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MAHSUS TA'LIM VAZIRLIGI

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MATEMATIKA KAFEDRASI

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EYLER INTEG'RALLARINING TADBIQLARI

5130100-matematika ta'lim yo'nalishi bo'yicha

bakalavr akademik darajasini olish uchun yozilgan

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Kirish

Farzandlarimiz bizdan ko'ra kuchli, bilimli,
dono va albatta baxtli bo'lishlari shart

I.A.Karimov.

Insoniyat ibtidoiy jamoa tuzumi davridanoq o'zining jismoniy mehnatini yengillatish maqsadida ko'plab texnologiyalarni vujudga keltirgan. Ya'ni ular qo'l mehnatini yengillatish uchun qurol yasab texnik mexanizmlar yaratishga asos soldi.

Jamiyat taraqqiyotining olg'a siljishi eng avvalo inson omiliga bog'liqdir. Shuning uchun ham inson o'z tafakkuri, aql-zakovatini ko'proq ijodiy ishlarga jalb qilishi shartligi e'tirof etilmoqda. Yangidan-yangi texnik qurilma va vositalarni kashf qilish insonni o'z yashash sharoitiga, qilayotgan ishiga, ilmiy-texnik izlanishlariga ijodiy yondashish samarasidir. XX asrga kelib insoniyat qo'l mexnatinigina emas, balki aqliy mexnatini ham yengillatish ustida anchagina izlanish olib bordi. Bu yo'lda ko'plab texnik qurilmalar yaratildi va amaliyotga tadbiq etildi.

O'zbekistonning iqtisodiy va ijtimoiy sohalarda yuqori natijalarga erishishi, jahon iqtisodiy tizimida to'laqonli natijalarga to'laqonli sheriklik o'rnini egallay borishi, inson faoliyatining barcha jabhalarida zamonaviy axborot texnologiyalaridan yuqori darajada foydalanishning ko'lamlari qanday bo'lishiga hamda bu tehnologiyalar ijtimoiy mehnat samaradorligining oshishida qanday rol o'ynashiga bog'liq. Demak, zamonaviy kompyuterlardan amalda keng foydalana oladigan yetuk kadrlar tayyorlash kechiktirib bo'lmaydigan vazifadir.

Yangi asr bo'sag'asida jamiyatimiz axborot jamiyati deb atala boshlandi. Hisoblash texnikasi va aloqasi vositalarining keng rivojlanishi axborotni ilgari xayolga ham keltirib bo'lmaydigan shunday hajm va tezkorlikda yig'ish, saqlash, qayta ishlash va uzatish imkonini beradi.

Mamlakatimiz mustaqil bo'lganiga ham 23 yildan oshgan bo'lsa, shu vaqt mobaynida davlatimiz rivojlangan davlatlar qatoridan o'rin olib bordi. Bunga erishishning asosiy omillaridan biri yuqori malakali kadrlar tayyorlashdir.

1997 yil 29 avgust O'zbekiston Respublikasi Oliy majlisida "Ta'lim to'g'risida"gi qonunning va "Kadrlar tayyorlash milliy dasturi"ning qabul qilinishi muqaddas zaminimizda yashayotgan har bir inson va uning baxtu-saodati, farzandini fazlu-kamolini ko'rish uchun ulkan imkoniyatlar yaratishga asos bo'ldi. Ta'lim tizimidagi mamlakatimizda olib borilayotgan islohotlarning mazmuni va amalga oshirish muddatlari ushbu qonunda o'z aksini topgan.

"Kadrlar tayyorlash milliy dasturi"da ta'kidlanganidek, "Kadrlar tayyorlash tizimi va mazmunini mamlakatning ijtimoiy va iqtisodiy taraqqiyoti istiqbollaridan, jamiyat ehtiyojlaridan, fan, madaniyat, texnika va texnologiyaning zamonaviy yutuqlaridan kelib chiqqan holda qayta qurish lozim".

