashikda qanchadan olma bo'lgan.
   A)110 dona B)90 dona C)100 dona D)120 dona
6. $a=5+\sqrt{3}$ va $b=3+\sqrt{14}$ ifodaning qiymatini taqqoslang.
   A) taqqoslab bo'lmaydi B) a>b C) a=b D) a<b
7. Hisoblang: $\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{100^2}\right)$
   A) $\frac{101}{200}$ B) $\frac{143}{200}$ C) $\frac{441}{600}$ D) $\frac{8751}{9900}$
8. $\frac{4}{x^2}=x+3$ tenglama ikkita turli ildizga ega. $\frac{4}{x^2}=x+\sqrt{\pi}$ tenglama nechta turli ildizga ega?
   A) 3 B) 2 C) 1 D) 0
9. To'rtta natural sonni *x,y,z* va *t* ni o'rniga *x+y+z+t=a* tenglamaga qo'yganda tenglamani sonli to'g'ri aylantirsa, bu sonlar tenglamaning natural yechimi deb ataladi. *x+y+z+t=10* tenglamaning nechta turli natural yechimi mavjud?
   A) 108 B) 84 C)132 D) 36
10. $ax^2+bx+c=0$ tenglama ikkita turli noldan farqli va har biri birdan katta bo'lgan ildizga ega bo'lsa, $(ax^2+bx+c)(cx^2+bx+a)=0$ nechta turli ildizga ega bo'lishi mumkin.
    A) 2 yoki 3 B) faqat 3 C) 2 yoki 3 yoki 4 D) faqat 4
11. $ax^2+bx+c=0$ tenglama uchun *c<0*; *a+b+c>0*; *4a+2b+c<0* ekani ma'lum. *abc* ko'paytmaning ishorasini aniqlang.
    A) *abc*>0 B) *abc<0* C) *abc=0* D) *abc<0* yoki *abc>0*
12. Agar $a^4+a^{-4}=5$ bo'lsa, $a^{12}+a^{-12}$ ni toping.
    A) 119 B) 121 C) 125 D) 110
13. *a* ning qanday eng katta qiymatida

$$y = \frac{a^2 - 18a + 65}{a^2 - 169} \cdot x^2 + (a^2 + 3a + 7) \cdot x + 6a - 1$$

funksiyaning grafigi parabola bo'lmaydi?

A) barcha a larda B) 13 C) 1/6 D) 5

14. $y = x^2 + bx + c$ parabolaning uchi *y=2x-8* va *y=5x+1* tenglamalar bilan berilgan to'g'ri chiziqlarning kesishish nuqtasi bo'ladi. Parabolaning birinchi koordinatasi 1 ga teng bo'lgan nuqtaning ikkinchi koordinatasini toping.

A) 1 B) 2 C) 3 D) -1

15. $y = ax^2 + bx + c$ tenglama bilan berilgan parabolaning uchi *B*(7;11) nuqtada. Parabola absissalar o'qini *A* va *C* nuqtalarda kesib o'tadi. *ABC* uchburchakning yuzi 66 ga teng. $ax^2 + bx + c = 0$ tenglamaning ildizlari ko'paytmasini toping.

A) 13 B) -13 C) 77 D) 28

16. Agar parabola *A*(3;1) nuqtadan o'tsa va uning uchi *B*(2;-1) nuqtada bo'lsa, parabolaning tenglamasini toping.

A) $y = x^2 - 2x + 3$ B) $y = 3x^2 - 5x - 11$

C) $y = 3x^2 + 8x - 7$ D) $y = 2x^2 - 8x + 7$

17. Birinchi koordinatalar choragi bissektrissasiga nisbatan $y = \frac{x}{3} + 2$ to'g'ri chiziqqa simmetrik bo'lgan to'g'ri chiziq tenglamasini yozing.

A) $y = -\frac{x}{3} + 2$ B) $y = 3x - 6$ C) $y = -3x + 6$ D) $y = -3x - \frac{2}{3}$

18. Ifodaning qiymatini toping: $\frac{\pi^{100} - 100^{25}}{|100^{25} - \pi^{100}|}$

A) 1 B) -1 C) 0 D) mavjud emas

19. Tenglamani yeching: $\sqrt[6]{x \cdot \sqrt[3]{x \cdot \sqrt[4]{x}}} \cdot \sqrt[12]{\sqrt[6]{x}} = 4$

Javob: ____

20. Yig'indini toping: $sin^2 1° + sin^2 2° + \cdots + sin^2 88° + sin^2 89°$

Javob:____

21. *k* ning qanday ratsional qiymatida $\sin(\pi \cdot k) = \sin \pi°$ tenglik bajariladi?

Javob:____

22. $\Delta ABC \ va \ \Delta A_1B_1C_1$ uchburchaklar berilgan. $AB = A_1B_1 \ va \ AC = A_1C_1$. A burchak A_1burchakdan ikki marta katta. Uchburchak yuzlari bir xil. A_1 burchak kattaligini toping.

Javob:____

23. *ACB* to'g'ri burchakli uchburchakda to'g'ri burchagidan *CD* balandlik o'tkazilgan. *ADC* uchburchakka ichki chizilgan aylananing radiusi $\sqrt{13}$ ga

teng, *BDC* uchburchakka ichki chizilgan aylananing radiusi $\sqrt{3}$ ga teng. *ACB* uchburchakka ichki chizilgan aylananing radiusi nimaga teng?
Javob: ____

24. *ABC* uchburchakning *AB* tomonida *M* nuqta, *BC* tomonidan *N* nuqta, *AC* tomonidan *K* nuqta qo'yilgan. *AN*, *BK*, *CM* kesmalar bir nuqtada kesishadi. *AM:MB*=3:5; *BN:NC*=7:2 bo'lsa, *CK:AK* ni toping.
Javob: ____

25. *ABC* to'g'ri burchakli uchburchakda $\frac{AB}{AC} = \frac{13}{5}$. *CK* kesma to'g'ri burchakning bissektrissasi. *ACK* uchburchakning yuzi 40 ga teng. *BCK* uchburchakning yuzini toping.
Javob: ____

6-variant

1. Integralni hisoblang: $\int_{-1}^{1} \frac{x}{cos^3x} dx$
   A) 0   B) $\frac{2}{cos1}$ C) $\frac{1}{cos2}$ D) 1
2. ABC uchburchakda $\cos(2A - B) + \sin(A + B) = 2$ va $AB = 4$ bo'lsa, $BC =$?
   A) 2   B) 1,5   C) 4   D) $3\sqrt{2}$
3. Tenglamaning barcha turli haqiqiy ildizlari ko'paytmasini toping.
$$\sqrt{2x^2 + 3x + 14} - \sqrt{2x^2 + 3x + 2} = 2$$
   A) -1   B) 1   C) $\frac{1}{3}$ D) $-\frac{1}{3}$
4. Hisoblang: $ctg\left\{\frac{\pi}{2}\right\} \cdot ctg\left[\frac{\pi}{2}\right]$ =? ([*x*] – *x* sonining butun qismi, {*x*} – *x* sonining kasr qismi)
   A) 1   B) 0   C) -1   D) π
5. Agar $(x^2 + 1)(y^2 + 1) + 9 = 6(x + y)$ bo'lsa, $x^2 + y^2$ =?
   A) 7   B) 8   C) 5   D) 4
6. Hisoblang: $tg15° \cdot tg25° \cdot tg35° \cdot tg85°$
   A) 0   B) 1   C) $\frac{\sqrt{2}}{2}$ D) $\frac{\sqrt{3}}{2}$
7. $f(x) + f\left(\frac{x-1}{x}\right) = x + 1$ bo'lsa, $f(2)$ ni toping.
   A) $\frac{3}{4}$ B) $\frac{5}{2}$ C) $\frac{4}{5}$ D) $\frac{3}{2}$
8. Integralni hisoblang: $\int_{e}^{e^2} \ln x^3 \, dx$
   A) e B) $3e^2$ C) $3e$ D) e^3

9. *ABC* uchburchakda *BH* balandlik va *BM* medianalar o'tkazilgan. Agar *AB*=1, *BC*=2 va *AM*=*BM* bo'lsa, $\angle MBH$ burchakni toping.
A) *arcsin0,8* B) *arccos0,8* C) *arcsin0,7* D) *arccos0,3*
10. Muntazam *ABCDEF* oltiburchakning tomoni 6 ga teng. Oltiburchakning *C* uchidan *AE* diagonalgacha bo'lgan masofani toping.
A) 8 B) 9 C) 12 D) 7
11. $3^{32} - 2^{41} - 5$ sonini 11 ga bo'lgandagi qoldiqni toping.
A) 6 B) 4 C) 2 D) 0
12. Tenglama nechta haqiqiy yechimga ega: $[x + [x]] = 2019$ (bunda [*x*] – *x* sonining butun qismi)
A) Cheksiz ko'p B) 0 ta C) 5 ta D) 19 ta
13. Hisoblang: $sin\left(2arctg\frac{1}{2}\right) + tg\left(\frac{1}{2}arcsin\frac{15}{17}\right)$
A) -1 B) 1 C) 2 D) 1,4
14. Agar $x^3 - y^3 - z^3 = 3xyz$ va $x^2 = 2(y + z)$ bo'lsa $(x, y, z \in N)$, $x + y + z =?$
A) 4 B) 2 C) 3 D) 1
15. *ABCD* parallelogramm ichida *O* nuqta shunday olinganki, bunda *COD* teng tomonli uchburchak. Agar *O* nuqtadan *AD*, *AB* va *BC* to'g'ri chiziqlargacha bo'lgan masofalar mos ravishda 3, 6 va 5 ga teng bo'lsa, parallelogrammning perimetrini toping.
A) $\frac{36\sqrt{3}}{2}$ B) $\frac{49\sqrt{3}}{2}$ C) $\frac{69}{2}$ D) $\frac{57}{2}$
16. Tenglamaning ildizlari ko'paytmasini toping:

$$2\left(1 + sin^2(x - 1)\right) = 2^{2x - x^2}$$

A) 1 B) 2 C) 6 D) 4
17. Hisoblang: $cos12° \cdot cos24° \cdot cos48° \cdot cos276° =?$
A) $\frac{1}{8}$ B) $\frac{1}{16}$ C) $\frac{\sqrt{3}}{8}$ D) $\frac{\sqrt{3}}{16}$
18. $4x^2 - 9x + a = 0$ tenglamaning ildizlari ko'paytmasi eng katta bo'lishi uchun *a* qanday bo'lishi kerak?
A) $\frac{9}{4}$ B) $\frac{81}{16}$ C) $\frac{81}{4}$ D) $\frac{48}{81}$
19. $(1 - ctg22°)(1 - ctg23°)$ ning qiymatini toping.
Javob: ____
20. x, y, z haqiqiy sonlar uchun $x^2 + y^2 + z^2 = 3(x + y + z)$ shart bajariladi. U holda $xy + yz + zx$ ifodaning eng kichik qiymatini toping.
Javob: ____
21. Tenglama nechta haqiqiy yechimga ega: $2^{[x]} - 1 = 2x$ (bunda [*x*] – *x* sonining butun qismi).

Javob: ____

22. Ifodani soddalashtiring: $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{2018}{2019!}$

Javob: ____

23. *ABC* uchburchakka ichki chizilgan aylana *AB* va *AC* tomonlarga mos ravishda *M* va *N* nuqtalarda urinadi. Agar *AB*=4, *AC*=3, *BC*=2 bo'lsa, *AMN* uchburchak yuzini toping.

Javob: ____

24. Aylananing *AB* diametridan olingan *K* nuqta orqali *MN* vatar o'tkazilgan. Agar $\angle ABM = 30°, \angle BMK = 15°$ va $MK = 3$ bo'lsa, AB ni toping.

Javob: ____

25. Agar $log_x 100 = a$ ekanligi ma'lum bo'lsa, $lg^2\sqrt{x}$ ni hisoblang.

Javob: ____

7-variant

1. $a+b=-1,\ a+c=6,\ b+c=1$ bo'lsa, $a^2(3b+3c+2a)+b^2(3a+3c+2b)+c^2(3a+3b+2c)$ ning son qiymatini toping.

   A) 45 B) 27 C) 100 D) 216

2. $2x^2-2ax+1=0$ kvadrat tenglama ildizlari uchun $\left(\frac{2x_1}{x_1+x_2+1}\right)^2+\left(\frac{2x_2}{x_1+x_2+1}\right)^2=3$ tenglik o'rinli bo'lsa, *a* ni toping.

   A) 1 B) -3 C) 7 D) 6

3. Ko'paytmani hisoblang: $cos12°cos24°cos36°cos48°cos60°cos72°cos84°$

   A) $\frac{1}{128}$ B) $\frac{1}{32}$ C) $\frac{1}{16}$ D) $\frac{1}{256}$

4. ABCD parallelogrammda AB=10, AD=16. BC tomonga AE va DF bissektrisalar o'tkazilgan bo'lib, ular G nuqtada kesishadi. E va F nuqtalar BC tomonda bo'lsa, AGD va FGE uchburchaklar yuzlarining nisbatini toping.

   A) 16 B) 4 C) 12 D) 64

5. $10^{98}\cdot 2^{23}\cdot 15^3\cdot 25^{11}$ son necha xonali?

   A) 121 B) 122 C) 123 D) 124

6. Ifodanisoddalashtiring:

$$\left(1-\frac{1}{(c+2)^2}\right)\left(1-\frac{1}{(c+3)^2}\right)\left(1-\frac{1}{(c+4)^2}\right)\ldots\left(1-\frac{1}{(c+28)^2}\right)$$

   A) $\frac{c^2+30c+29}{c^2+29c+56}$ B) $\frac{c^2+29c+28}{c^2+30c+56}$ C) $\frac{c^2+32c+87}{c^2+30c+56}$ D) $\frac{c^2+30c+29}{c^2+30c+56}$

7. Hisoblang: $\frac{\sqrt{7+3\sqrt{4,(4)}}}{\sqrt{2}}+\frac{\sqrt{7-3\sqrt{4,(4)}}}{\sqrt{2}}$

A) $2\sqrt{5}$ B) $\sqrt{10}$ C) 2 D) $2\sqrt{2}$

8. Agar $\begin{cases} x+y-\sqrt{xy}=13 \\ x^2+y^2+xy=481 \end{cases}$ bo'lsa, $\sqrt{xy}$ ning qiymatini toping.

A) 12 B) 36 C) 52 D) 8

9. Agar geometrik progressiyani tashkil etuvchi 3 ta son ko'paytmasi 64 ga, o'rta arifmetigi esa $\frac{14}{3}$ ga teng bo'lsa, $b_1^2+b_2^2+b_3^2$ ni toping.

A) 83 B) 80 C) 84 D) 69

10. Tengsizlikni yeching. $27^{\frac{1}{x}+1}+3^{\frac{3}{x}}>252$

A) $x>1{,}5$ B) $x<1{,}5$ C) $0<x<1{,}5$ D) $x>1$

11. $y=9sin^2x+cos^2x$ funksiyaning butun qiymatlari yig'indisini toping.

A) 45 B) 44 C) 17 D) 26

12. Hisoblang. $\int_0^1 \frac{6x}{x+1}dx$

A) –ln2 B) 6(1-ln2) C) 6-ln2 D) 1-6ln2

13. To'g'ri burchakli uchburchakning perimetri $2p$ va gipotenuzasiga tushirilgan balandligi h ga teng. Uchburchakning gipotenuzasini toping.

A) $\frac{2p^2}{2p+h}$ B) $\frac{p^2}{2p+2h}$ C) $\frac{p^2}{2p+h}$ D) $\frac{p^2}{2p+h}$

14. Asoslari 8 va 12 bo'lgan trapetsiya yuzi 30 ga teng. Shu trapetsiyaning $150°$ li burchagiga yopishgan yon tomoni uzunligini toping.

A) $2\sqrt{3}$ B) 5 C) 6 D) 9

15. $x<|x|$, $x\cdot y^3<0$ shartga ko'ra, $|x-y|-|-y|-|-x|$ ifoda nimaga teng?

A) -2y-2x B) -2y C)-2x D)0

16. 10 so'mlik tanga 5 marta dumalaganda bosib o'tgan masofani 50 so'mlik tanga 2 marta dumalaganda bosib o'tadi. 10 so'mlik tanga 50 so'mlik tanganing ustiga qo'yilsa, 50 so'mlik tanga yuzasining necha foizi ochiq qoladi?

A) 80% B)75% C)84% D) 88%

17. $\begin{cases} 2x^2+2xy+y=6 \\ x+2y=6 \end{cases}$ tenglamalar sistemasini qanoatlantiruvchi barcha x larning yig'indisini toping.

A) $16\frac{1}{2}$ B) $8\frac{3}{4}$ C) -3 D) $-5\frac{1}{2}$

18. 4 ning barcha manfiy butun darajalari yig'indisi 4 dan qanchaga kam?