Prezidentimiz Islom Karimov tashabbusi bilan ishlab chiqilgan Kadrlar tayyorlash milliy dasturining hayotga tadbiq etilishi tufayli uzluksiz ta'lim tizimi muntazam yangilanayotgani va takomillashayotgani, ta'lim muassasalarining zamonaviy moddiy-texnik va o'quv bazasini shakllantirish va mustahkamlash, ta'lim-tarbiya jarayoniga yangi standartlar, ilg'or pedagogik va axborot tehnologiyalarini joriy etish borasida keng ko'lamli ishlar amalga oshirilayotir.

Prezidentimiz "Yuksak ma'naviyat-engilmas kuch" asarida ta'kidlagandek "Vatanimizning kelajagi, xalqimizning ertangi kuni, mamlakatimizning jahon hamjamiyatidagi obro'-e'tibori, avvalambor, farzandlarimizning unib-o'sib, ulg'ayib, qanday inson bo'lib hayotga kirib borishiga bog'liqdir. Biz bunday haqiqatni hech qachon unutmasligimiz kerak."

Xalqimiz mustaqillikka erishganidan buyon o'tgan yillar mobaynida mamlakatimiz o'zining mustaqil siyosiy va ijtimoiy-iqtisodiy rivojlanish yo'lini belgilab oldi hamda huquqiy davlat va fuqarolik jamiyatini shakllantirish borasida ulkan muvaffaqiyatlarga erishdi.

Bugungi kundagi erishgan yutuqlarimiz va amalga oshirilgan islohotlardan ko'rinib turibdiki, mamlakatimizda amalga oshirilayotgan islohotlarning mohiyatini "shaxs-jamiyat-davlat" tizimi aks ettiradi. Bugungi kunda mamlakatimizda olib borilayotgan islohotlar ham tasodifan yuz berayotgani yo'q, balki mustaqil O'zbekiston Respublikasi o'z taraqqiyot yo'lini belgilab olgan, adolatli fuqarolik jamiyati va demokratik huquqiy davlat qurishni o'z oldiga maqsad qilib qo'yganligida namoyon bo'ladi.

O'zbekiston mustaqillikka erishgan kundan boshlab o'tgan qisqa vaqt ichida o'zbek xalqi siyosiy-ijtimoiy, iqtisodiy va madaniy sohalarda katta yutuqlarga erishdi. O'z tarixiga yangicha tafakkur asosida yondoshish, ulug' ajdodlar qoldirgan boy madaniy, ma'naviy merosni o'rganish sharafiga muyassar bo'ldi, milliy g'ururi qayta tiklandi. Respublikada ilm-fan, jumladan, matematika fani taraqqiyot bosqichiga ko'tarilmoqda. O'tmishdagi riyoziyot (matematika) daholarining shuxratini tiklash, ularning g'oyalarini xalq hayotiga tadbiq etishdek ulug' ishlar amalga oshirilmoqda.

Matematika fani taraqqiy etishida o'rta asr olimlaridan Muso al-Xorazmiy, Axmad al-Farg'oniy, Abu Rayxon Beruniy, Mirzo Ulug'bek va boshqalar juda katta xissa qo'shganlar.

O'zbek matematiklarining matematika fani soxasidagi xizmatlarini yuqori baholab, O'zbekiston Respublikasi Prezidinti I. A. Karimovning "O'zbekiston XXI asr bo'sag'asida" asarida shunday deyiladi: "Matematikaning " ehtimollar nazariyasi va matematika statistika", "Differentsial tenglamalar nazariyasi", "Matematika-fizika tenglamalari", "Funktsional analiz" sohalari bo'yicha erishilgan natijalar Respublikadan tashqarida ham ma'lum"

Yuqorida aytilgan natijalarga ega bo'lishda matematiklardan V. I. Romonovskiy, T. N. Qori-Niyoziy, T. A. Sarimsakov, S. X. Srojiddinov, I. S. Artaix, M.S. Saloxiddinov, Sh. A. Achilov, T. A. Azlarov, Sh. A. Alimov, D. X. Xojiyev va boshqalarning xizmatlari nihoyatta kattadir.

Hamma sohalarda matematik qonuniyatlarga asoslangan zamonaviy kompyuterlarning muvaffaqiyat bilan tatbiq etilishi hamda uning kundan-kunga rivojlanib borayotganligi, yosh mutaxassislarning tegishli sohalar, masalalarining matematik modellarini tuza bilishi va unda hisoblash texnikasini joriy etish vazifalarini qo'ymoqda. Bu masalalarni modellashtirish matematik amallar va usullar yordamida amalga oshiriladi.