Javob: ________________________________

19. Tengsizlikni yeching. $2^{log_3x+2}+4^{log_9x+3}<17$

Javob: ___________________________________

20. Soddalashtiring: $\frac{(ctg44° + tg226°)cos406°}{cos316°} - tg(-405°)$

Javob: ___________________________________

21. Hisoblang: $cos\left(arcsin\frac{2}{5} - arccos\frac{1}{4}\right)$

Javob: ___________________________________

22. Teng yonli uchburchakning yon tomoniga tushirilgan medianasi uzunligi 39 sm, medianalar kesishish nuqtasidan asosgacha bo'lgan masofa esa 10 sm. Uchburchak yuzini toping.

Javob: ___________________________________

23. Radiusi *r* ga teng aylana markazidan *a* masofada ($a>r$) joylashgan nuqtadan aylanaga o'tkazilgan ikki urinma aylanadan qanday uzunlikdagi yoy ajratadi?

Javob: ___________________________________

24. Teng yonli trapetsiyaning o'rtalarini tutashtiruvchi kesma 6 ga teng. Yon tomoni esa 10. Trapetsiyaning balandligini toping.

Javob: ___________________________________

25. $y = 1 + x^2$ $va\ y = 9 - x^2$ funksiyalar grafiklari orasidagi soha yuzasini toping.

Javob: ___________________________________

8-variant

1. 1,2,2,3,3,3,4,4,4,4,5,…. ketma-ketlikning dastlabki 100 tasining yig`indisini toping.

   A) 755   B) 845   C) 927   D) 945

2. $(x+1)^{65}$ ko`phadning nechta koeffitsiyenti 65 ga bo'linmaydi?

   A) 20   B) 18   C) 16   D) 3

3. x_1 va x_2 sonlari $x^2+5x-7=0$ tenglamaning ildizlar bo`lsa $x_1^3+5x_1^2-4x_1+x_1^2x_2-4x_2$ ni toping.

   A) -15   B) $175+25\sqrt{53}$ C) -50   D) 20

4. Nechta *p* - tub son uchun $|p^4-86|$ ifoda ham tub son bo`ladi?

   A) 1   B) 2   C) 3   D) 4

5. $0 \le x, y, z < 2011$ sonlari uchun, $xy + yz + zx \equiv 0 (\mathrm{mod}\ 2011)$ va $x + y + z \equiv 0 (\mathrm{mod}\ 2011)$ shartlarni qanoatlantiradigan nechta (x,y,z) butun sonlar uchligi bor?

   A) 2011   B) 2012   C) 4021   D) 4023

6. $2011^{2011^{2011^{2011}}}$ sonini 19 ga bo`lgandagi qoldiqni toping.

A) 5 B) 4 C) 3 D) 2

7. Aylanaga *ABCD* to'rtburchak ichki chizilgan. Agar *AB*=3, *BC*=4, *CD*=7, *AD*=5 bo'lsa uning diognallari ko'paytmasini toping.

A) 47 B) 43 C) 41 D) 42

8. *ABCD* - trapetsiyada (*AD*//*BC*) *AB*=6, *BC*=7, *CD*=8, *AD*=17 va *AB* va *DC* lar *E* nuqtada kesishsa $\angle AED$ ni toping (gradusda)

A) 60 B) 90 C) 150 D) 120

9. *AC* va *BD* diognallari o'zaro perpendikulyar bo'lgan *ABCD* to'rtburchakka radiusi 2 ga teng bo'lgan aylana tashqi chizilgan, agar *AB*=3 bo'lsa *CD* ni toping.

A) $\sqrt{3}$ B) 3 C) $\sqrt{5}$ D) $\sqrt{7}$

10. $a = 5^{56}, b = 10^{51}, c = 17^{35}, d = 31^{28}$ o'sib borish tartibida joylashtiring

A) *a,c,d,b* B) *a,d,c,b* C) *a,d,b,c* D) *a,b,c,d*

11. $|\sin x| > |\cos x|$ tengsizlikni yeching

A) $(\frac{\pi}{4} + \pi n; \frac{3\pi}{4} + \pi n)$ B) $(\frac{\pi}{4} + 2\pi n; \frac{3\pi}{4} + 2\pi n)$ C) $(-\frac{\pi}{4} + \pi n; \frac{\pi}{4} + \pi n)$ D) $(\pi n; \frac{\pi}{4} + \pi n)$

12. f(0)=0, f(1)=1 va har bir $n \geq 1$ uchun, f(3n-1)=f(n)-1, f(3n)=f(n), f(3n+1)=f(n)+1 bo`lsa f(2011) ni toping:

A) 7 B) 5 C) 3 D) 1

13. $1^4 + 2^4 + 3^4 + ... + 2011^4$ ifodani 16 ga bo`lgandagi qoldiqni toping.

A) 14 B) 11 C) 8 D) 5

14. *x,y,z* sonlari $\frac{xyz}{x+y} = -1$, $\frac{xyz}{y+z} = 1$ va $\frac{xyz}{z+x} = 2$ ni qanoatlantirsa (*xyz*) ni toping.

A) $\frac{8}{\sqrt{5}}$ B) $-\frac{8}{\sqrt{15}}$ C) $-8\sqrt{\frac{3}{5}}$ D) $\frac{7}{\sqrt{15}}$

15. a_n ketma-ketlikda a_1=1, a_3=4 va har bir $n \geq 2$ uchun $a_{n+1} + a_{n-1} = 2a_n + 1$ bo`lsa a_{2011} ni toping.

A) 2^{2010} B) 2021056 C) 1010528 D) 3016

16. *ABCD* to'g'ri to'rtburchak ichidagi *P* nuqta uchun *PA*=2, *PB*=3, *PC*=10 bo'lsa *PD* ni toping

A) $\sqrt{105}$ B) 11 C) $\sqrt{91}$ D) $\sqrt{95}$

17. $i^2 + j^2 + k^2 = 2011$ tenglikni qanoatlantiruvchi butun sonlar uchun $(i + j + k)_{max}$ ni toping.

A) 73 B) 74 C) 76 D) 77

18. $y=cos(cosx)$ funksiyaning eng kichik musbat davrini toping.
Javob: ________________________________
19. ABC uchburchakda $AB=12$, $BC=24$ va $AC=18$.Uchburchakga ichki chizilgan aylana markazidan BC ga parallel qilib o`tkazilgan to`g`ri chiziq AB ni M nuqtada va AC ni N nuqtada kesadi. ANM uchburchakning perimetrini toping.
Javob: ________________________________
20. $P(x)$ ko`phadda barcha $x$ uchun $(x-1)P(x+1)-(x+2)P(x)=0$ va $P(2)=6$ bo`lsa $P\left(\frac{3}{2}\right)-?$
Javob: ________________________________
21. $tgx+tgy=4$, $cosx+cosy=5$ bo'lsa $tg(x+y)$ ni hisoblang
Javob: ________________________________
22. a,b,c uchlik $x^3-x+1=0$ ning ildizlari bo'lsa $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}$ ni toping
Javob: ________________________________
23. ABC da $AB=7, BC=6$ va $CA=5$ bo'lsin. ABC uchburchakka ichki chizilgan aylana AB, BC, CA tomonlarga mos ravishda C_1, A_1, B_1 nuqtalarda urinsa $A_1B_1C_1$ uchburchak yuzini toping.
Javob: ________________________________
24. $\frac{1+\cos x}{tg\frac{x}{3}}=0$ tenglamaning $[0;9\pi]$ oraliqdagi ildizlar sonini toping.
Javob: ________________________________
25. x va y sonlari uchun $x^2+y^2-3y-1=0$ bo'lsa $(x+y)_{max}$ ni toping.
Javob: ________________________________

9-variant

1. k ning qanday qiymatida $3x^2-10x+3k=0$ tenglamaning x_1 va x_2 ildizlari orasida $6x_1+13x_2=48$ munosabat o'rinli bo'ladi?
A) $-\frac{10}{3}$ B) $\frac{8}{3}$ C) $\frac{10}{3}$ D) $-\frac{8}{3}$
2. n ning qanday qiymatida $\vec{a}(n;-2;1)$ va $\vec{b}(n;3n;8)$ vektorlar perpendikulyar bo'ladi?
A) 1 B) 3 C) 4; 2 D) 2
3. $4a^2+9b^2+4c^2-4a-6b-4c+3=0$ bo'lsa, $a\cdot b\cdot c$ ko'paytmaga teskari sonni toping.
A) $\frac{1}{24}$ B) 12 C) 24 D) $\frac{1}{12}$

4. $\sqrt{\left(\frac{\pi}{2}-\sqrt{3}\right)^2}+\sqrt{\left(\frac{\pi}{3}-\sqrt{2}\right)^2}-\sqrt{5-2\sqrt{6}}$ ni soddalashtiring.

A) $\frac{5\pi}{6}-2(\sqrt{2}+\sqrt{3})$ B) $2\sqrt{2}-\frac{5\pi}{6}$ C) $\frac{5\pi}{6}$ D) $-\frac{5\pi}{6}$

5. (x+3)(x+1)(x+2)x ko'paytmaning eng kichik qiymatini toping.

A) 3 B) 2 C) 1 D) -1

6. Agar x=71,8 va y=69,8 bo'lsa, x^3-y^3-5y^2-12y-8+x^2-2xy ni hisoblang.

A) 1 B) 21 C) 79 D) 4

7. Agar $a^3+27b^3=90$ va $3ab^2+a^2b=14$ bo'lsa, $a+3b$ ning qiymatini toping.

A) 7 B) 8 C) 6 D) 10

8. Agar $A^2+B^2+C^2=AB+BC+AC$ bo'lsa, $\frac{A+B}{C}+\frac{B+C}{A}$ qiymati nechaga teng?

A) 1 B) 2 C) 3 D) 4

9. Agar $8(x^4+y^4)-\frac{1}{4}(x^2+y^2)+\frac{1}{256}=0$ bo'lsa, $|x|+|y|$ ning qiymatini toping.

A) 1 B) 0,5 C) 0,25 D) 2

10. Agar $\frac{4b+a}{5a-7b}=\frac{7}{8}$ bo'lsa, $\frac{3a^2-2ab+b^2}{5a^2+2b^2}$ ning qiymati nimaga teng bo'ladi?

A) 2 B) $\frac{22}{47}$ C) 0,5 D) $\frac{9}{22}$

11. Agar $n-m=(a-2)^2, p-n=(b-4)^2, m-p=(c-5)^2$ bo'lsa, $a+b+c$ yig'indi nechaga teng bo'ladi?

A) 8 B) 10 C) 11 D) 7

12. $9(x^4+y^4)-12(x^2+y^2)+8=0$ ekanligini bilgan holda, x^2+y^2 ning qiymatini hisoblang.

A) $\frac{1}{3}$ B) 1 C) $\frac{2}{3}$ D) $\frac{4}{3}$

13. $\frac{(x+4)(3-x)}{(x-2)^2}>0$ tengsizlikning eng katta va eng kichik butun yechimlari yig'indisini toping.

A) 1 B) -2 C) -1 D) 2

14. $(x^2-25)(x-3)(x+6)\sqrt{x+4}=0$ tenglama ildizlari o'rta arifmetigini toping.

A) $4\frac{1}{3}$ B) $1\frac{1}{3}$ C) $\frac{2}{3}$ D) $4\frac{1}{2}$

15. a ning qanday qiymatlarida $2ax+3y=3$ va $6x+3y=7$ to'g'ri chiziqlar kesishish nuqtasining absissasi manfiy bo'ladi?

A) $a<3$ B) $a>3$ C) $a<2$ D) $a>2$

16. Ushbu $-\sqrt{8}$; $-\sqrt{2}$; ... arifmetik progressiyaning dastlabki sakkizta hadi yig'indisini toping.

A) $12\sqrt{2}$ B) 12 C) $-12\sqrt{2}$ D) $5\sqrt{2}$

17. Yig'indini hisoblang: $1 - \frac{1}{3} + \frac{1}{9} - \cdots$

A) 0,75 B) 0,45 C) 0,55 D) 0,85

18. Ketma-ket kelgan oltita natural sonning yig'indisi 435 ga teng. shu sonlarning eng kichigini toping.

Javob: ________________________________

19. Tomonlari 8; 15 va 17 sm bo'lgan uchburchakka tashqi chizilgan aylananing radiusi necha sm?

Javob: ________________________________

20. $tg\left(\frac{\pi}{4} - \alpha\right) = \frac{1}{3}$ bo'lsa, $tg\alpha$ ni toping.

Javob: ________________________________

21. $\frac{cos3x}{cosx} - \frac{sin3x}{sinx}$ ni soddalashtiring.

Javob: ________________________________

22. Agar $cosx = -\frac{1}{3}$ bo'lsa, $\frac{2sinx+sin2x}{2sinx-sin2x}$ ni hisoblang.

Javob: ________________________________

23. $cos930°$ ning qiymatini aniqlang.

Javob: ________________________________

24. Agar $\begin{cases} sin^2x = cosxcosy + \frac{1}{4} \\ cos^2x = sinxsiny + \frac{1}{4} \end{cases}$ bo'lsa, $\cos(x - y)$ ni toping.

Javob: ________________________________

25. $y = cos\left(5x - \frac{5}{2}\right)$ funksiyaning eng kichik musbat davrini toping.

Javob: ________________________________

10-variant

1. Tenglama ildizlarining ko'paytmasini toping:

$$|x - 2| + |x + 3| + |x| = 7$$

A) -4 B) 4 C) 0 D) 6

2. Agar $m^2 + \frac{4}{m^2} = 8$ bo'lsa, $\frac{m^2-2}{m}$ ning qiymatini toping.

A) -4; 4 B) -2; 2 C) -6 D) 0

3. Agar $|\vec{a}| = 4$, $|\vec{b}| = 8$ va $|\vec{a} - \vec{b}| = 10$ bo'lsa, $|\vec{a} + \vec{b}|$ ning qiymatini toping.

A) $\sqrt{26}$ B) $\sqrt{39}$ C) $2\sqrt{15}$ D) $2\sqrt{9}$

4. $y = \sqrt{8^{5x-2} - 8}$ funksiyaning aniqlanish sohasini toping.

A) $[6;\infty)$ B) $[0{,}6;\infty)$ C) $[1;\infty)$ D) $(-\infty;\infty)$

5. Tenglama ildizlarining ko'paytmasini toping:

$$2x^4+3x^3-5x^2+x+7=0$$

A) 3 B) -1,5 C) 3,5 D) 5

6. $p=n^5+n^4+1,\ n\in N\ \ p$ tub bo'ladigan *n* larning yig'indisini toping.

A) 1 B) 4 C) 5 D) 19

7. Hisoblang: $\sqrt{2018^2+2018^2\cdot 2019^2+2019^2}$

A) 4074343 B) 4074353 C) 4074363 D) 4074323

8. Ushbu $y=cos(\sqrt{2}x)+cos\frac{x}{\sqrt{2}}$ funksiyaning eng kichik davrini toping.

A) 2π B) 3π C) $2\sqrt{2}\pi$ D) $3\sqrt{3}\pi$

9. Ushbu $2sin^2x+4cos^2x+6sinxcosx$ ifodaning eng katta qiymatini toping.

A) 6 B) $3+\sqrt{10}$ C) 4 D) $3-\sqrt{5}$

10. Arifmetik progressiyaning 6- hadi 3 ga teng, ayirmasi 0,5 dan katta. Progressiyaning ayirmasi qanday bo'lganda birinchi, to'rtinchi va beshinchi hadlar ko'paytmasi eng katta bo'ladi?

A) 1 B) 2,4 C) 1,8 D) 2

11. Tengsizlikni yeching: $x^{\frac{2x-1}{3-x}}>1$

A) $(0;0{,}5)\cup(1;3)$ B) $(0;0{,}5)\cup(1;2)$ C) $(0{,}5;1)\cup(1;2)$ D) $(0;1)\cup(1;3)$

12. Integralni hisoblang: $\int_0^{\frac{7}{3}}\frac{x+1}{\sqrt[3]{3x+1}}$

A) $\frac{16}{5}$ B) $\frac{56}{15}$ C) $\frac{46}{15}$ D) $\frac{26}{5}$

13. Tenglamaning ildizlari yig'indisini toping: $\frac{27x^3+125}{3x+5}=-(5+48x)$

A) $-\frac{11}{3}$ B) $-\frac{10}{3}$ C) -2 D) -3

14. Agar $\begin{cases}x^3-3xy^2=46\\ y^3-3x^2y=9\end{cases}$ bo'lsa, $x^2+y^2=?$

A) 25 B) 13 C) 29 D) 19

15. Uchburchak burchaklari kosinuslari kvadratlarining yig'indisi 1 ga teng. Uchburchakka ichki va tashqi chizilgan aylana radiuslari mos ravishda $\sqrt{3}$ va $3\sqrt{2}$ bo'lsa, uchburchak yuzini toping.

A) $6\sqrt{6}+3$ B) $6\sqrt{6}$ C) $12+\sqrt{3}$ D) $6\sqrt{3}$

16. O'tkir burchakli uchburchakning balandliklari asoslarini tutashtirishdan hosil bo'lgan kesmalarning uzunliklari 5, 12 va 13. Berilgan uchburchak yuzini toping.

A) $120\sqrt{3}$ B) $90\sqrt{6}$ C) 180 D) 195

17. *ABCD* qavariq to'rtburchakda $BC=\sqrt{3},\ BD=1,\ \angle CBD=120°$ va $\angle BAC=\angle CAD=30°$ bo'lsa, AB ni toping.

A) $\sqrt{1,8}$ B) 1 C) $\sqrt{1,2}$ D) $\sqrt{1,5}$

18. To'g'ri burchakli uchburchakning o'tkir burchagining bissektrisasi gipotenuzaga tushirilgan balandlik bilan kesishish nuqtasida uchidan boshlab hisoblaganda $(1+\sqrt{2}):1$ nisbatda bo'linadi. Uchburchakning eng kichik burchagini toping.

A) $arctg\frac{1}{2}$ B) $30°$ C) $arctg\frac{1}{\sqrt{2}}$ D) $45°$

19. Aylanaga tashqi chizilgan oltiburchakning ketma-ket tomonlari 5,6,7,8 va 9 ga teng bo'lsa, oltinchi tomonini toping.

Javob: ____

20. Aylanaga tashqi chizilgan teng yonli trapetsiyaning yuzi S ga teng bo'lib, balandligi yon tomonining yarmiga teng bo'lsa, aylana radiusini toping.

Javob: ____

21. $2\cdot|x-4|+|3x+5|\geq 16$ tengsizlikning eng katta manfiy yechimini toping.

Javob: ____

22. $\frac{x-2}{\sqrt{2x-3}-1}<4$ tengsizlikning butun yechimlari yig'indisini toping.

Javob: ____

23. Ushbu ifodaning eng kichik qiymatini toping

$$\sqrt{x^2-6x+13}+\sqrt{x^2-14x+58}$$

Javob: ____

24. Hisoblang: $sin\frac{\pi}{14}\cdot sin\frac{3\pi}{14}\cdot sin\frac{5\pi}{14}$

Javob: ____

25. Ushbu $(x-y)^3+(y-z)^3+(z-x)^3=30$ tenglamani qanoatlantiruvchi (x, y, z) butun sonlar uchliklari nechta?

Javob: ____

11- variant

1. $x^2-6x-91<0$ tengsizlikning barcha yechimlarini OX o'qida tasvirlangan. Qanday uzunlikdagi kesma hosil bo'lgan?

A) 20 B) 5 C) 17 D) 6

2. 140 g suvga 60 g tuz qo'shilgan. Eritmada tuz necha foizni tashkil qiladi?

A) 46 B) 20 C) 25 D) 30

3. 459 sonini o'zaro 1:2:6 nisbatda bo'lgan uchta qo'shiluvchi ko'rinishida yozing. Eng katta va eng kichik qo'shiluvchilar orasidagi farqni toping.

A) 245 B) 255 C) 235 D) 275

4. Ifodaning qiymatini toping: $\frac{1}{\sqrt{5}-2}-\frac{2}{\sqrt{5}+\sqrt{3}}+\frac{1}{2+\sqrt{3}}$

A) 4 B) $4-2\sqrt{3}$ C) 0 D) $-2\sqrt{3}$

5. Shartnoma bo'yicha bank qo'yilgan omonatlarga yiliga 10% ustama to'laydi. Ikki yilda omonat necha foizga ko'payadi?
   A) 20 B) 22 C) 23 D) 21
6. Nechta natural son tenglamaning ildizi bo'ladi: $\sqrt[4]{(2x-7)^4}=7-2x$
   A) 3 B) $\emptyset$ C) 1 D) cheksiz ko'p
7. $y=x^2+ax+b$ parabola uchining koordinatalari $x_0=5;\ y_0=-1.$ Parabolaning OY o'qi bilan kesishish nuqtasining ordinatasini toping.
   A) -19 B) 19 C) -24 D) 24
8. Agar ikki xonali sonning o'ng tomoniga 4 raqami yozilsa, berilgan sonning yarmi bilan 821 ning yig'indisiga teng son hosil bo'ladi. Qanday son berilgan?
   A) 44 B) 68 C) 86 D) 89
9. 120 g suvga 80 g tuz qo'shilgan. Eritmada tuz necha foizni tashkil etadi?
   A) 30 B) 40 C) 50 D) 60
10. $f=tg213°\cdot tg141°\cdot sin38°\cdot tg291°$ ifodani nol bilan taqqoslang.
   A) $f>0$ B) aniqlab bo'lmaydi C) $f<0$ D) $f=0$
11. $x_1\ va\ x_2$ sonlar $3x^2+4x-2=0$ tenglama ildizlari. U holda ildizlari $\frac{1}{x_1}\ va\ \frac{1}{x_2}$ bo'lgan kvadrat tenglama tuzing.
   A) $x^2-4x-3=0$ B) $2x^2-4x-3=0$ C) $2x^2+4x-3=0$ D) $x^2-4x+3=0$
12. 6 sonining qanday foizini u bilan 9 ning 4% i ayirmasi tashkil qiladi?
   A) 72% B) 88% C) 94% D) 96%
13. $x\le y\le z;\ \ x,y,z\in N;\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$ Barcha shartlarni qanoatlantiruvchi *(x,y,z)* sonlar uchligiga yechim deyiladi. Nechta yechim mavjud?
   A) 3 B) yechim yo'q C) 2 D) 1
14. $\left(\sqrt{4+\sqrt{15}}\right)^x+\left(\sqrt{4-\sqrt{15}}\right)^x=7$ tenglamaning ildizlari nisbatini toping.
   A) 3 B) 1 C) -1 D) -3
15. A shahar C va B shaharlar orasida joylashgan. Bir vaqtda A shahardan C shaharga 29 km/soat tezlik bilan motosiklchi, A shahardan B shaharga 15 km/soat tezlik bilan velosipedchi va B shahardan A shaharga 7 km/soat tezlik bilan piyoda yo'lga chiqishdi. A va B shaharlar orasidagi masofa 88 km. Velosipedchi va piyoda uchrashgan vaqtda motosiklchi C shaharga yetib keldi. A dan C gacha bo'lgan masofani toping.
   A) 87 B) 100 C) 116 D) 212

16. Agar $x^2+8x+y^2-6y+25=0$ bo'lsa, $5x+4y$ ifodaning qiymatini toping.

A) 16 B) -8 C) aniqlab bo'lmaydi D) -12

17. Tenglamani yeching: $\frac{49x^2-25}{49x^2-70x+25}=\frac{33}{23}$

A) $3\frac{11}{13}$ B) $5\frac{1}{3}$ C) 3 D) 4

18. A sonini 5 ga bo'lganda qoldiqda 3 qoladi, 7 ga bo'lganda esa qoldiqda 1 qoladi. A sonini 35 ga bo'lganda qoldiq nimaga teng?

A) 0 B) 27 C) 2 D) 8

19. Ifodani soddalashtiring: $\frac{a\sqrt{2}+a-\sqrt{2}-1}{a\sqrt{2}-2-\sqrt{2}+2a}$

A) $\sqrt{2}$ B) $\frac{1}{\sqrt{2}}$ C) $\frac{\sqrt{3}}{2}$ D) $\frac{1}{2}$

20. $41\cdot 42\cdot ...\cdot 69\cdot 70$ ko'paytma 2^N ga qoldiqsiz bo'linadi. N ning eng katta qiymati nimaga teng?

A) 20 B) 30 C) 29 D) 28

21. Agar ikki xonali sonning o'ng tomoniga 4 raqami yozilsa, berilgan sonning yarmi bilan 821 ning yig'indisiga teng son hosil bo'ladi. Qanday son berilgan?

A) 68 B) 44 C) 89 D) 86

22. Uchburchakning perimetri 66 sm. Uchburchakning burchaklari quyidagicha munosabatda: $(A-B)\cdot(A-C)\cdot(B-C)=0$. AB tomon uzunligini toping.

A) 22 B) 23 C) 10 D) 12

23. A(1;6) nuqtaga OY o'qiga nisbatan simmetrik bo'lgan B nuqtadan C(19;27) nuqtagacha bo'lgan masofani toping.

A) $6\sqrt{14}$ B) 53 C) $10\sqrt{8}$ D) 29

24. Teng yonli uchburchakning asosi $20\sqrt{3}$, asosiga tushurilgan balandligi asos va yon tomoni o'rtalarini tutashtiruvchi kesma uzunligiga teng bo'lsa, uchburchakning yuzini toping.

A) $100\sqrt{3}$ B) 100 C) $10\sqrt{3}$ D) 10

25. Ishni birinchi ishchi ikkinchisidan 3 marta tezroq bajaradi. Agar ular birgalikda ishlasalar ishni 3 soatda tugatishadi. Tez ishlaydigan ishchi bu ishni qancha vaqtda bajaradi.

A) 4 B)5 C)6 D)3

26. Parallelogramm diagonali orqali ikkita uchburchakka ajratilgan. Parallelogram perimetri 78 sm. Bitta uchburchak perimetri 58 sm. Diagonal uzunligini toping.

A) 20 B)19 C) aniqlab bo'lmaydi D)21

27. Aylana vatari 10 sm. Vatarning bir uchidan aylanaga urinma, ikkinchisidan esa urinmaga parallel ravishda kesuvchi o'tkazilgan. Kesuvchining aylana ichidagi kesmasi 12 sm bo'lsa, aylana radiusini aniqlang.

A) 6 B) 6,25 C) 6,5 D) 6,75

28. Tenglamani yeching: (-7x-8)(5x+7)-(x+2)(-35x-8)=-73

A) 2 B) 6 C) 3 D) 4

29. Bitta kateti 27 sm, gipotenuzasi 45 sm bo'lgan to'g'ri burchakli uchburchakka ichki chizilgan aylananing radiusini toping.

A) 11 B) 8 C) 9 D) 18

30. Radiusi 11 sm bo'lgan aylana markazidan 7 sm uzoqda P nuqta berilgan. Bu nuqta orqali uzunligi 18 sm bo'lgan vatar o'tkazilgan. Vatardan P nuqta orqali ajralgan kichik kesma uzunligi qanday?

A) 10 B) 8 C) 4 D) 6

31. Uchburchakning asosi uzunligi 36 sm. Asosiga parallel to'g'ri chiziq uchburchak yuzini teng ikkiga bo'ladi. Shu to'g'ri chiziqning uchburchak tomonlari orasidagi kesmasi uzunligini toping.

Javob ________________________________

32. ABC uchburchakda AB=8, BC=6, sinA+sinC=7/6 ekani ma'lum. ABC uchburchakka tashqi chizilgan aylana diametrini toping.

Javob ________________________________

33. Tenglamalar sistemasini yeching: $\begin{cases} \frac{3x+y}{x-1} - \frac{x-y}{2y} = 2 \\ x - y = 4 \end{cases}$

Javob ________________________________

34. a ning qanday qiymatida $5x^2 + 6x + 8a - 2 = 0$ tenglama ikkita haqiqiy ustma-ust tushadigan ildizga ega?

Javob ________________________________

35. Agar $x^2 + (-2b + 8)x - 27 = 0$ va $x^2 + 6x + (-2c - 19) = 0$ tenglamalar teng kuchli bo'lsa, b+c ni toping.

Javob ________________________________

36. Tengsizlikni yeching: $\sqrt{9x^2 + 6x + 1} > \sqrt{25x^2 + 30x + 9}$

Javob ________________________________

37. Tengsizlikni yeching: $(3x + 5)^7 < (7x + 6)^7$

Javob ________________________________

38. Cheksiz yig'indini hisoblang:
$sin\frac{\pi}{6} + cos^2\frac{\pi}{3} + sin^3\frac{13\pi}{6} + cos^4\frac{7\pi}{3} + sin^5\frac{25\pi}{6} + cos^6\frac{13\pi}{3} + \cdots$

Javob ________________________________

39. Kirish imtihoni bo'layotgan auditoriyada abituriyentlarning 68% i qizlar, qolganlari o'g'il bolalar. Auditoriyada eng kamida nechta abituriyent bo'lishi mumkin?
Javob ________________________________

40. Radiuslari R ga teng bo'lgan uchta aylananing har biri qolgan ikkitasi bilan urinadi. Umumiy tashqi urinmalar tashkil qilgan uchburchak yuzi bilan aylanalar markazlarini tutashtirish natijasida hosil bo'lgan uchburchak yuzi orasidagi bog'lanishni toping.
Javob ________________________________

12-variant

1. Agar $x = 2{,}5$ va $y = -1{,}5$ bo'lsa, $x^3 - x^2y - xy^2 + y^3$ ni hisoblang.
A) 16 B) 25 C) 8 D) 10
2. $\sqrt{5+x} - \sqrt{x+4} = a$ bo'lsa, $\sqrt{5+x} + \sqrt{x+4}$ ni toping
A) $\frac{1}{a}$ B) $\frac{a-1}{a^2}$ C) $\frac{a^2-1}{a}$ D) $a - 1$
3. $x \cdot y = x + y$ ekani ma'lum, y butun son bo'ladigan x ning nechta butun qiymati mavjud
A) 2 B) 1 C) bitta ham yo'q D) cheksiz ko'p
4. Agar $\big((a+3)x + b\big)(2x-5) = 14x^2 - 29x - 15$ ayniyat bo'lsa, $a + b$ ni hisoblang.
A) 9 B) 10 C) 2 D) 7
5. Ifodani soddalashtiring: $\frac{729a+1}{81\sqrt[3]{a^2}-9\sqrt[3]{a}+1} - \frac{729a-1}{81\sqrt[3]{a^2}+9\sqrt[3]{a}+1}$
A) 2 B) 1 C) 3 D) $a + 2$
6. Ifodaning eng kichik qiymatini toping. $f(x,y) = x^2 - 6x + y^2 - 8y + 36$
A) 0 B) 11 C) -10 D) 24
7. $y = \frac{7}{3+sinx}$ funksiyaning qiymatlaridan tashkil topgan kesmaning uzunligini toping.
A) 1,75 B) 2 C) 5,25 D) 3,5
8. $1 * 490$ yozuvda yulduzchani shunday raqam bilan almashtiringki, hosil bo'lgan son 45 ga qoldiqsiz bo'linsin.
A) 6 B) 5 C) 4 D) 9
9. x_1 va x_2 $13x^2 + 12x + 1 = 0$ tenglamaning ildizlari bo'lsin.

$ax^2+bx+c=0$ ildizlari x_1-1 va x_2-1 bo'lgan kvadrat tenglama. a, b, c sonlar eng katta umumiy bo'luvchisi 1 ga teng va $a>0$ bo'lgan butun sonlar bo'lsa, $100a+10b+c$ ifodaning qiymatini toping.

A) 411 B) 232 C) 256 D) 224

10. Ikki natural sonning eng kichik umumiy karralisi 240 ga teng, shu sonlarning eng katta umumiy bo'luvchisi 16 ga teng. Bu sonlarning ko'paytmasini toping.

A) 15 B) 3840 C) 1706 D) 322

11. Yog'liligi 2% bo'lgan 80 litr sut bilan yog'liligi 5% bo'lgan necha litr sut aralashtirilsa, yog'liligi 2,6% bo'lgan sut olish mumkin?

A) 50 B) 30 C) 20 D) 40

12. Tenglamaning ildizlari kvadratlarining yig'indisini toping:

$$x^2-\frac{2^{10}\cdot 4^{10}\cdot 8^5\cdot 16^7}{32^{14}}x+15=0$$

A) 41 B) $28+4\sqrt{2}$ C) 34 D) 42

13. Soddalashtiring: $4+9\sqrt{2}-\frac{\sqrt{27}}{\sqrt{6}+\sqrt{3}}$

A) -5 B) 13 C) $6\sqrt{2}+7$ D) $13-6\sqrt{2}$

14. Yo'lning birinchi qismida turist soatiga *x* km tezlik bilan *x* soat yurdi. Ikkinchi qismida tezligini 7 km/soatga, vaqt esa 4 soatga oshirildi. Umumiy bosib o'tilgan yo'lni ifodalovchi *x* ga bog'liq ifoda tuzing.

A) $2x^2+11x+28$ B) $11x+8$ C) $x^2+11x+28$ D) $x^2+11x+8$

15. *x* sonining necha foizini *x* ning 30% ining 30% i tashkil qiladi?

A) 30 B) 90 C) 60 D) 9

16. Soddalashtiring: $\left(1-\frac{1}{42}\right)\left(1-\frac{1}{43}\right)\left(1-\frac{1}{44}\right)\left(1-\frac{1}{45}\right)$

A) $\frac{4}{43\cdot 45}$ B) $\frac{42}{45}$ C) $\frac{1}{45}$ D) $\frac{41}{45}$

17. Tenglamaning ildizlari yig'indisini toping:

$$(x-1)(x-2)(x-3)\ldots(c-99)(x-100)=0$$

A) 5000 B) 5050 C) 2500 D) 3880

18. Ifodaning oxirgi raqamini toping: $3^{279}\cdot 7^{298}-3^{178}\cdot 7^{197}$

A) 0 B) 1 C) 3 D) 6

19. x_1 va x_2 $7x^2+2x-2=0$ tenglamaning ildizlari bo'lsin. $x_1^3+x_2^3$ ifodani qiymatini toping.