Ma'lumki, matematikadagi mavjud, natural sonlar, arifmetik amallardan boshlab, hozirgi zamonaviy, chiziqli algebra va analitik geometriya, differentsial va integral hisob hamda differentsial tenglamalargacha tushunchalar real dunyoning modellaridir. Bu tushunchalarning hammasi insoniyat ehtiyojlaridan- narsalarni sanash, xo'jalik hisobi kabi tirikchilik uchun zarur masalalardan kelib chiqqan va rivojlanib bormoqda.

Matematika o'z rivojlanish tarixida mexanika, fizika, biologiya kabi fanlardan tashqari ijtimoiy fanlarga ham jadal kirib, rivojlanib bormoqda. Matematikani insoniyat taraqqiyotida vujudga kelgan va uning rivojlanishida katta ahamiyatga ega bo'lgan fanlarning yetakchilaridan desak xato qilmagan bo'lamiz. Bu fikrimizning isbotini matematika iborasi yunoncha "matema" - "bilim, ilm, fan" deyilishi bilan ham izohlasa bo'ladi.

Ushbu bitiruv malakaviy ish "Eyler integrallarining tatbiqlari" deb nomlangan bo'lib, asosan “Gamma” funksiyani o‘rganish, bu funksiyaning a parametrga bog‘liq integral shaklida tasvirlanishidan foydalanib, uning chuqur xossalarinigina emas, balki hisoblash usullari ham yoritilgan.

Ishning maqsad va vazifalari. Mavzu doirasini keng va asosli yoritib berish va misollar keltirishdan iborat.

Obyekti va predmeti. Ob'yekti tabiatdagi hodisa va voqealarni amaliy masalalarda yoritilishi va predmeti nazariy bilimlar va adabiyotlar hisoblanadi.

Amaliy ahamiyati. Amaliyotdagi masalalarda qo'llash mumkinligini asoslash.

Ishning tuzilishi. Ushbu BMI mundarija, kirish, 2 ta bob, hulosa va foydalanilgan adabiyotlardan iborat.

I bob Eyler integrallari haqida umumiy ma'lumotga bag'ishlangan bo'lib, 3 ta paragrafdan iborat. 1-paragrafda birinchi tur eyler integrallarii bayon qilingan. 2-paragrafda ikkinchi tur Eyler integrallari haqida yozilgan. 3-paragrafda Gamma funksiyasining bazi xossalari bayon qilingan.

II-bobda Eyler integralining tadbiqlari ko'rib chiqilgan. Bu bob 3ta paragrafdan iborat bo'lib, ularda bazi aniq integrallar va yuzalarni xisoblash, Stirling fo'rmulasi, Asimtotik qatorlar va bazi masalalar bayon qilingan.

I bob.Eyler integrallari haqida umumiy malumot

1.1-§ Birinchi tur Eyler integrali.

Lejandrning taklifi bilan:

$$B(a,b) = \int_0^1 x^{a-1}\,(1-x)^{b-1}dx \qquad (1)$$

ko‘rinishdagi integral birinchi tur Eyler integrali deyiladi, bu yerda $a, b > 0$. Bu integral B ("$beta$") funksiyaning ikkita: a va b o‘zgaruvchi parametrlarning funksiyasidan iborat.

Biz bilganimizdek, ko‘rilayotgan integral a va b ning musbat (aqalli birdan kichik bo‘lgan) qiymatlari uchun yaqinlashadi, va demak, haqiqatan ham, B funksiyaning ta’rifiga asos bo‘la oladi. Bu funksiyaning ba’zi bir xossalarini aniqlaymiz.

1° Eng avval, bevosita ($x = 1 - t$ almashtirish bilan) ushbuni hosil qilamiz:

$$B(a,b) = B(b,a)$$

demak, B funksiya a va b ga nisbatan simmetrikdir.

Hozir biz B uchun boshqa analitik ifodani beramiz, bu ifoda a bilan b ni almashtirilganda, tashqi ko‘rinish jihatdan ham o‘zgarmaydi.

Bu maqsadda avval $x = \frac{y}{1+y}$ almashtirishni bajaramiz, bu yerda u yangi o‘zgaruvchi bo‘lib, 0 dan ∞ gacha o‘zgaradi. Biz

$$B(a,b) = \int_0^\infty \frac{y^{a-1}}{(1+y)^{a+b}}dy \qquad (2)$$

formulaga ega bo‘lamiz va undan kelgusida ko‘p marta foydalanamiz.

Agar $\int\limits_0^\infty$ integralni $\int\limits_0^1 + \int\limits_1^\infty$ yig‘indi shaklida tasvirlasak, u holda $y = \frac{1}{z}$ almashtirish bilan ikkinchi integral ham $[0, 1]$ oraliqqa keltiriladi:

$$\int_1^\infty \frac{y^{a-1}}{(1+y)^{a+b}} dy = \int_0^1 \frac{z^{b-1}}{(1+z)^{a+b}} dz,$$

$$1O$$

demak, natijada

$$B(a,b) = \int_0^1 \frac{x^{a-1} + x^{b-1}}{(1+x)^{a+b}} dx.$$

2° Bo‘laklab integrallash yordami bilan, (1) formuladan $b > 1$ da, quyidagini topamiz:

$$B(a,b) = \int_0^1 (1-x)^{b-1} d\frac{x^a}{a} =$$

$$= \frac{x^a(1-x)^{b-1}}{a} \Big|_0^1 + \frac{b-a}{a} \int_0^1 x^a (1-x)^{b-2} dx =$$

$$= \frac{b-1}{a} \int_0^1 x^{a-1} (1-x)^{b-2} dx - \frac{b-1}{a} \int_0^1 x^{a-1} (1-x)^{b-1} dx =$$

$$= \frac{b-1}{a} B(a, b-1) - \frac{b-1}{a} B(a,b),$$

bundan

$$B(a,b) = \frac{b-1}{a+b-1} B(a, b-1). \qquad (3)$$

$b > 1$ bo‘lganda, b ni kamaytirish maqsadida bu formulani qo‘llanish mumkin; shunday qilib, doim ikkinchi argumentning ≤ 1 bo‘lishiga erishish mumkin.

Ikkinchi argumentga nisbatan ham shunga erishish mumkin, chunki B simmetrik funksiya bo‘lganidan, yana ushbu

$$B(a,b) = \frac{a-1}{a+b-1} B(a-1,b) \qquad (a > 1)$$

keltirish formulasiga ega bo‘lamiz.

Agar b parametr n natural songa teng bo‘lsa, u holda (3) formulani ketma-ket qo‘llanish bilan

$$B(a,n) = \frac{n-1}{a+n-1} \cdot \frac{n-2}{a+n-2} \cdots \frac{1}{a+1} B(a,1)$$

formulaga kelamiz. Lekin

$$B(a,1) = \int_0^1 x^{a-1} dx = \frac{1}{a}.$$

Shu sababli $B(a,n)$ uchun, va bir paytda, $B(n,a)$ uchun ham:

$$B(n,a) = B(a,n) = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)}{a \cdot (a+1) \cdot (a+2) \cdot \ldots \cdot (a+n-1)} \qquad (4)$$

ifodani hosil qilamiz.

Agar a ham natural m songa teng bo‘lsa, ushbuni topamiz:

$$B(m,n) = \frac{(n-1)!\,(m-1)!}{(m+n-1)!}.$$

Agar 0! simvolni 1 deb tushunsak, bu formulani $m = 1$ yoki $n = 1$ bo‘lganda ham qo‘llanish mumkin.

3° (2) formulada $0 < a < 1$ hisoblab, $b = 1 - a$ faraz qilamiz; u holda:

$$B(a, 1-a) = \int_0^\infty \frac{y^{a-1}}{1+y} dy.$$

Uning qiymatini o‘rniga qo‘yib, ushbu formulaga kelamiz:

$$B(a, 1-a) = \frac{\pi}{sina\pi} \qquad (0 < a < 1). \qquad (5)$$

Agar, xususiy holda, $a = 1 - a = \frac{1}{2}$ desak,

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$$

hosil bo‘ladi.