A) $-\frac{92}{343}$ B) $\frac{91+7\sqrt{14}}{3}$ C) $-\frac{91-7\sqrt{14}}{3}$ D) $\frac{125}{8}$

20. a ning qanday eng kichik qiymatida $y=(a^2-6a+8)x^2+(a-5)x-8$ funksiya grafigi parabola bo'lmaydi.

A) 0 B) 2 C) 5 D) 4

21. $f(x)$ funksiya aniqlanish sohasiga tegishli ixtiyoriy x uchun $\frac{f(-x)\cdot f(x)}{f^2(x)}=-1$ bo'lsa, $f(5)=13$ ekani ma'lum. $3\cdot f(-5)+2\cdot f(5)$ ni hisoblang.
A) 5 B) -13 C) 13 D) 65

22. $a=5+\sqrt{3}$ va $b=3+\sqrt{14}$ ifodalarning qiymatlarini taqqoslang.
A) Taqqoslab bo'lmaydi B) a>b C) a=b D) a<b

23. Funksiyaning aniqlanish sohasiga nechta butun son tegishli bo'ladi?

$$f(x)=\frac{\sqrt{x-4}\cdot\sqrt[3]{x+5}\cdot\sqrt[4]{36-x^2}}{\sqrt[5]{x^2-11x+30}}$$

A) 1 B) 2 C) 3 D) butun sonlar yo'q

24. Doiraga tomoni 3 sm ga teng kvadrat ichki chizilgan. Shu doira yuzini toping.
A) 18π B) $2{,}25\pi$ C) $4{,}5\pi$ D) $\frac{3\sqrt{2}\pi^2}{2}$

25. Birinchi uchburchakning perimetri 28 sm. Ikkinchi uchburchakning tomonlari birinchi uchburchakning o'rta chiziqlari, uchinchi uchburchakning tomonlari ikkinchi uchburchakning o'rta chiziqlaridir. Uchinchi uchburchakning perimetrini toping.
A) 21 B) 14 C) 4 D) 7

26. Radiusi 1 ga teng doiraning yuzi o'zining qismi bo'lgan doiraviy sektor yuzidan $\frac{5\pi}{6}$ ga ko'p. Shu sektor burchagining gradus o'lchovini toping.
A) $120°$ B) $30°$ C) $60°$ D) $45°$

27. Ikki to'g'ri chiziq kesishganda 4 ta burchak hosil bo'ldi. Bir burchakning kattaligi ikkinchisidan $100°$ ga katta. Ikkita eng katta burchakning yig'indisini toping.
A) $80°$ B) $280°$ C) $140°$ D) $300°$

28. ABC uchburchakda AA_1 va BB_1 kesmalar O nuqtada kesishuvchi medianalardir. BOA_1 uchburchakning yuzi 12. ABA_1B_1 to'rtburchakning yuzini toping.
A) 36 B) 54 C) 48 D) 72

29. Tenglamaning ildizlari yig'indisini toping: $|4x-7|=|-10x-6|$
A) $\frac{13}{84}$ B) $-\frac{13}{84}$ C) $\frac{21}{20}$ D) $-\frac{21}{20}$

30. Teng yonli uchburchakning perimetri 57 m, yon tomoni asosidan 9 marta katta. Uchburchakning asosini toping.
A) 27 B) 19 C) 6 D) 3

31. Agar $x^2+y^2+z^2-10x-6y+10z+59=0$ bo'lsa, $5x+4y+6z$ ifodaning qiymatini toping.
Javob: ________________________________

32. ABC uchburchakda AB=10, BC=3, $AC = \sqrt{89}$. B burchakning kosinusini toping.

Javob: ________________________________

33. Agar $a - b = -5$ va $(a^2 - b^2)(a + b) = -605$ bo'lsa, ab ni toping.

Javob: ________________________________

34. To'g'ri burchakli uchburchakka tashqi chizilgan aylana radiusining shu uchburchakka ichki chizilgan aylana radiusiga nisbati 5:2 kabi. Ichki chizilgan aylana radiusi 4 ga teng bo'lsa, uchburchak yuzini toping.

Javob: ________________________________

35. Hisoblang: $\left(\frac{7}{\sqrt{55}+\sqrt{48}} + \frac{20}{\sqrt{48}+\sqrt{28}}\right) \cdot \left(\sqrt{55} + \sqrt{28}\right)$

Javob: ________________________________

36. $\begin{cases} -3 < x < 0 \\ -5 < y < 0 \end{cases}$ $z = 4x - 2y$ ning qiymatini hisoblang.

Javob: ________________________________

37. Tarkibida 19% tuz bo'lgan 48 kg eritmani, tarkibida 39% tuz bo'lgan ertima bilan aralashtirildi. Agar tarkibida 31% tuz bo'lgan eritma hosil bo'lgan bo'lsa, ikkinchi eritmadan necha kg olingan?

Javob: ________________________________

38. $7x^2 - 2x - 2 = 0$ tenglamaning ildizlari x_1, x_2 bo'lsa, $\frac{1}{x_1} + \frac{1}{x_2}$ ifodaning qiymatini toping.

Javob: ________________________________

39. Tenglamani yeching: $\frac{4x^2-100}{2x-10} = 14$

Javob: ________________________________

40. $|x + 5| < 3$ tengsizlikning barcha butun yechimlarining o'rta arifmetigini toping

Javob: ________________________________

13-variant

1. Natural m va n sonlar uchun $\frac{m}{4} + n = 8$ bo'lsa, m qabul qilishi mumkin bo'lgan qiymatlar ichida eng kattasini toping?

A) 16 B) 20 C) 24 D) 28

2. $133^2 - 129^2 = 2x$ bo'lsa, x ni toping?

A) 131 B) 262 C) 346 D) 524

3. Agar $\sqrt{25 - x^2} - \sqrt{15 - x^2} = 2$ bo'lsa, $\sqrt{25 - x^2} + \sqrt{15 - x^2}$ yig'indi nimaga teng?

A) -5 B) 3 C) 5 D) $\frac{2}{3}$

4. $\sqrt{9-x}(x^2-21x+110)=0$ tenglamaning ildizi nechta?

A) 1 B) 2 C) 3 D) 0

5. $ctg\alpha=8$ bo'lsa, $\frac{-7\sin\alpha-8\cos\alpha}{-6\sin\alpha+5\cos\alpha}$ ni hisoblang?

A) $-\frac{71}{34}$ B) $\frac{71}{34}$ C) $-\frac{34}{71}$ D) $\frac{34}{71}$

6. $4x^2-9x+a=0$ tenglamaning ildizlari ko'paytmasi eng katta bo'lishi uchun a qanday bo'lishi kerak.

A) $\frac{9}{4}$ B) $\frac{81}{16}$ C) $\frac{81}{4}$ D) $\frac{48}{81}$

7. Agar 20^x ni 20%i $2^6\cdot5^a$ bo'lsa, a ni toping?

A) 2 B) 3 C) 4 D) 5

8. Agar $a>0$, $b>0$, $c>0$ bo'lsa, $\frac{a\sqrt{bc}+b\sqrt{ac}+c\sqrt{ab}}{\sqrt{abc}}$ ifodani soddalashtiring?

A) $\sqrt{a+b+c}$ B) $a\sqrt{b}+b\sqrt{c}+c\sqrt{a}$ C) $\sqrt{abc}$ D) $\sqrt{a}+\sqrt{b}+\sqrt{c}$

9. Agar $m+\frac{6}{n}=11$ va $m-\frac{6}{n}=5$ bo'lsa, n ni toping?

A) 2 B) 3 C) 4 D) 6

10. a,b va c manfiy sonlar bo'lib, $ab=\frac{5}{4}$, $bc=\frac{6}{5}$, $ca=\frac{7}{6}$ bo'lsa, ularni o'sish tartibida joylashtiring?

A) $a<b<c$ B) $a<c<b$ C) $c<a<b$ D) $c<b<a$

11. $\sin x+\cos x=2a$ bo'lsa, u holda $\sin 2x$ ni toping?

A) $2a^2-1$ B) $2a^2+1$ C) $4a^2-1$ D) $4a^2+1$

12. $\begin{cases}-\frac{a}{5}=\frac{4}{b}=c\\ a+b+c=0\end{cases}$ bo'lsa, c^2 ni toping?

A) 1 B) 0 C) $-\frac{5}{4}$ D) $\sqrt{2}$

13. Agar $0\le m\le 6$ va $3\le n\le 9$ bo'lsa, $\frac{m-n}{m+n}$ ning eng kichik qiymatini toping?

A) -3 B) -1 C) $-\frac{1}{4}$ D) 0

14. Arifmetik progressiyada $a_1=4$, $a_2=7$ bo'lsa a_9 ni toping?

A) 22 B) 24 C) 25 D) 28

15. a ning qanday qiymatlarida $\begin{cases}3|x|+y=2\\ |x|+2y=a\end{cases}$ yagona yechimga ega.

A) 0 B) 4 C) 2 D) -2

16. Soddalashtiring: $\cos\alpha-\frac{1}{2}\cos3\alpha-\frac{1}{2}\cos5\alpha$

A) $8\cos^3\alpha\cdot\sin^2\alpha$ B) $2\sin^2 2\alpha$ C) $8\sin^2\alpha$ D) $8\cos^2 2\alpha$

17. Agar $\lg 2=m$, $\lg 5=n$, $\lg1400=p$ bo'lsa, $\lg 7$ ni hisoblang.

A) $p+3m+2n$ B) $p-3m-2n$ C) $p-2m-3n$ D) $p+2m+3n$

18. BE kesma ABC uchburchakni o'xshashlik koeffitsiyenti $\sqrt{3}$ ga teng bo'lgan ikkita o'xshash uchburchaklarga ajratsa, B burchakni toping.

A) 60° B) 30° C) 90° D) 45°

19. $ABCD$ parallelogrammning A o'tkir burchagidan BC va CD tomonlarga mos ravishda AH_1 va AH_2 perpendikulyarlar o'tkazilgan. Agar $\angle H_1AH_2=130°$ bo'lsa, A burchakni toping.

A) 40° B) 45° C) 50° D) 55°

20. $a=3^{100}+4^{100}$ va $b=5^{100}$ sonlari orasidagi munosabatni aniqlang.

A) $a=b+2^{100}$ B) $a<b$ C) $a>b$ D) $a\geq b+1$

21. $y=\frac{9}{\pi}\arccos\frac{3\sqrt{2}+Sinx-Cosx}{4\sqrt{2}}$ funksiyaning qiymatlar to'plamini toping.

A) $[0;3)$ B) $[0,1;3)$ C) $[0;3]$ D) $[0;1)$

22. Agar $y=f(x)$ funksiyaning aniqlanish sohasi $[-1;2]$ dan iborat bo'lsa, $y=f(2x)$ funksiyaning aniqlanish sohasini toping.

A) $\left[-\frac{1}{2};1\right]$ B) $\left[-\frac{1}{2};0\right]$ C) $[0;1]$ D) $[-1;2]$

23. Hisoblang: $Sin47°+Sin61°-Sin11°-Sin25°$

A) $Sin7°$ B) $Cos7°$ C) $tg7°$ D) 0

24. Muntazam $ABCDEF$ oltiburchakning tomoni 6 ga teng. Oltiburchakning C uchidan AE diagonalgacha bo'lgan masofani toping.

A) 8 B) 9 C) 12 D) 7

25. Uchburchakka tashqi chizilgan aylana radiusi 2 ga teng bo'lsa, shu uchburchak medianalari kvadratlari yig'indisining eng katta qiymatini toping.

A) 24 B) 4 C) 16 D) 27

26. Arifmetik progressiyada $S_3=6, S_6=-42$ bo`lsa, S_9 ni toping.

A) -288 B) -144 C) -171 D) -285

27. Tomonlarining uzunliklari 39, 65 va 52 bo'lgan uchburchakka tashqi chizilgan aylana markazi bilan ichki chizilgan aylana markazi orasidagi masofani toping.

A) $\frac{13}{2}\sqrt{5}$ B) $\frac{15}{2}\sqrt{5}$ C) 4 D) $\frac{\sqrt{845}}{3}$

28. $4^x = 125$, $8^y = 5$ bo'lsa, $\frac{2x-y}{y}$ ni toping?

A) 8 B) -6 C) 1 D) 6

29. Ikki son yig'indisi 15 ga teng, ularning o'rta arifmetigi o'rta geometrigidan 25 % ko'p. Bu sonlarni toping.

A) 5,10 B) 3,12 C) 7,8 D) 10,5

30. $|x-3| + 2\,|x+1| = 4$ tenglamani qanoatlantiruvchi x ning eng kichik butun qiymatini toping.

A) $\{-1\}$ B) $\{1\}$ C) $\{2\}$ D) $\{3\}$

31. Uchi $\frac{2\pi}{3}$ ga teng bo'lgan *BAC* burchakning *AB* va *AC* tomonlarini diametr qilib ikkita yarim aylanalar chizilgan. Bu yarim aylanalarning umumiy sohasiga katta radiusli aylana ichki chizilgan. Agar *AB*=4 sm va *AC*=2 sm ga teng bo'lsa bu aylana radiusini toping.

Javob: ____

32. *ABCD* parallelogramning *BAD* uchi $\frac{\pi}{3}$ ga teng, *AB* tomoning uzunligi 3 sm. *A* uchining bissektrissasi *BC* tomonni *E* nuqtada kesadi. *ABE* uchburchak yuzini toping.

Javob: ____

33. Balandligi *H* ga teng bo'lgan trapetsiyaga aylana tashqi chizilgan. Trapesiyaning asosi aylana markazidan $\alpha\ va\ \beta$ burchaklar ostida ko'rinadi. Trapesiya yuzini toping.

Javob: ____

34. *MPQF* trapetsiyaning *MF* va *PQ* asoslari mos ravishda 24 sm va 4 sm ga teng. Trapesiyaning balandligi 5 sm. *N* nuqta *MP* tomonni *MN* va *NP* kesmalarga ajratadi. *MN* kesma uzunligi *NP* kesma uzunligidan 3 marta katta. *NQF* uchburchak yuzini toping.

Javob: ____

35. *ABC* to'g'ri burchakli va teng yonli uchburchakning *AC* katetida *P* nuqta shunday olinganki *PC* kesmani diametr qilib chizilgan yarim aylana *AB* gipotenuzaga urinadi. Bu yarim aylana *PB* kesmani qanday kesmalarga ajratadi.

Javob: ____

36. Tenglamani yeching (x-1)(x-2)(x-3)(x-4)=15

Javob: ____

37. Ratsional tengsizlikni yeching: $\frac{(x+2)(x^2-2x+1)}{4+3x-x^2} \geq 0$

Javob: ______________________________________

38. Ikkita avtomashina bitta punktdan bir vaqtda bir xil yo'nalishda yo'lga chiqishdi. Birinchi avtomashina 40 km/soat tezlik bilan, ikkinchisi esa birinchisining tezligining 125 % ni tashkil etadigan tezlikda bormoqda. 30 minutdan keyin shu punktdan shu yo'nalish bo'yicha uchinchi avtomashina harakat boshladi va birinchisiga yetib olishdan 1,5 soat keyin ikkinchisini quvib yetdi. Uchinchi avtomashinaning tezligini toping.

Javob: ______________________________________

39. Birinchi qotishma ikki xil metaldan 1:2 nisbatda berilgan. Ikkinchi qotishma xuddi shu metallardan 2:3 nisbatda tuzilgan. Ikki qotishmadan qanday nisbatda olinganda yangi 17:27 nisbatdagi qotishma hosil bo'ladi.

Javob: ______________________________________

40. Tengsizlikni yeching $(0,3)^{2+4+6+8+\cdots+2x} > (0,3)^{72}$ x$\in N$

Javob:______________________________________

14-variant

1. Agar $\begin{cases} 2p^2 + k^2 - 2pk = 25 \\ 2pq - q^2 = 25 \end{cases}$ bo'lsa, $\frac{p+q}{k^2}$ ni toping.

A) Aniqlab bo'lmaydi. B) 1 C) 0,25 D) 0,4

2. 13^{20} ni 19 ga bo'lganda qoldiqni toping.

A) 5 B) 12 C) 13 D) 17

3. $\varphi(n)$ orqali n dan kichik va n bilan o'zaro tub sonlar sonini belgilasak, $\varphi(1996)$ ni toping.

A) 1995 B) 1996 C) 996 D) 1001

4. $f_1(x) = x^2$ va $f_2(x) = x - 1$ funksiyalar grafiklari orasidagi eng qisqa masofani toping.

A) 1 B) $\frac{\sqrt{3}}{4}$ C) $\frac{3\sqrt{2}}{3}$ D) $\frac{3\sqrt{2}}{8}$

5. Muntazam tetraedrning qirrasi 1 ga teng. Tetraedr ichidagi ixtiyoriy nuqtadan uning yoqlarigacha bo'lgan masofalar yig'indisini toping.

A) 1 B) $\frac{\sqrt{6}}{3}$ C) $\frac{\sqrt{6}}{2}$ D) $\sqrt{12}$

6. ABC uchburchakning medianalari 3, 4 va 5 ga teng. Uchburchakning yuzini toping.

A) 5 B) 10 C) 8 D) 25

7. ABC uchburchakning yuzi 48 ga teng. Uning 10 va 12 teng medianalari bu uchburchakni 3 ta uchburchakka va bitta to'rtburchakka ajratadi. Hosil bo'lgan to'rtburchakning yuzini toping.

A) 12 B) 8 C) 10 D) 16

8. $f(x) = \frac{cosx}{1+sinx}$ bo'lsa, $f'(\frac{\pi}{2})$ =?

A) 0 B) $-\frac{1}{2}$ C) 1 D) $\frac{1}{2}$

9. $f(x) = |x^2 - 3x - 4|$, $f'(0)$ =?

A) 3 B) -3 C) mavjud emas D) 4

10. $AC\ va\ BD$ diagonallari o'zaro perpendikulyar bo'lgan $ABCD$ to'rtburchakka radiusi 2 ga teng bo'lgan aylana tashqi chizilgan. Agar $AB = 3$ bo'lsa, CD ni toping.

A) $\sqrt{3}$ B) 3 C) $\sqrt{5}$ D) $\sqrt{7}$

11. Agar $f(x) = x^2 + 14x + 42$ bo'lsa, $f\left(f\left(f\left(f(x)\right)\right)\right) = 0$ tenglamani yeching.

A) Ildizi yo'q B) $\pm\sqrt[16]{7} - 7$ C) $\pm\sqrt[32]{7} + 7$ D) $\pm\sqrt[16]{7} + 7$

12. Agar $a + \frac{1}{a} = 2cos\alpha$ bo'lsa, $a^n + \frac{1}{a^n}$ qiymati nimaga teng.

A) $2cosn\alpha$ B) $cosn\alpha$ C) $2^n cos^n\alpha$ D) $2^n cosn\alpha$

13. Agar $0 < x < \frac{\pi}{2}$ bo'lsa, $(tgx)^{sinx} + (ctgx)^{cosx}$ ifodaning eng kichik qiymatini toping.

A) 2 B) 3 C) 1 D) $\frac{3}{2}$

14. Tenglamani yeching: $\left[\sqrt[3]{1}\right] + \left[\sqrt[3]{2}\right] + \left[\sqrt[3]{3}\right] + \cdots + \left[\sqrt[3]{x^3 - 1}\right] = 400$

A) 5 B) 4 C) 3 D) 2

15. $x^2 + px - \frac{1}{2p^2}$ ko'phadning ildizlari $x_1\ va\ x_2$ bo'lsa, $min(x_1^4 + x_2^4)$–? Bunda $p \in R,\ p \neq 0$.

A) $2 + \sqrt{2}$ B) $2 - \sqrt{2}$ C) 2 D) $\sqrt{2}$

16. Teng yonli uchburchakda $\frac{r}{R}$ munosabat eng katta qiymatga ega bo'lsa, burchaklar qanday qiymatga ega bo'ladi. (r, R – ichki, tashqi chizlgan aylanalar radiuslari)

A) teng tomonli B) to'g'ri burchakli C) aniqlab bo'lmaydi D) uchidagi burchagi 120°

17. $|x| \cdot (x^2 - 4) = -1$ tenglama nechta ildizga ega.

A) 1 B) 2 C) $\emptyset$ D) 4

18. Agar $a, b > 0\ va\ a + b = 1$ bo'lsa, $a^4 + b^4$ ning eng kichik qiymatini toping.

A) 1 B) $\frac{1}{2}$ C) $\frac{1}{4}$ D) $\frac{1}{8}$

19. Berilgan kvadrat ichiga uchlari tomonlarida yotuvchi kvadrat chizilgan. Ularning yuzlarining nisbati 3: 2 ga teng. Tomonlar orasidagi burchakni toping.

A) 30° B) 15° C) 22,5° D) 60°

20. Hisoblang: $arcsin\frac{5}{13}+2arctg\frac{2}{3}$

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{6}$ D) $\frac{\pi}{2}$

21. a ning qanday qiymatlarida $ax^2+2(a+3)x+a+2=0$ tenglama ildizlari nomanfiy?

A) $[-2{,}25;-2]$ B) $[-2{,}1;-1]$ C) $[1;2]$ D) $(-\infty;2]$

22. $\begin{cases} x^2+y^2+2x\le 1 \\ x-y+a=0 \end{cases}$ sistema yagona yechimga ega bo'ladigan a ning barcha qiymatlarini toping.

A) 3; -1 B) 3; 1 C) -1 D) 1

23. Hisoblang: $1+2x+3x^2+4x^3+\cdots+(n+1)x^n,\ \ (x\ne 1)$

A) $\frac{1-x^{n+1}}{(1-x)^2}-\frac{(n+1)x^{n+1}}{1-x}$ B) $\frac{(n+1)x^{n+1}}{(1-x)^2}$ C) $\frac{1-x^{n+1}-(n+1)x^{n+1}}{(1-x)^2}$ D) $\frac{1-x^{n+1}}{(1-x)^2}$

24. $y=0,\ \ x^2+y^2=1$ chiziqlar orasidagi yuzani toping.

A) π B) $\frac{\pi}{4}$ C) $\frac{\pi}{2}$ D) $\frac{\pi}{6}$

25. $x^2+y^2+ay=0,\ \ (a>0)$ aylana markazidan $y=2(a-x)$ to'g'ri chiziqqacha bo'lgan masofani toping.

A) $\frac{a\sqrt{5}}{4}$ B) $\frac{a\sqrt{3}}{2}$ C) $\frac{a\sqrt{5}}{2}$ D) $\frac{\sqrt{5}}{2a}$

26. Agar $|x|<1$ bo'lsa $1+2x+3x^2+4x^3+\cdots$ yig'indini hisoblang.

A) $\frac{1}{(1-x)^2}$ B) $-\frac{1}{(1-x)^2}$ C) $\ln(1-x)$ D) $\ln(1+x)$

27. Agar $f'(sin^2x)=1+cos^2x$ bo'lsa $f(x)$ ni toping.

A) $2x+\frac{x^2}{2}$ B) $2x-\frac{x^2}{2}$ C) $\frac{x^2}{2}-2x$ D) $6x^3$

28. Quyidagi yig'indini hisoblang: $1+2\cdot 2+3\cdot 2^2+4\cdot 2^3+\cdots+2010\cdot 2^{2009}$

A) $2009\cdot 2^{2010}+1$ B) $\frac{1}{2}(1003\cdot 2^{2010}-1)$ C) $\frac{1}{2}(1003\cdot 2^{2010}+1)$ D) 2010

29. $y=2x-6,\ \ y=0,\ \ x=5$ chiziqlar bilan chegaralangan figurani OY o'qi atrofida aylanishdan hosil bo'lgan jismning hajmini toping.

A) 34π B) $34\frac{2}{3}\pi$ C) 33π 36π

30. Uchlari $A(1;2;3),\ \ B(5;2;1),\ \ C(0;4;4)$ nuqtalarda bo'lgan uchburchakning yuzasini hisoblang.

A) $\sqrt{21}$ B) 5 C) 1 D) $\sqrt{12}$

31. $2^{19}+1\ va\ 2^{98}-1$ sonlarining eng katta umumiy bo'luvchisini toping.

A) 3 B) 17 C) 31 D) 7

32. $f(x)=\frac{1}{\sqrt[3]{1-x^3}}$ bo'lsa, $\underbrace{f\big(\ldots f\big(f(19)\big)\ldots\big)}_{95\ marta}=?$

A) $\frac{1}{\sqrt[3]{1-\frac{1}{19^3}}}$ B) 19 C) $\frac{1}{\sqrt[3]{1-19^3}}$ D) $\sqrt[3]{19}$

33. Agar $p = \underbrace{11 \dots 11}_{1997}\underbrace{22 \dots 22}_{1998}5$ bo'lsa, $\sqrt{p}$ ni toping.

A) $\underbrace{33 \dots 33}_{1998}5$ B) $\underbrace{33 \dots 33}_{1996}5$ C) $\underbrace{33 \dots 33}_{1997}5$ D) $\underbrace{33 \dots 33}_{1994}5$

34. $|4 - x| = ax^2$ tenglama $a \in R$ ning qanday qiymatlarida 3 ta turli haqiqiy ildizga ega bo'ladi.

A) $(0; +\infty)$ B) $[\frac{1}{16}; \infty)$ C) $[\frac{1}{16}; 1)$ D) to'g'ri javob yo'q

35. $\vec{p} = \vec{s} + 2\vec{t}$ va $\vec{a} = 5\vec{s} - 4\vec{t}$ vektorlar o'zaro perpendikulyar bo'lsa, $\vec{s}$ va $\vec{t}$ birlik vektorlar orasidagi burchakni toping.

A) $-\frac{\pi}{3}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$

36. $\left(\frac{4}{5}\right)^x = 4$ tenglamaning yechimi qaysi oraliqda yotadi.

A) $(-\infty; -1)$ B) $(0; 1)$ B) $[2; \infty)$ D) $(-1; 0)$

37. Hisoblang: $\left(\frac{81 \cdot 2}{567} + \frac{33}{77}\right) \cdot 24{,}5 + \frac{2}{3} : 0,(3)$

A) 7,5 B) 15,5 C) 19,5 D) 20,5

38. $y = arctg\left(arcsin\frac{sinx - cosx}{sinx + cosx}\right)$ funksiyaning aniqlanish sohasini toping.

A) $\left(\pi k; \frac{\pi}{2} + \pi k\right]$, $k \in Z$ B) $\left[\pi k; \frac{\pi}{2} + \pi k\right]$, $k \in Z$ C) $\left(\pi k; \frac{\pi}{2} + \pi k\right)$, $k \in Z$ D) $\left[\pi k; \frac{\pi}{2} + \pi k\right)$, $k \in Z$

39. $cos\left(lg\left(2 - 3^{x^2}\right)\right) = 3^{x^2}$ tenglama nechta ildizga ega.

A) $\emptyset$ B) cheksiz ko'p C) 1 D) 2

40. a ning qanday qiymatlarida $-3 < \frac{x^2 + ax - 2}{x^2 - x + 1} < 2$ tengsizlik x ning barcha qiymatlarida o'rinli bo'ladi.

A) $-1 < a < 2$ B) $-3 < a < 2$ C) $-2 < a < 1$ D) $a > 0$

15-variant

1. $(x-1)(x-2) + (x-2)(x-3) - (x-3)(x-1) = 2$ tenglamani yeching.

   A) 1; 3 B) 2 C) -2 D) $\forall x \in R$

2. $x^3(x^3+1)(x^3+2)(x^3+3)$ ifodaning eng kichik qiymatini toping.

   A) -1 B) 2 C) -2 D) 1

3. $x^{100} - 2x^{51} + 1$ ni $x^2 - 1$ ga bo'lgandagi qoldiqni toping.

   A) 0 B) -2*x*+2 C) -4*x* D) 4*x*

4. $|\vec{a}| = 11, |\vec{b}| = 23$ va $|\vec{a} - \vec{b}| = 30$ bo'lsa, $|\vec{a} + \vec{b}|$ ni toping.

   A) 16 B) 15 C) 20 D) 18

5. $\frac{x+4}{x+1} > 2 - x$ tengsizlikni yeching.

A) (-6;-3) B) (0;1) C) $\forall x \in R$ D) $(-1;\infty)$

6. $P^2(x+1) = P(x^2) + 2x + 1$ ayniyatni qanoatlantiradigan *P(x)* ko'phadni toping.

A) $P(x) = x$ B) $P(x) = 1$ C) $P(x) = -x$ D) $P(x) = x^2 + 1$

7. $a_0 = 0,\ a_{n+1} = \sqrt{6 + a_n}$ ketma-ketlikning a_{2006} hadini toping.

A) 1 B) 2 C) 3 D) to'g'ri javob yo'q.

8. $|sinx| > |cosx|$ tengsizlikni yeching.

A) $\frac{\pi}{4} + \pi n; \frac{3\pi}{4} + \pi n$ B) $\frac{\pi}{4} + 2\pi n; \frac{3\pi}{4} + 2\pi n$

C) $-\frac{\pi}{4} + \pi n; \frac{3\pi}{4} + \pi n$ D) $\pi n; \frac{3\pi}{4} + \pi n$

9. $f(x) = (x^2 + x)^{100}$ ko'phadning barcha koeffitsiyentlari yig'indisini toping.

A) 200 B) 3 C) 2^{100} D) 2^{200}

10. $(x^2 - x - 3)^4$ ifoda yoyilmasida x ning juft darajalari oldidagi koeffitsiyentlari yig'indisini toping.

A) 40 B) 41 C) 42 D) 43

11. $f(x) = x^3 - 3x + \lambda$ ko'phad λ ning qanday qiymatlarida karrali ildizga ega?

A) ± 3 B) ± 1 C) ± 2 D) ± 4

12. $x^5 - 7x^2 - 5x - 8$ ko'phadni $x - 2$ ga bo'lgandagi qoldiqni toping.

A) -8 B) -5 C) -7 D) -14

13.Hisoblang: $cos\frac{\pi}{5} - cos\frac{2\pi}{5}$

A) $-\frac{1}{2}$ B) $\frac{1}{2}$ C) $\frac{1}{4}$ D) $\frac{1}{8}$

14. $\varphi(x) = sin^6x + cos^6x$ funksiyaning eng katta qiymatini toping.

A) 1 B) $\frac{1}{4}$ C) $\frac{1}{2}$ D) $\frac{1}{3}$

15. Tengsizlikni yeching: $|x^2 - 2x - 3| < 2x - 3$

A) (2;4) B) (2;5) C) (1;6) D) (3;4)

16. Hisoblang: $\ln tg1° + \ln tg2° + \ln tg3° + \cdots + \ln tg89°$

A) 1 B) 0 C) $\frac{1}{2}$ D) $-\frac{1}{2}$

17. *ABC* uchburchakda *AC* tomonga tushirilgan balandligi 2 ga *AB* tomoni 5 ga, *ABC* uchburchakka tashqi chizilgan aylana radiusi 5 ga teng bo'lsa, *BC* tomonining uzunligini toping.

A) 2 B) 5 C) 4 D) $\sqrt{21}$

18. Gipotenuzasi c ga teng bo’lgan to’g’ri burchakli uchburchakning o’tkir burchaklarining kosinuslari yig’indisi q ga teng. Uchburchakning yuzini hisoblang.

A) $\frac{c^2(q^2-1)}{4}$ B) cq C) c^2q^2 D) $\frac{c^2(q^2+1)}{4}$

19. Uchlari A(1;1), B(2;2), C(3;1) nuqtalarda bo’lgan uchburchakda medianalar kesishgan nuqtaning koordinatalarini toping.

A) (1;2) B) $(2;\frac{4}{3})$ C) $(\frac{1}{3};3)$ D) (2;1)

20. Yon tomoni a ga teng bo’lgan teng yonli uchburchakning asosi qandey bo’lganda uning yuzi eng katta bo’ladi?

A) $a\sqrt{2}$ B) $a\sqrt{3}$ C) $2a$ D) a

21. $\vec{a}$ vektor $\vec{b}(1,2,3)$ vektorga koolinear bo’lib, $\vec{a}\cdot\vec{b}=28$ bo’lsa, $|\vec{a}|$ ni toping.

A) $\sqrt{44}$ B) $\sqrt{56}$ C) 3 D) 14

22. $y=\frac{2}{x-3}$ chiziqning asimptotalarini toping.

A) $y=0$ B) $x=0$ C) $x=3$ D) $x=3, y=0$

23. $M(-3;-5)$ nuqtadan o’tib, $7x+4y+3=0$ to’g’ri chiziqqa parallel bo’lgan chiziq tenglamasini tuzing.

A) $-3x-5y+1=0$ B) $7x+4y+41=0$ C) $4x+4y+41=0$ D) $4x+4y-41=0$

24. *ABC* uchburchakning *B* va *C* burchaklari 3:1, *A* burchagining bissekrisasi uchburchakning yuzini 2:1 nisbatda bo’lsa, uchburchakning burchaklarini toping.

A) $\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}$ B) $\frac{\pi}{2},\frac{\pi}{2},0$ C) $\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{4}$ D) $\frac{\pi}{3},\frac{\pi}{6},\frac{\pi}{2}$

25. *ABC* uchburchakda *AM* va *BN* – bissektrisalari *O* nuqtada kesishadi. Agar $AO:OM=\sqrt{3}:1,\ BO:ON=1:(\sqrt{3}-1)$ bo’lsa, uchburchakning *A,B,C* burchaklarini toping.

A) $\frac{\pi}{4},\frac{\pi}{3},\frac{7\pi}{2}$ B) $\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4}$ C) $\frac{\pi}{3},\frac{\pi}{2},\frac{\pi}{6}$ D) $\frac{\pi}{6},\frac{\pi}{6},\frac{2\pi}{3}$

26. Ikkinchi darajali haqiqiy o’zgaruvchili shunday *f(x)* ko’phad topingki, uning uchun *f(1)=8, f(-1)=2, f(2)=14*

A) x^2+3x+4 B) x^2-2x+3 C) x^2+3x-4 D) x^2-4x-3

27. Agar $f(x)=\frac{x}{\sqrt{1+x^2}}$ bo’lsa, $\underbrace{f\left(f\left(f(f\ldots f(2008)\ldots)\right)\right)}_{2008\,ta}$ ni hisoblang.