1.2-§ Ikkinchi tur Eyler integrali.

Lejandr, ushbu ajoyib

$$\Gamma(a) = \int_0^{\infty} x^{a-1}\, e^{-x} dx \qquad (6)$$

integralni ikkinchi tur Eyler integrali deb atagan, bu integral istalgan $a > 0$ qiymatlarda yaqinlashadi va Γ („Gamma”) funksayani aniqlaydi. Elementar funksiyalardan keyin, Γ funksiya analiz va uning tatbiqi uchun muhim funksiyalardan biri hisoblanadi. Γ funksiyaning xossalarini (6) integral ta’rifiga asosan mufassal o‘rganish, bir paytda, parametrga bog‘liq integrallar nazariyasining tatbiqotiga ajoyib misol bo‘ladi.

(6) da $x = log\frac{1}{z}$ deb

$$\Gamma(a) = \int_0^1 (log\frac{1}{z})^{a-1}\, dz$$

ni topamiz.

Ma’lumki,

$$log\frac{1}{z} = \lim_{n\to\infty} n(1 - z^{\frac{1}{n}})$$

bunda n o‘sganda, $n(1 - z^{\frac{1}{n}})$ ifoda ham o‘sa borib, o‘z limitiga intiladi. Bu holda,

$$\Gamma(a) = \lim_{n\to\infty} n^{a-1} \int_0^1 (1 - z^{\frac{1}{n}}) a^{-1}\, dz$$

tenglik, yoki $z = y^n$ almashtirishdan foydalansak,

$$\Gamma(a) = \lim_{n\to\infty} n^{a} \int_0^1 y^{n-1}(1-y)^{a-1}\, dy$$

tenglik o‘rinlidir. Lekin, (4) ga asosan

$$\int_0^1 y^{n-1}(1-y)^{a-1}dy = B(n,a) = \frac{1\cdot 2\cdot 3\dots(n-1)}{a\cdot(a+1)\cdot(a+2)\dots(a+n-1)}.$$

Shunday qilib, xulosada, Eyler-Gaussning

$$\Gamma(a) = \lim_{n\to\infty} n^a \cdot \frac{1\cdot 2\cdot 3\dots(n-1)}{a\cdot(a+1)\cdot(a+2)\dots(a+n-1)} \qquad (7)$$

formulasiga kelamiz.

Biz B va Γ funksiyalarning yuqorida eslatib o‘tilgan bog‘lanishini tayinlashdan boshlaymiz. Shu maqsadda, $x = ty (t > 0)$ almashtirish bilan (6) ni

$$\frac{\Gamma(a)}{t^a} = \int_0^\infty y^{a-1}e^{-ty}dy \qquad (8)$$

shaklga keltiramiz. Bu yerda a ni $a+b$ bilan va, bir paytda t ni $1+t$ bilan almashtirib, quyidagini hosil qilamiz:

$$\frac{\Gamma(a+b)}{(1+t)^{a+b}} = \int_0^\infty y^{a+b-1}e^{-(1+t)y}dy$$

Endi bu tenglikning ikkala tomonini t^{a-1} ga ko‘paytiramiz va t bo‘yicha 0 dan ∞ gacha integrallaymiz:

$$\Gamma(a+b)\int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt = \int_0^\infty t^{a-1}dt \int_0^\infty y^{a+b-1}e^{-(1+t)y}dy.$$

Chap tomondagi integral $B(a,b)$ funksiyadir o‘ng tomonda esa, integrallarning o’rinlarini almashtiramiz. Natijada:

$$\Gamma(a+b)\cdot B(a,b) = \int_0^\infty y^{a+b-1}e^{-y}dy \int_0^\infty t^{a-1}e^{-ty}dt =$$

$$\int_0^\infty y^{a+b-1}e^{-y}\frac{\Gamma(a)}{y^a}dy = \Gamma(a)\int_0^\infty y^{b-1}e^{-y}dy = \Gamma(a)\cdot\Gamma(b)$$