A) 0 B) $\frac{1}{\sqrt{2009}}$ C) $\frac{2008}{\sqrt{1+2008^3}}$ D) $\frac{2008}{\sqrt{1+2008^2}}$

28. Agar $a+b=1$ tenglik o’rinli bo’lsa, a^4+b^4 ning eng kichik qiymatini toping.

A) $\frac{1}{64}$ B) $\frac{1}{8}$ C) $\frac{1}{128}$ D) $\frac{1}{256}$

29. ABC uchburchakning medianalari 6, 8 va 10 ga teng. Uchburchakning yuzini toping.

A) 32 B) 12 C) 25 D) 24

30. $f(x) = |x^2 - 3x - 4|,\ f'(0) = ?$

A) 3 B) -3 C) mavjud emas D) 4

31. $\sqrt{x}^x = x^{\sqrt{x}}$ tenglama yechimlarining yig'indisini toping.

A) 1 B) 4 C) 3 D) 5

32. "Qonuniy Ma'sudiy" asari muallifining ismi kim?

A) Beruniy B) Abu Rayhon C) Muhammad D) Ahmad

33. Tenglamani yeching. $sin\left(2x + \frac{\pi}{2}\right) = \frac{\pi}{3}$

A) $(-1)^{n+1}\frac{1}{2} - \frac{\pi}{4} + \frac{\pi n}{2}, n\epsilon Z$ B) $(-1)^{n+1}arcsin\frac{\pi}{3} - \frac{\pi}{4} + \frac{\pi n}{2}, n\epsilon Z$

C) $(-1)^{n+1}\frac{\sqrt{3}}{2} - \frac{\pi}{4} + \frac{\pi n}{2}, n\epsilon Z$ D) yechimga ega emas

34. $sinx = \frac{x}{100}$ tenglama nechta yechimga ega?

A) 31 B) 32 C) 61 D) 63

35. Soddalashtiring. $\left(\frac{1}{\sqrt{2}-1}\right)^{\frac{log_6 log_6\left(\frac{1}{\sqrt{2}-1}\right)}{log_6\left(\frac{1}{\sqrt{2}-1}\right)}}$

A) 1 B) $log_6(\sqrt{2}+1)$ C) $log_6(\sqrt{2}-1)$ D) $\frac{1}{\sqrt{2}-1}$

36. Tengsizlikni yeching. $\frac{4x^2}{\left(1-\sqrt{1+2x}\right)^2} < 2x + 9$

A) $-\frac{1}{2} < x < 5\frac{5}{8}$ B) $-\frac{1}{2} \le x < 0, 0 < x < 5\frac{5}{8}$

C) $0 \le x \le 5$ D) $-\frac{1}{2} < x \le 5\frac{5}{8}$

37. "Ziji ko'ragoniy" asari muallifining ismi kim?

A) Muhammad Tarag'ay B) Ulug'bek C) Mirzo D) Ali Qushchi

38. Agar $f\left(\frac{x}{x+1}\right) = x^2$ bo'lsa, *f(x)* ni toping.

A) $f(x) = \left(\frac{x}{1-x}\right)^2$ B) $f(x) = \left(-\frac{1}{1-x}\right)^2$ C) $f(x) = \left(\frac{x}{x+1}\right)^2$ D) $f(x) = x^2$

39. Agar $a^2 + b^2 = 1$ bo'lsa, $a^6 + 3a^2b^2 + b^6$ ning qiymatini toping.

A) 0 B) 2 C) 1 D) 3

40. Tomonlarining uzunligi 13, 14 va 15 bo'lgan uchburchakka ichki va tashqi chizilgan aylanalar markazlari orasidagi masofani toping.

A) $\frac{\sqrt{65}}{8}$ B) $\frac{\sqrt{65}}{4}$ C) $\frac{\sqrt{65}}{12}$ D) $\frac{\sqrt{65}}{10}$

16-variant

1. Ifodanisoddalashtiring. $\sqrt[4]{19x^2+6\sqrt{2}x^2}\cdot\sqrt{3\sqrt{2}x-x}$

   A) $\sqrt{17}x$ B) $\sqrt{x}$ C) $\frac{\sqrt{x}}{17}$ D) x

2. Arifmetik progressiyaning dastlabki $3n$ ta hadi yig'indisi 111 ga teng. Uning dastlabki $2n$ ta hadi yig'indisi 50 ga teng bo'lsa, dastlabki n ta hadi yig'indisini toping.

   A) 25 B) 37 C) 13 D) Aniqlab bo'lmaydi

3. 2222^{7777} sonini 7 ga bo'lgandagi qoldiqni toping.

   A) 3 B) 1 C) 5 D) 0

4. 3, 4 va 5 ga bo'lganda 1 qoldiq qoladigan dastlabki 3 ta natural son yig'indisini toping.

   A) 363 B) 183 C) 333 D) 219

5. Radiusi 3 ga teng bo'lgan aylanaga asoslari 3 va 7 ga teng bo'lgan teng yonli trapetsiya ichki ichizilgan. Bir yon tomon o'rtasidan ikkinchi yon tomongacha bo'lgan masofani toping.

   A) 6 B) 5 C) 4 D) 8

6. Trapetsiyaning o'tkirburchaklaridan biri $30°$, o'rta chizig'i 10, asosi 8 ga teng. Trapetsiyaning yon tomonlari davom ettirganda to'g'ri burchak ostida kesishadi. Trapetsiya kichik yon tomonini toping.

   A) 4 B) $2\sqrt{3}$ C) 2 D) 3

7. $4x+3y+22=0,\ x=-1,\ y=-2$ to'g'ri chiziqlar hosil qilgan uchburchakka ichki chizilgan aylana radiusini toping.

   A) 1 B) 1,5 C) $2\sqrt{3}$ D) $\sqrt{2}$

8. $\left(1+n+\frac{1}{n}\right)^4$ ifoda yoyilmasining ozod hadini toping.

   A) 25 B) 31 C) 17 D) 19

9. $\begin{cases}2xy=z^2+4\\x+y-z=2\end{cases}$ tenglamalar sistemasi nechta ildizga ega?

   A) 0 B) 3 C) 2 D) 1

10. Agar $2<x\le 5,\ 3\le y<6$ bo'lsa, $\frac{y-x}{y+x}$ ifodaning qiymati qaysi oraliqda o'zgaradi?

    A) $\left(-1;\frac{7}{11}\right)$ B) $\left(\frac{1}{5};\frac{1}{3}\right)$ C) $\left[-\frac{1}{4};\frac{1}{2}\right)$ D) $\left(-\frac{2}{5};\frac{4}{5}\right)$

11. $x+\frac{1}{x}=3$ bo'lsa, $x^4+x^8+\frac{1}{x^4}+\frac{1}{x^8}$ ni hisoblang.

    A) 6642 B) 2254 C) 2258 D) 2450

12. $y=sinx$ funksiyaning 67- tartibli hosilasini toping.

    A) $-sinx$ B) $-cosx$ C) $cosx$ D) $sinx$

13. Agar $a = \lg 2$ va $b = \lg 3$ bo'lsa, $log_2 360$ ni a va b orqali ifodalang.

A) $\frac{2(a+b)-1}{a}$ B) $\frac{2(a-b)+1}{a}$ C) $\frac{2(b-a)+1}{a}$ D) $\frac{2(a+b)+1}{a}$

14. $x + \frac{1}{y+\frac{1}{z}} = \frac{15}{8}$, $x, y, z \in N$ bo'lsa, $x+y+z$ ning yarmini toping,

A) 9 B) 4,5 C) 5 D) 10

15. Muntazam uchburchak ichidagi P nuqta uchun $BP = 2\sqrt{21}$, $AP = 5\sqrt{3}$, $PC = 3$ bo'lsa, u holda unga ichki chizilgan aylana radiusini toping.

A) 4 B) $\frac{\sqrt{63}}{3}$ C) $\frac{\sqrt{21}}{2}$ D) $\frac{\sqrt{43}}{2}$

16. Toshkentdan Samarqandga 4 xil transport bilan, Samarqanddan Jizzaxga 2 xil transport bilan borish mumkin. Toshkentdan Jizzaxga necha xil usul bilan borish mumkin?

A) 2 B) 16 C) 8 D) 6

17. Ikki parallel to'g'ri chiziqlarning birida 2 ta nuqta, ikkinchisida 7 ta nuqta bor. Uchlari shu nuqtalarda bo'lgan nechta uchburchak yasash mumkin?

A) 64 B) 49 C) 40 D) 24

18. Soddalashtiring: $\frac{sin34°-sin222°+cos16°+cos8°}{4cos4°cos20°cos32°}$

A) 1 B) -1 C) 2 D) 0,5

19. *ABC* uchburchakning *AB* tomonidan *M* va *N* nuqtalar, *BN*=*NM*=*MA*. *AC* tomonidan *K* va *L* nuqtalar shunday tanlab olindiki, *AL*=2*LC* va *AK*=3*KL* bo'lsa, $\frac{S_{MNLK}}{S_{BCLN}}$ ning qiymatini toping.

A) 2 B) 0,5 C) 3 D) 1

20. Diagonallari soni tomonlari soniga teng bo'lgan ko'pburchakning ichkiburchaklari yig'indisining tashqi burchaklari yig'indisiga nisbatini toping.

A) 1,5 B) 2 C) 1 D) 4

21. To'g'ri chiziqqa nisbatan simmetrik almashtirish … deyiladi.

A) Gamotetiya B) harakat C) gipoteza D) burish

22. $x^2 + y^2 + |z - 2xy| - 2x + 4y - 5$ ifoda eng kichik qiymatga erishganda, *xyz* ni toping.

A) 8 B) 10 C) -8 D) -10

23. 50! ni tub ko'paytuvchilarga ajratganda ko'paytmada $2^{n+1}, 3^{m+2}, 7^{k+3}$ lar ishtirok etsa, *m*+*n*+*k* ning eng katta qiymatini toping.

A) 71 B) 73 C) 72 D) 70

24. Teng yonli uchburchakning yon tomoni 20, asosi 24 ga teng. Uchburchakning medianalari kesishish nuqtasidan ichki chizilgan aylana markazigacha bo'lgan masofani toping.

A) 1 B) $\frac{1}{3}$ C) $\frac{2}{3}$ D) $\frac{1}{2}$

25. Maxrajni irratsionallikdan qutqaring: $\left(\frac{\sqrt[5]{4\sqrt[5]{4\sqrt[5]{4\ldots}}}}{\sqrt[5]{2:\sqrt[5]{2:\sqrt[5]{2:\ldots}}}}\right)^{-0,6}$

A) $\frac{\sqrt[5]{2}}{2}$ B) $\frac{\sqrt[5]{16}}{2}$ C) $\frac{\sqrt[5]{4}}{2}$ D) $\frac{\sqrt[5]{8}}{2}$

26. $f(x) = \begin{cases} 3-4x, & x>2 \\ x^2-8, & x \le 2 \end{cases}$ $g(x) = \begin{cases} 2x^2+3, & x>-3 \\ 6x-4, & x \le -3 \end{cases}$ bo'lsa, $f(g(2)) - g(f(2))$ ni toping.

A) -78 B) -79 C) -13 D) 0

27. $P(x,y,z) = (x^3 - 2y^3 + 3z^3)^3 - (2x^2 - 3y^2 + z^2 - 1)^2 -$ $-2x^3 - 4y - z - 2$ ko'phad soddalashtirilgandan so'ng barcha o'zgaruvchilar oldidagi koeffitsiyentlar yig'indisini toping.

A) -3 B) 1 C) -6 D) 6

28. Eritma tarkibida 12 g tuz bor. Unga 800 g toza suv qo'shilsa, tuz konsentratsiyasi 1,5 marta kamayadi. Dastlabki eritma necha gramm?

A) 1240 B) 1360 C) 2160 D) 1600

29. Hisoblang: $cos\left(4arcct\frac{3}{4}\right)$

A) $-\frac{608}{625}$ B) $-\frac{527}{625}$ C) $\frac{527}{625}$ D) $\frac{608}{625}$

30. $2cosx + sinx = -2$ tenglamaning $[-\pi;\pi]$ kesmada ildizlari nechta?

A) 4 B) 2 C) 3 D) 1

31. Ko'phadni ko'paytuvchilarga ajrating. $x^5 + x^4 - 2x^3 + 3$

Javob:______________________________

32. $f\left(\frac{x-2}{x-1}\right) + 2f\left(\frac{3-x}{x}\right) = \frac{x+1}{1-x}$ bo'lsa, $f(x)$ ni toping.

Javob:______________________________

33. $M = 1 + 1\cdot3 + 1\cdot3\cdot5 + \cdots + 1\cdot3\cdot5\cdot\ldots\cdot97\cdot99$ va $N = 2 + 2\cdot4 + 2\cdot4\cdot6 + \cdots + 2\cdot4\cdot6\cdot\ldots\cdot98\cdot100$ bo'lsa, $M+N$ yig'indining oxirgi raqamini toping.

Javob:______________________________

34. $\frac{c}{a+b} + \frac{b}{a+c} + \frac{a}{b+c} = 7$ va $\frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{b+c} = 2$ bo'lsa, $a+b+c$ ni hisoblang.

Javob:______________________________

35. $P(x) = (x^2 + 2014x + 2015)\cdot(x^2 + 2015x + 2016) +$ $(x^2 + 2013x + 2014)\cdot(x^2 + 2017x + 2018) + 3x + 2$ ko'phadni $Q(x) = x^2 + 2016x + 2017$ ko'phadga bo'lgandagi qoldiqni toping,

Javob:___________________________________

36. $1^3 + 2^3 + 3^3 + \cdots + 12^3$ yig'indi qaysi sonning kvadratiga teng.

Javob:___________________________________

37. *ABCD* parallelogrammda *AB*=10, *AD*=16. *BC* tomonga *AE* va *DF* bissektrisalar o'tkazilgan bo'lib, ular *G* nuqtada kesishadi. *E* va *F* nuqtalar *BC* tomonda bo'lsa, *AGD* va *FGE* uchburchaklar yuzlari nisbatini toping.

Javob:___________________________________

38. Uchburchakning ichidan olingan ixtiyoriy nuqtadan tomonlariga parallel chiziqlar o'tkazilgan. Hosil bo'lgan to'rtburchaklar yuzlari mos ravishda 4, 9, 16 ga teng bo'lsa, uchburchak yuzini toping.

Javob:___________________________________

39. Yig'indini hisoblang:

$$\frac{2^4+2^2+1}{2^7-2}+\frac{3^4+3^2+1}{3^7-3}+\cdots+\frac{2017^4+2017^2+1}{2017^7-2017}+\frac{1}{2\cdot 2017\cdot 2018}$$

Javob:___________________________________

40. Ushbu 2*xy*+*x*+*y*=83 tenglamani qanoatlantiruvchi *x* va *y* ning barcha butun qiymatlari yig'indisini toping.

Javob:_________________________________

17-variant

1. Agar $a = 3^{100} + 4^{100}$ va $b = 5^{100}$ bo'lsa, quyidagi munosabatlardan qaysi biri to'g'ri?

   A) $a < b$ B) $a > b$ C) $a \geq b + 1$ D) $a = b + 2^{100}$

2. Ifodani soddalashtiring: $cosx - \frac{1}{2}cos3x - \frac{1}{2}cos5x$

   A) $8sin^2xcos^3x$ B) $2sin^2 2x$ C) $8cos^2 2x$ D) $8sin^2x$

3. Agar $a + b + c = 7$ va $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} = \frac{7}{10}$ bo'lsa, $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}$ ifodaning qiymatini hisoblang.

   A) $\frac{49}{10}$ B) $\frac{1}{2}$ C) $-\frac{49}{10}$ D) $\frac{19}{10}$

4. $(x^4 - x + 1)^{1999} + (x^5 + x - 1)^{1999}$ ko'phadning x ning toq darajalari oldidagi koeffitsiyentlari yig'indisini toping.

   A) -1 B) 0 C) 1 D) 2

5. Agar $\begin{cases} x^2 + 2z = -1 \\ y^2 + 2x = -1 \\ z^2 + 2y = -1 \end{cases}$ bo'lsa, $x + y + z$ ning qiymatini toping.

   A) -6 B) -3 C) 3 D) 6

6. Uzunliklari $|\vec{a}| = 3, |\vec{b}| = 4$ bo'lgan $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak $\frac{\pi}{3}$ bo'lsa, $|\vec{c}| = 3\vec{a} + 2\vec{b}$ vektorni uzunligini hisoblang.

A) $\sqrt{217}$ B) 12 C) $\sqrt{221}$ D) 13

7. $y = \frac{9}{\pi} arccos\frac{3\sqrt{2}+sinx-cosx}{4\sqrt{2}}$ funksiyaning qiymatlar to'plamini toping.

A) $[0; 3)$ B) $[0,1; 3)$ C) $[0; 3]$ D) $(0; 1)$

8. $y = \frac{2x+2}{x^2-x+2}$ funksiyaning eng kichik qiymatini toping.

A) $-\frac{3}{11}$ B) $-\frac{1}{4}$ C) $\left[-\frac{2}{7}\right]$ D) $-\frac{3}{14}$

9. Ushbu $\frac{x+5}{x+3} + \frac{x}{x+1} = \frac{x+1}{x-1} + \frac{2}{x+3}$ tenglamaning haqiqiy ildizlari yig'indisini toping.

A) -3 B) 3 C) 0 D) 6

10. Hisoblang: $2^{\sqrt{log_2 3}} - 3^{\sqrt{log_3 2}}$

A) -1 B) 0 C) 1 D) 3

11. Tenglama nechta yechimga ega $(1 - 2sin^2x)log_7(18 + x - 4x^2) = 0$

A) 1 ta B) 2 ta C) 3 ta D) 4 ta

12. Integralni hisoblang:

$$\int_1^e \frac{\sin(lnx)}{x} dx$$

A) 1 B) $cosx$ C) $1 - cos1$ D) $cos0,5$

13. $\frac{\sqrt{2}+1}{\sqrt{2}-1} + 1 + \frac{\sqrt{2}-1}{\sqrt{2}+1} + \cdots$ ni hisoblang.

A) $\frac{\sqrt{2}+1}{2}$ B) $\frac{\sqrt{2}-1}{2}$ C) $\frac{(\sqrt{2}+1)^2}{2}$ D) $\frac{(\sqrt{2}+1)^3}{2}$

14. Agar $x \in [arctg0,5; arctg3]$ bo'lsa, $y = sin2x$ funksiyaning qiymatlar sohasini toping.

A) $(-0,5; 0,5)$ B) $[0,1; 2)$ C) $[0,6; 1]$ D) $[0; 2)$

15. Bir xonada o'rtacha yoshlari 25 bo'lgan 9 kishi, ikkinchi xonada esa o'rtacha yoshlari 45 bo'lgan 11 kishi dam olmoqda. Ularning barchasining o'rtacha yoshini toping.

A) 40 B) 36 C) 35 D) 32

16. Ushbu $(x - y)^3 + (y - z)^3 + (z - x)^3 = 30$ tenglamani qanoatlantiruvchi (x,y,z) butun sonlar nechta?

A) 3 B) 4 C) 2 D) 0

17. Tengsizlikni yeching: $2^x + 2^{|x|} \geq 2\sqrt{2}$

A) $(-\infty; log_2(\sqrt{2}-1)] \cup \left[\frac{1}{2}; \infty\right)$ B) $[log_2(\sqrt{2}-1); \infty)$

C) $(-\infty; log_2(9-\sqrt{2})] \cup [4; \infty)$ D) $(-\infty; log_2(9-\sqrt{2})]$

18. Agar $m, n \in N$ va $40 \cdot n = m^3$ bo'lsa, $m+n$ ning qabul qilishi mumkin bo'lgan eng katta qiymatini toping.

A) 35 B) 100 C) 125 D) 53

19. Hosilasini hisoblang: $y = \sqrt{arctgx}$

A) $y = \frac{1}{2\sqrt{arctgx}}$ B) $y = \frac{1}{2(1+x^2)\sqrt{arctgx}}$

C) $y = -\frac{1}{2(1+x^2)\sqrt{arctgx}}$ D) $y = \frac{1}{2(1+x^2)}$

20. To'g'ri burchakli parallelepipedning bitta uchidan chiquvchi ikkita qirralari 72 va 18 ga teng. Agar uning diagonali 78 ga teng bo'lsa, uning hajmini toping.

A) 31104 B) 3104 C) 3110 D) 3100

21. To'g'ri burchakli parallelepiped radiusi va balandligi 6 ga teng bo'lgan silindrga tashqi chizilgan. Parallelepiped hajmini toping.

A) 764 B) 864 C) 264 D) 462

22. To'g'ri burchakli parallelepipedning diagonalining uzunligi $\sqrt{8}$ ga teng bo'lib, uning yoqlari bilan $30°, 30°$ va $45°$ burchak tashkil etadi. Parallelepiped hajmini toping.

A) 1 B) 2 C) 3 D) 4

23. Hajmi 8 ga teng bo'lgan *SABC* uchburchakli piramida oltiburchakli *SABCDEF* piramidaning qismi bo'lsa, oltiburchakli *SABCDEF* piramidaning hajmini toping.

A) 48 B) 28 C) 38 D) 84

24. Tetraedrning qirrasi 16 ga teng. Uning 4 ta qirrasining o'rtasidan o'tuvchi kesim yuzini toping.

A) 23 B) 44 C) 64 D) 84

25. Yuzasi 169π bo'lgan aylanaga to'g'ri burchak ichki chizilgan. Agar uning bir tomoni $2\sqrt{105}$ ga teng bo'lsa, ikkinchi tomonini toping.

A) 10 B) 16 C) 7 D) 5

26. Uchburchakning tomonlari 39, 65 va 52 ga teng bo'lsa, unga tashqi va ichki chizilgan aylanalar markazlari orasidagi masofani toping.

A) $\frac{13}{2}\sqrt{5}$ B) $\frac{15}{2}\sqrt{5}$ C) 4 D) $\frac{\sqrt{845}}{3}$

27. Konusga radiusi 2 ga teng bo'lgan shar ichki chizilgan. Agar konusning yasovchisi bilan balandligi orasidagi burchak $30°$ ga teng bo'lsa, konusning yon sirtini toping.

A) 24π B) 4π C) 16π D) 20π

28. Muntazam to'rtburchakli piramidaning hajmi 48 ga, balandligi 4 ga teng. Piramidaning yon sirtining yuzini toping.

A) 144 B) 1205 C) 60 D) 72

29. Uchburchakli to'g'ri prizma $ABCA_1B_1C_1$ ning asosining tomonlari 2 ga, yon yog'i diagonali $\sqrt{5}$ ga teng. Prizma asosi va A_1BC tekislik orasidagi burchak topilsin.

A) $30°$ B) $45°$ C) $60°$ D) $135°$

30. Konusning hajmi 6 sm^3 ga teng. konusning asosi va balandligi bilan bir xil o'lchovga ega silindrning hajmini toping.

A) 10 B) 16 C) 18 D) 22

31. Tenglamani yeching: $x^2\sqrt{x} - x\sqrt{x} = 3x^2 - 3x + 6\sqrt{x} - 18$

Javob: ________________________________

32. $3^{100}33^{100}333^{100} + 8^{100}88^{100}888^{100}$ son 5 ga bo'lgandagi qoldiqni toping.

Javob: ________________________________

33. $3a^2 + 4ab + 4b^2 + 4a + 3$ ifodaning eng kichik qiymatini toping.

Javob: ________________________________

34. Tenglamani yeching: $log_2(2^x + 1 - x^2) = log_2(2^{x-1} + 2 - 2x) + 1$

Javob: ________________________________

35. Ifodaning qiymatini toping: $log_2 sin\frac{\pi}{12} + log_2 sin\frac{\pi}{6} + log_2 sin\frac{5\pi}{12}$

Javob: ________________________________

36. *ABCDE* qavariq beshburchakning *CE* diagonali *BD* va *AD* diagonallari bilan mos ravishda *F* va *G* nuqtalarda kesishadi. Agar *BF:FD*=5:4, *AG:GD*=1:1, *CF:FG:GE*=2:2:3 munosabatlar o'rinli bo'lsa, *CFD* va *ABE* uchburchaklar yuzlarining nisbatini toping.

Javob: ________________________________

37. *ABCD* to'rtburchakda $\angle DAB = 150°, \angle DAC + \angle ABD = 120°$, $\angle DBC - \angle ABD = 60°$ lar berilgan. $\angle BDC$ ni toping.

Javob: ________________________________

38. Tomoni *a* ga teng bo'lgan *ABCD* kvadrat berilgan. *BC* va *CD* tomonlarda *M* va *N* nuqtalar mos ravishda shunday olinganki, bunda *BM*=3*MC*, 2*CN*=*ND*. *AMN* uchburchakka ichki chizilgan aylana radiusini toping.

Javob: ________________________________

39. *ABC* o'tkir burchakli uchburchakda *AH* va *CK* balandliklar o'tkazilgan. Agar $HK = 2\sqrt{2}$, *ABC* va *BHK* uchburchaklarning yuzlari mos ravishda 18 va 2 ga teng bo'lsa, *ABC* uchburchakka tashqi chizilgan aylana radiusini toping.

Javob: ________________________________

40. Rombning tomoni diagonallarining o’rtacha geometrik qiymatiga teng bo’lsa, rombning o’tkir burchagini toping.
Javob: ____________________________________

18-variant

1. Ifodaning oxirgi raqamini toping: $3^{266} \cdot 7^{294} - 3^{237} \cdot 7^{187}$
A) 9 B) 2 C) 8 D) 0
2. $x^2 - (2c - 3)x + c = 0$ tenglamaning bitta ildizi nolga teng. Ikkinchi ildizini toping.
A) -3 B) 3 C) -1 D) 4
3. Tenglamani yeching: $\sqrt{14 + \sqrt{192}} - \sqrt{x} = \sqrt{6}$
A) 8 B) 12 C) 4 D) yechimi yo’q
4. x ning qanday qiymatida $y = x^2 + 14x + 52$ ifoda eng kichik qiymatga ega bo’ladi.
A) 0 B) -7 C) -14 D) eng kichik qiymatga ega emas
5. Soddalashtiring: $\frac{16x^4+4x^2+1}{4x^2-2x+1} - 4x^2$
A) x+2 B) 2x-1 C) x-1 D)2x+1
6. $\frac{5}{8}$ sonining taqribiy qiymati uchun $\frac{3}{5}$ soni olindi. Yaqinlashtirishning absolut xatoligini toping.
A) $\frac{1}{40}$ B) 0,04 C) 3% D) $\frac{2}{3}$
7. $(x^8 \cdot x^2 \cdot x^7 \cdot x^6)^6$ ni $(x^6)^5 \cdot (x^7 \cdot x^6)^4$ ga bo’lgandagi bo’linma x^A ga teng. A ni toping.
A) -60 B) 28 C) 56 D) 98
8. Radiusi 5 ga teng bo’lgan markazi 5x+4y=39 va -2x-7y=-21 tenglamalar bilan berilgan to’g’ri chiziqlar kesishgan nuqtada joylashgan. Bu aylana tenglamasini ko’rsating.
A) $x^2 - 14x + y^2 - 2y + 25 = 0$ B) $x^2 - 2x + y^2 - 14y + 25 = 0$
C) $x^2 + 14x + y^2 - 2y - 25 = 0$ D) $x^2 - 2x + y^2 - 14y - 16 = 0$
9. Tengsizlikning barcha butun yechimlarining o’rta arifmetigini toping:
$$|x - 2| < 3$$
A) 3 B) $2\frac{1}{6}$ C) 2,5 D) 2
10. Tarkibida 12% tuz bo’lgan 48kg aralashmani tarkibida 38% tuz bo’lgan bo’lsa, aralashmaga qo’shildi. Agar tarkibida 26% tuz bo’lgan aralashma hosil bo’lgan bo’lsa, ikkinchi aralashmadan necha kg olishgan?
A) 52 B) 56 C) 48 D) 44

11. Birinchi qor tozalovchi mashina ko'chani 11 soatda qordan tozalashi mumkin, ikkinchisi esa 1soatda tozalaydi. Ikki mashina bir vaqtda ko'chani tozalashga kirishdi va 50 minut ishlab , birinchi mashina garajga ketdi, ikkinchi mashina ishni oxiriga yetkazdi. Birinchi mashina ketganidan keyin ikkinchi mashina necha soat ishlagan?

A) $\frac{1}{10}$ B) $\frac{2}{13}$ C) $\frac{1}{11}$ D) $\frac{1}{13}$

12. Agar x_1 va x_2 $5x^2 - 3x - 1 = 0$ tenglamaning ildizlari bo'lsa, $x_1^2 \cdot x_2 + x_1 \cdot x_2^2$ ifodaning qiymatini toping.

A) $-\frac{3}{25}$ B) -3 C) $\frac{7+\sqrt{29}}{4}$ D) $6 - \sqrt{29}$

13. Hisoblang: $\frac{1\frac{1}{8}\cdot 0,0114\cdot\frac{8}{9}+7\cdot 0,0264\cdot\frac{1}{7}}{0,6\cdot 0,00636+0,00064\cdot 0,6}$

A) 0,8 B) 8 C) 0,9 D) 9

14. Tenglamani yeching: $\frac{64x^3+27}{16x^2-12x+9} = 39$

A) Yechimi yo'q B) 9; 3+$\sqrt{2}$ va 3-$\sqrt{2}$ C) 3+$\sqrt{2}$ va 9 D) 9

15.Tenglamani yeching: $\frac{x}{2} - \frac{x+3}{4} = \frac{x+3}{6} - \frac{x-4}{4}$

A) 6,75 B) 6,5 C) 6 D) 5,5

16. *ABC* uchburchakning tomonlari: *AB*=13, *BC*=7, *AC*=8. *A* uchidan *AD* bissektrisa o'tkazilgan.$|BD - CD|$ ni toping.

A) $2\frac{1}{4}$ B) $2\frac{1}{3}$ C) $1\frac{2}{3}$ D) 2

17. *A*(-5;-3) nuqtaga *OY* o'qiga nisbatan simmetrik bo'lgan *B* nuqtadan *C*(19;45) nuqtagacha masofani toping.

A) 25 B) 50 C) $\sqrt{1960}$ D) 24

18. *ABCD* to'g'ri to'rtburchakda *AB*=26, *BC*=18. *M* nuqta *BC* tomonda shunday joylashganki, bunda *BM:MC*=5:4. *N* nuqta *CD* tomonda shunday joylashganki, bunda *CN:ND*=10:3. *AMN* uchburchakning yuzini toping.

A) 204 B) 224 C) 214 D) 196

19. Gipotenuzasi uzunligi 29, yuzi 210 bo'lgan to'g'ri burchakli uchburchakning o'tkir burchaklarining tangenslari yig'indisini toping.

A) $\frac{\sqrt{29}}{2}$ B) $2\frac{1}{7}$ C) $2\frac{1}{420}$ D) 1

20. Parallelogramm diagonali orqali ikkita uchburchakka ajratilgan. Parallelogrammning perimetri 62 sm. Uchburchaklardan birining perimetri 53 sm. Diagonalning uzunligini toping.

A) $\frac{13\sqrt{3}}{2}$ B) 24 C) 9 D) 22

21. Ikki to'g'ri chiziqning kesishishidan 4 ta burchak hosil bo'ldi, ulardan biri ikkinchisidan 24^0 ga katta. Katta burchakning kattaligini graduslarda toping.

A) 78^0 B) 102^0 C) 96^0 D) 100^0

22. Rombning diagonallari mos ravishda 80 va 84. Rombning perimetrini toping.

A) 232 B) 104 C) 116 D) $4\sqrt{840}$

23. To'g'ri burchakli uchburchakning katetlari yig'indisi 82 sm ga, gipotenuzasi 58 sm ga teng bo'lsa, unga ichki chizilgan aylananing radiusini toping.

A) 24 B) $12\sqrt{2}$ C) 12 D) 16

24. To'g'ri burchakli uchburchakning katetlari yig'indisi 16 sm ga teng. Bunday uchburchaklardan eng katta yuzasi qanday bo'lishi mumkin?

A) 24 B) $8\sqrt{2}$ C) 16 D) 32

25. Aylana tashqarisidagi *M* nuqtadan aylanaga ikkita kesuvchi o'tkazilgan. Birinchi kesuvchi aylanani avval *A* nuqtada keyin *B* nuqtada kesib o'tadi. Ikkinchi kesuvchi aylanani *C* nuqtada keyin *D* nuqtada kesib o'tadi. *BD* yoyga tiralgan, *BMD* burchakning ichida joylashgan markaziy burchak 195^0 ga teng. *AC* yoyga tiralgan, *BMD* burchakning ichida joylashgan markaziy burchak 23^0 ga teng. *BMD* burchakning kattaligini toping.

A) 86^0 B) 30^0 C) 109^0 D) 90^0

26. *ABC* uchburchakning tomonlari *AB*=48, *BC*=80, *AC*=72. *BD* – uchburchakning bissektrissasi. *AD* ni toping.

A) 24 B) 27 C) 21 D) 26

27. *ABC* uchburchakning tomonlari: *AB*=20, *BC*=16, *AC*=26. *BD* –medilananing uzunligini toping.

A) $\sqrt{119}$ B) $\sqrt{167}$ C) $\sqrt{159}$ D) $\sqrt{157}$

28. Tomonlari mos ravishda 3 sm va 4 sm bo'lgan to'g'ri to'rtburchakning diagonallari orasidagi o'tkir burchakning kosinusini toping.