hosil bo‘ladi, nihoyat, bundan

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a,b)} \qquad (9)$$

Eyler munosabatining bu ajoypb isbotini Dirixle bergan. Biroq, bu yo‘lni asoslash uchun, integrallarning o‘rinlarini almashtirishning qonuuniy ekanini isbotlash kerak. Ahvol shu bilan murakkablashadiki, y o‘zgaruvchi uchun $+\infty$ nuqtalardan tashqari, 0 nuqta ham ($a + b < 1\, da$) maxsus bo‘lishi mumkin; t o‘zgaruvchi uchun ham shunga o‘xshash, $+\infty$ nuqtalardan tashqari, 0 nuqta ham $a < 1$ da maxsus bo‘lishi mumkin. Lekin

$$\int_0^\infty dt \int_0^\infty t^{a-1} y^{a+b-1} e^{-(1+t)y} dy$$

ifodada integrallashning o‘rinlarini almashtirish masalasi (masalan, $a + b < 1$ qiymatda) ushbu

$$\int_0^1 dt \int_0^1 \dots dy, \int_0^1 dt \int_1^\infty \dots dy, \int_1^\infty dt \int_0^1 \dots dy, \int_1^\infty dt \int_1^\infty \dots dy$$

ifodalarga nisbatan integrallashning o‘rinlarini almashtirishga osongina keltiriladi.

Γ funksiyaning eng sodda hosilalari.

1° Γ(a) funksiya uzluksiz bo‘lib, $a > 0$ uchun uzluksiz $\Gamma'(a)$ hosilaga egadir. Ikkinchi tasdiqni isbotlash kifoya, integral belgisi ostida differensiallab, quyidagini hosil qilamiz:

$$\Gamma'(a) = \int_0^\infty x^{a-1} logx e^{-x} \qquad (10)$$

Bu integral a ga nisbatan tekis yaqinlashganidan $a \geq a_0 > 0$ uchun $x = 0$ qiymatda (majoranta $x^{a_0-1}|logx|$) va $a \leq A < +\infty$ uchun $x = \infty$ qiymatda (majoranta $x^{A}e^{-x}$), shuning uchun Leybnits formulasini qo‘llanish qonuniydir. Shunga asosan $\Gamma'(a)$ ning uzluksizligi haqida ham natija chiqaramiz.

Shu yo‘l bilan keyingi hosilalarning ham mavjudligiga ishonch hosil qilish mumkin.

2° Bo‘laklab integrallash bilan (6) dan ushbuni

$$a\int_0^\infty x^{a-1}e^{-x}dx = x^{a}e^{-x}\Big|_0^\infty + \int_0^\infty x^{a}e^{-x}dx$$

ya’ni quyidagini birdaniga topamiz:

$$\Gamma(a+1) = a\cdot\Gamma(a). \qquad (11)$$

Bu formulani qayta-qayta qo‘llanish ushbuni beradi:

$$\Gamma(a+n) = (a+n-1)(a+n-2)\dots(a+1)a\Gamma(a). \qquad (12)$$

Shu yo‘l bilan argumentning istalgancha katta qiymatlari uchun Γ ni hisoblash — argumen $<$ 1 bo‘lgan Γ ni hisoblashga keltiriladi.

Agar (12) da $a = 1$ desak va

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1 \qquad (13)$$

bo‘lishini e’tiborga olsak, u holda

$$\Gamma(n+1) = n! \qquad (14)$$

kelib chiqadi. Ba’zi bir ma’noda Γ funksiya n ning faqat natural qiymatlari uchun tayinlangan $n!$ argumentning istalgan musbat qiymatlari sohasi uchun umumlashtirishdan iboratdir.

(11) dan (va 1° dan) $a \to 0$ da

$$\Gamma(a) = \frac{\Gamma(a+1)}{a} \to +\infty$$

ekanligi ravshan. Γ funksiyaning cheksiz o‘sish tartibini ham tayinlash mumkin:

$$\lim_{a\to+0} \frac{\Gamma(a)}{1/a} = \lim_{a\to+0} \Gamma(a+1) = \Gamma(1) = 1. \qquad (15)$$

3° To‘ldirish formulasi.

Agar (9) formulada ($0 < a < 1$ hisoblab) $b = 1 - a$ desak, u holda, (5) va (13) ga asosan,

$$\Gamma(a) \cdot \Gamma(1-a) = \frac{\pi}{sina\pi} \qquad (16)$$

munosabatga ega bo‘lamiz, bu to‘ldirish formulasi deyiladi.