A) $\frac{11}{25}$ B) $\frac{6}{25}$ C) $\frac{8}{25}$ D) $\frac{7}{25}$

29. Radiusi 1 ga teng doira yuzi bu doiraning qismi bo'lgan doiraviy sektor yuzasidan $\frac{9\pi}{10}$ ga katta. Bu sektor burchagining gradus o'lchovini toping.

A) 36 B) 26 C) 34 D) 38

30. Aylana uzunligi 66π ga teng. kattaligi radian o'lchovda $\frac{2\pi}{3}$ bo'lgan markaziy burchak tiralgan yoy uzunligini toping.

A) 20π B) 22π C) 32π D) 24π

31. $x^2 - 6x + |x-4| + 8 = 0$ tenglama ildizlari yig'indisini toping.

A) 8 B) 7 C) 4 D) -4

32. $\frac{2}{|x+2|} \geq 1$ tengsizlikni butun yechimlari nechta?

A) 5 B) 4 C) 3 D) 2

33. $a(a+1)(a+2)(a+3)+1$ ifodani ko'paytuvchilarga ajrating.

A) $(a^2+3a+1)^2$ B) $(a^2+2a+1)^2$ C) $(a^2+1)(a^2+3)$ D) $(a^2+a+1)^2$

34. Agar $a=\frac{1}{2}(\sqrt{3}+1)$ bo'lsa, $4a^3+2a^2-8a+7$ ni qiymatini hisoblang.

A) $2\sqrt{3}+1$ B) 10 C) $\frac{\sqrt{3}}{2}$ D) 8

35. y=6x-19 va y=kx-35 to'g'ri chiziqlar (4;5) nuqtada kesishsa k ni toping.

Javob: ________________________________

36. Tengsizlikning barcha yechimlar to'plamini ko'rsating: $\frac{1}{x-9} < \frac{1}{x}$.

Javob: ________________________________

37. Hovuzdan bir tekis suv chiqarila boshlagandan 5 soat keyin yana 975 m^3, yana 5 soatdan keyin 825 m^3 suv qoldi. Hovuzda qancha suv bo'lgan edi?

Javob: ________________________________

38. Qopga 100 kg qum solishgandan keyin u avvalgi og'irligidan 25% og'irroq bo'ldi. Qopning dastlabki og'riligi qanday bo'lgan?

Javob: ________________________________

39. Kuchuklar ko'rgazmasida barcha ishtirokchilarning 65% i ovcharkalar, 27% i rotveylerlar va qolgan 24 tasi pudellar edi. Ko'rgazmada qancha kuchuk ishtirok etgan?

Javob: ________________________________

40. Hisoblang: $\frac{\sqrt{405}}{\sqrt{245}}+\frac{\sqrt{48}}{\sqrt{3}}$

Javob: ________________________________

19-variant

1. Agar $f(g(x))=-x+13$ bo'lsa, $f(x)$ va $g(x)$ to'g'ri chiziqlar orasidagi burchakni toping.

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) 0

2. $a+b=-1, a+c=6,\ b+c=1$ bo'lsa, quyidagi ifodaning qiymatini toping.

$$a^2(3b+3c+2a)+b^2(3a+3c+2b)+c^2(3a+3b+2c)$$

A) 45 B) 27 C) 100 D) 216

3. $abc=1$ bo'lsa, $\frac{2019}{1+a+ab}+\frac{2019}{1+b+bc}+\frac{2019}{1+c+ac}$ ifodaning qiymatini toping.

A) 2 B) 0 C) 1 D) 2019

4. $\left|x^2+2x-8\right|=3a$ tenglama a ning qanday qiymatlarida 3 ta haqiqiy yechimga ega bo'ladi?

A) $a=3$ B) $(0;3)$ C) $(1;3)$ D) $a>3,\ a=0$

5. Tenglamani yeching: $x^{x^{2019}}=2019$

A) 2019 B) $\sqrt{2019}$ C) $\sqrt[2018]{2019}$ D) $\sqrt[2019]{2019}$

6. Tenglamani yeching. $\dfrac{ctg^2x+4ctgx}{5\cos^2x-4\cos x}=0$

A) $\pi k;\ -arcctg4+\pi k,\ k\in Z$ B) $\arccos 0{,}8+2\pi k,\ k\in Z$

C) $\pi k;\ \pi-arcctg4+\pi k,\ k\in Z$ D) $\pi-arcctg4+\pi k,\ k\in Z$

7. $y=f(x)$ funksiyaning grafigini absissa o'qidan 2019 marta cho'zish, ordinata o'qiga 2019 marta siqish natijasida hosil bo'lgan funksiyani toping.

A) $y=2019f\left(\dfrac{x}{2019}\right)$ B) $y=2019f(x)+2019$

C) $y=2019f(2019x)$ D) $y=\dfrac{1}{2019}f(2019x)$

8. Agar $2^a=27,\ 3^c=16$ bo'lsa, $a\cdot c$ ning qiymatini toping.

A) 10 B) 13 C) 11 D) 12

9. *GH* kesmani *O* nuqtada, *G* nuqtadan boshlab hisoblaganda 5:7 kabi, *P* nuqta esa 5:11 kabi nisbatda bo'ladi. *O* va *P* nuqtalar orasidagi masofa 30 sm bo'lsa, *GH* kesmaning uzunligini toping.

A) 288 B) 18 C) 72 D) 324

10. Radiusi 6 ga teng bo'lgan uchta aylana o'zaro tashqi urinishdan hosil bo'lgan egri chiziqli uchburchakka ichki chizilgan aylana radiusini toping.

A) $6-2\sqrt{3}$ B) $4\sqrt{3}-6$ C) $2\sqrt{2}-1$ D) 1

11. *ABCD* tetraedrning *D* uchidagi barcha yassi burchaklari to'g'ri. Shu tetraedrga kub shunday ichki chizilganki, kubning bitta uchi D nuqtada, unga qarama-qarshi uchi esa *ABC* yoqda yotibdi. Agar *DA*=5, *DB*=6 va *DC*=10 bo'lsa, kub qirrasining uzunligini toping.

A) $\dfrac{25}{12}$ B) $\dfrac{15}{7}$ C) 2 D) $2\sqrt{2}$

12. $f(x)=\dfrac{x^3}{3}-x^2-35x+2$ funksiya uchun $f'(x)=0$ bo'lsa, x ni toping.

A) 5; 7 B) – 7 ; 5 C) – 5 ; 7 D) – 7 ; – 5

13. $y=\log_{\frac{\pi}{3}}|6x-8|$ funksiyaning aniqlanish sohasini toping.

A) $\left(\dfrac{\pi}{3};\infty\right)$ B) $\left(-\infty;\dfrac{\pi}{3}\right)\cup\left(\dfrac{\pi}{3};\infty\right)$ C) $\left(-\infty;\dfrac{4}{3}\right)\cup\left(\dfrac{4}{3};\infty\right)$ D) $\left(-\infty;\dfrac{4}{3}\right)$

14. Perimetri 4 ga o'tkir burchagi $30°$ ga va shu burchak qarshisidagi tomoni $\sqrt{3}$ ga teng bo'lgan uchburchakka ichki chizilgan aylana radiusini toping.

A) $4\sqrt{3}+7$ B) $\frac{\sqrt{3}-1}{2}$ C) $\frac{\sqrt{3}+1}{2}$ D) $7-4\sqrt{3}$

15. $y=\frac{1}{\sqrt[3]{x^2}}+x\sqrt[3]{x^2}-5^x+2$ funksiyaning boshlang'ich funksiyasini toping.

A) $3\sqrt[3]{x}+\frac{3}{8}x^2\sqrt[3]{x^2}-5^x+2x+C$ B) $3\sqrt[3]{x}+\frac{3}{8}x^2\sqrt[3]{x^2}-5^x\ln 5+2x+C$

C) $\sqrt[3]{x}+3x^2\sqrt[3]{x^2}-5^x+2x+C$ D) $3\sqrt[3]{x}+\frac{3}{8}x^2\sqrt[3]{x^2}-\frac{5^x}{\ln 5}+2x+C$

16. Agar $\lg 5=a$ va $\lg 3=b$ bo'lsa, $\log_{30}8$ ni toping.

A) $b+1$ B) $3(1-a)$ C) $(1-a)(1+b)$ D) $\frac{3(1-a)}{1+b}$

17. $f(x)=70\cos x\cos 6x$ funksiya uchun boshlang'ich funksiyani toping.

A) $7\cos 5x-5\cos 7x+C$ B) $-7\cos 5x-5\cos 7x+C$

C) $7\sin 5x+5\sin 7x+C$ D) $7\sin 5x-5\sin 7x+C$

18. $y=6\sin 2x+\sin 12x$ funksiyaning hosilasini toping.

A) $24\sin 5x\cos 7x$ B) $-24\sin 7x\cos 5x$

C) $24\cos 5x\cos 7x$ D) $24\sin 5x\sin 7x$

19. Tengsizlikni yeching: $\log_{\sqrt{3}}(2x-1)<\log_{\sqrt{3}}(x^2+6x+9)$

A) $\left(\frac{2}{3};2\right)$ B) $\left(\frac{5}{3};2\right)$ C) $\left(\frac{1}{2};\infty\right)$ D) $\left(\frac{4}{3};3\right)$

20. Tenglamani yeching: $3\cos 2x-3\sqrt{3}\sin 2x=0$

A) $\frac{\pi}{12}+\frac{\pi k}{2}, k\in Z$ B) $\frac{\pi}{6}+\pi k, k\in Z$ C) $\frac{\pi}{12}+2\pi k, k\in Z$ D) $\frac{\pi}{12}+\pi k, k\in Z$

21. Ifodani soddalashtiring: $\frac{1+\cos 3x+\cos 2x+\cos x}{2\cos^2 x+\cos x-1}$

A) $\cos x$ B) 1 C) $2\cos x$ D) $2\sin x$

22. Hisoblang: $\frac{5\cdot(-2)^{-2}+\left(\frac{1}{2}\right)^{-4}-\left(\frac{2}{3}\right)^{-2}}{2^{-2}+3^0}$

A) – 13 B) 13 C) – 12 D) 12

23. Poyezd 3 minutda 7 km ni, motosiklchi 5 minutda 7 km ni bosib o'tdi. Motosiklchi tezligi poyezd tezligining necha foizini tashkil qiladi?

A) 60% B) 70% C) 61% D) 80%

24. Agar $f(x)=(a+b-4)x^3+2x^2+(b-3)x$ juft funksiya bo'lsa, $f(b)-f(a)$ ning qiymatini toping.

A) 27 B) 24 C) 7 D) 20

25. Agar $a-b=|x|+2$ bo’lsa, a va b lar uchun to’g’ri munosabatni aniqlang.

A) $a>b$ B) $a\le b$ C) $a<b$ D) $a=b+1$

26. Agar $x^2-8x+20z+4z^2+41=0$ bo’lsa, xz ni toping.

A) – 10 B) – 20 C) – 40 D) – 5

27. Hisoblang: $\cos 10°-2\cos 50°-\cos 70°$

A) $\sin 40°$ B) $\cos 40°$ C) $-\sin 40°$ D) $-\cos 40°$

28. 20 dan 250 gacha sonlar ichida 4 ga bo’linadigan, lekin 6 ga bo’linmaydigan nechta natural son bor?

A) 48 B) 30 C) 39 D) 33

29. $\frac{1}{x}+\frac{1}{z}=\frac{1}{2}$ tenglamaning butun sonlardan iborat yechimlari juftligi nechta?

A) 6 B) 4 C) 7 D) 5

30. Tenglamani yeching: $\left(\sqrt{\sqrt{2019}+\sqrt{2018}}\right)^x+\left(\sqrt{\sqrt{2019}-\sqrt{2018}}\right)^x=2$

A) 0 B) 1 C) 2019 D) 2018

31. Agar $a+b+c=2019$ va $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c}=1$ bo’lsa, u holda $a+b+c-\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$ ifodaning qiymatini toping.

Javob: ________________________________

32. Kasrning surati va maxraji musbat sonlar bo’lsin. Agar surati 14% ga orttirsak, maxrajini 20% kamaytirsak, kasrning qiymati qanday o’zgaradi?

Javob: ________________________________

33. $P(x-2)$ ko’phadni $x+3$ ga bo’lsak 1 qoldiq qoladi. $P(x)$ ko’phadni $x+5$ ga bo’lgandagi qoldiqni toping.

Javob: ________________________________

34. Yon tomoni 8 ga teng bo’lgan teng yonli uchburchakning asosiga tushirilgan medianasi 6 ga teng bo’lsa, *ABC* uchburchakning asosini toping.

Javob: ________________________________

35. *ABC* uchburchakning medianasi hosil qilgan uchburchaklardan biri muntazam bo’lsa, *ABC* uchburchakning eng katta tashqi burchagini toping.

Javob: ________________________________

36. $\sqrt{20-\sqrt{20-\sqrt{20-...}}}=a$ bo’lsa, $\sqrt{14a+\sqrt{14a+\sqrt{14a+...}}}$ ning qiymatini toping.

Javob: ________________________________

37. *A, B, C, D* nuqtalar aylanada ketma-ket joylashgan nuqtalar. *CD* aylana diametri. *DB* vatar *ADC* burchakni teng ikkiga bo’ladi. Agar *DCB* burchak $65°$ bo’lsa, *DCA* burchakni toping.

Javob: ________________________________

38. Hisoblang: $2017\cdot\left(2018^{10}+2018^{9}+...+2018^{2}+2019\right)+1$

Javob: ________________________________

39. Integralni hisoblang: $\int_{0}^{1}\frac{2019x^{2018}-2019}{x^{2019}-2019x+2019}dx$

Javob: ________________________________

40. Tenglamani yeching: $\begin{vmatrix}2019 & 2018\\ 2017 & 2016\end{vmatrix}=2019x$

Javob: ________________________________

Faollashtiruvchi savollar.

1. *Irratsional sonlar deb nimaga aytiladi?*
2. *O'zaro teskari sonlar deb nimaga aytiladi?*
3. *Funksiyaning o'suvchi bo'lish bo'ladi?*
4. *Funksiyaning kamayuvchi bo'lish sharti?*
5. *Kvadrat funksiyaning aniqlanish sohasi nimadan iborat?*
6. *Kvadrat funksiyaning qiymatlar sohasi nimadan iborat?*
7. *Teskari finksiya ta'rifini ayting?*
8. *Ko'rsatkichli funksiyaning teskari funksiyasini ayting?*
9. *Trigonometrik funksiyalarning teskari funksiyalarini ayting?*
10. *Murakkab foiz formulasini ayting?*

GLOSSARIY

1. *n* **faktorial –** 1 dan *n* gacha natural sonlar ko'paytmasi (n!).

2. **Lemma –** isbot talab etiladigan matematik jumla.

3. **Evklid algoritmi –** bir nechta natural sonlarning EKUBini topish.

4. **EKUB –** eng katta umumiy bo'luvchi $(\mathrm{B}(a,b))$.

5. **EKUK –** eng kichik umumiy karrali $(K(a,b))$.

6. **Deduksiya –** umumiy xulosalardan xususiy xulosalar chiqarish.

7. **Induksiya –** xususiy xulosalardan umumiy xulosalar chiqarish.

8. $\tau(n)$ – *n* natural sonning natural bo'luvchilar soni $\tau(n)=(\alpha_1+1)(\alpha_2+1)...(\alpha_n+1)$.

9. **Eyler funksiyasi –** 1 dan *n* gacha bo'lgan natural sonlar orasida shu n soni bilan o'zaro tub bo'lgan sonlar soni $\varphi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)...\left(1-\frac{1}{p_k}\right)$.

10. $\sigma(n)$ – *n* natural sonning natural bo'luvchilar yig'indisi $\sigma(n)=\frac{p_1^{\alpha_1+1}-1}{p_1-1}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2-1}\cdot...\cdot\frac{p_k^{\alpha_k+1}-1}{p_k-1}$.

11. **Mod** – qoldiqli bo'lishda ishlatiladigan musbat qoldiq tushunchasi.

12. **Summa** – yig'indi $\sum_{k=1}^{n}k=1+2+3+...+n$.

13. $\prod_{k=1}^{n}k=1\cdot2\cdot3\cdot...\cdot n$ - ko'paytma.

14. $[x]$ – x sonning butun qismi.

15. $\{x\}$ – x sonning kasr qismi.

16. **Diofant tenglama** – butun sonlarda yechiladigan tenglama.

17. **Nyuton binomi** – Nyuton ikkihadi.

18. **AM-GM** – o'rta arifmetik va o'rta geometrik.

19. **K.B.S** – Koshi-Bunyakovskiy-Shvars.

20. **Og’irlik markaz** – medianalar kesishgan nuqta.

21. **Ortomarkaz** – balandliklar kesishgan nuqta.

22. **Imarkaz** – bissektrisalar kesishgan nuqta.

23. **Cheviana** – uchburchakning uchlaridan chiquvchi bir nuqtada kesishuvchi to’g’ri chiziqlar.

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Umumiy o'rta ta'lim maktablari va akademik litseylar uchun o'quv qo'llanma