Bundan $a = \frac{1}{2}$ bo‘lganda ushbuni topamiz:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

(chunki $\Gamma(a) > 0$).

Agar

$$\int_0^\infty \frac{e^{-z}}{\sqrt{z}} dz = \sqrt{\pi}$$

integralda $z = x^2$ almashtirishni bajarsak, u holda yana Eyler-Puasson integralining qiymatini hosil qilamiz:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

4° To‘ldirish formulasini qo‘llanish sifatida,

$$E = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-2}{n}\right)\Gamma\left(\frac{n-1}{n}\right)$$

ko'paytmaning qiymatini aniqlaymiz (bu yerda n -istalgan natural son). Bu ko'paytmani teskari tartibda yozib, ya'ni:

$$E = \Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{n-2}{n}\right)\ldots\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right)$$

ikkala ifodani bir-biriga ko'paytiramiz:

$$E^2 = \prod_{v=1}^{n-1}\Gamma\left(\frac{v}{n}\right)\Gamma\left(\frac{n-v}{n}\right)$$

va ko'paytuvchilarning har bir $\Gamma\left(\frac{v}{n}\right)\Gamma\left(\frac{n-v}{n}\right)$ juftiga to'ldirish formulasini qo'llanamiz. Quyidagini hosil qilamiz:

$$E^2 = \frac{\pi^{n-1}}{sin\frac{\pi}{2}\cdot sin2\frac{\pi}{2}\ldots\sin(n-1)\frac{\pi}{2}}.$$

Endi sinuslarning ko'paytmasini hisoblash uchun

$$\frac{z^n-1}{z-1} = \prod_{v=1}^{n-1}\left(z - cos\frac{2v\pi}{n} - isin\frac{2v\pi}{n}\right)$$

ayniyatni tekshiramiz va unda z ni 1 ga intiltiramiz. Limitda ushbuni

$$n = \prod_{v=1}^{n-1}\left(1 - cos\frac{2v\pi}{n} - isin\frac{2v\pi}{n}\right)$$

yoki modullarini tenglashtirib, quyidagini hosil qilamiz:

$$n = \prod_{v=1}^{n-1}\left|1 - cos\frac{2v\pi}{n} - isin\frac{2v\pi}{n}\right| = 2^{n-1}\prod_{v=1}^{n-1}sin\frac{v\pi}{n},$$

demak,

$$\prod_{v=1}^{n-1} sin\frac{v\pi}{n} = \frac{n}{2^{n-1}}.$$

Buni E^2 ning ifodasiga qo‘yib, oxirgi natijani hosil qilamiz:

$$E = \prod_{v=1}^{n-1} \Gamma\left(\frac{v}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} \qquad (18)$$

5° Raabe integrali. Quyidagi muhim va mavjudligi ravshan bo‘lgan integralni hisoblash ham to‘ldirish formulasi bilan bog‘langan:

$$R_0 = \int_0^1 log\Gamma(a)da.$$

Bunda a ni $1 - a$ bilan almashtirib, bunday yozish:

$$R_0 = \int_0^1 log\Gamma(1-a)da$$

va ikkalasini qo‘shib, ushbuni hosil qilish mumkin:

$$2R_0 = \int_0^1 log\Gamma(a)\Gamma(1-a)da = \int_0^1 log\frac{\pi}{sina\pi}da =$$

$$= log\pi - \frac{1}{\pi}\int_0^{\pi} logsinxdx = log\pi - \frac{2}{\pi}\int_0^{\frac{2}{\pi}} log\pi sinxdx.$$

Bunga integralning qiymatini qo‘ysak,

$$R_0 = \int_0^1 log\Gamma(a)da = log\sqrt{2\pi} \qquad (19)$$

topiladi.

Raabe ($a > 0$ da)

$$R_0 = \int_0^{a+1} log\Gamma(a)da = \int_0^{a+1} - \int_0^{a}.$$

integralni tekshirgan. Ravshanki,

$$R'(a) = log\Gamma(a+1) - log\Gamma(a) = loga$$

bo‘lganidan, $a > 0$ uchun integrallab, mana buni topamiz:

$$R(a) = a\,(loga - 1) + C.$$

Lekin $a = 0$ bo‘lganda ham $R(a)$ uzluksizligini saqlaydi; bu yerda $a \to 0$ da limitga o‘tib, $C = R_0$ ni hosil qilamiz. Bu yerga (19) ning qiymatini qo‘yib, Raabe formulasiga kelamiz:

$$R(a) = \int_a^{a+1} log\,\Gamma(a)da = a(loga - 1) + log\sqrt{2\pi}. \qquad (20)$$

6° Lejandr formulasi. Agar

$$B(a,a) = \int_0^1 x^{a-1}(1-x)^{a-1}\,dx = \int_0^1 \left[\frac{1}{4} - \left(\frac{1}{2} - x\right)^2\right]^{a-1} dx =$$

$$= 2\int_0^{\frac{1}{2}} \left[\frac{1}{4} - \left(\frac{1}{2} - x\right)^2\right]^{a-1} dx$$

integralda $\frac{1}{2} - x = \frac{1}{2}\sqrt{t}$ almashtirishni bajarsak, u holda

$$B(a,a)=\frac{1}{2^{2a-1}}\int_0^1 t^{-\frac{1}{2}}(1-t)^{a-1}dt=\frac{1}{2^{2a-1}}B\left(\frac{1}{2},a\right)$$

hosil bo‘ladi.

Ikkala holda B funksiyani, uning Γ orqali berilgan (9) ifodasi bilan almashtiramiz:

$$\frac{\Gamma(a)\Gamma(a)}{\Gamma(2a)}=\frac{1}{2^{2a-1}}\cdot\frac{\Gamma(\frac{1}{2})\Gamma(a)}{\Gamma(a+\frac{1}{2})}$$

Endi $\Gamma(a)$ ga qisqartib va $\Gamma(\frac{1}{2})$ o‘rniga uning $\sqrt{\pi}$ qiymatini qo‘yib, Lejandr formulasini hosil qilamiz:

$$\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2a-1}}\cdot\Gamma(2a). \qquad (21)$$

Xossalari yordami bilan *Γ* funksiyani bir qiymatli aniqlash.

Argumentning musbat qiymatlari uchun Γ funksiya o‘zining hosilasi bilan birga uzluksiz ekanini bilamiz. Undan tashqari, Γ funksiya quyidagi munosabatlarni qanoatlantiradi:

$$(I)\ \Phi(a+1)=a\Phi(a), \quad (II)\ \Phi(a)\Phi\left(a+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{a-1}}\Phi(2a),$$

$$(III)\ \Phi(a)\Phi(a-1)=\frac{\pi}{sina\pi}.$$

Bu xossalar Γ funksiyani to’la ravishda xarakterlashini ko‘rsatamiz (demak bu xossalarga ega bo‘lgan har bir funksiya Γ ga aynan tengdir).

(I) va (II) xossalarning o‘zi buning uchun yetarli emas, chunki, Γ bilan birga, bu xossalarga

$$\Phi(a)=\Gamma(a)[4sin^2 a\pi]^{\mu} \quad (\mu>0\ da)$$

funksiya ham egadir.

Xuddi shuningdek, (II) va (III) xossalarning o‘zi ham yetarli emas, chunki ularga

$$\Phi(a) = \Gamma(a) \cdot z^{a-\frac{1}{2}} \qquad (z > 0 \ da)$$

funksiya ham egadir.

(I) va (III) xossalar $0 < a < \frac{1}{2}$ uchun $\Phi(a)$ funksiyaning qiymatlarini ixtiyoriy qilib qoldiradi. Uchala xossa birdaniga mavjud bo‘lsa, ish boshqachadir. Biroq, (III) xossani kuchsizroq talab bilan, ya’ni $a > 0$ qiymatlarda $\Phi(a)$ funksiya 0 ga teng bo‘lmasin degan talab bilan almashtirish mumkin, bu talab esa xuddi (III) xossadan kelib chiqadi.

Shunday qilib, $a > 0$ uchun $\Phi(a)$ funksiya o‘zining hosilasi bilan birga uzluksiz, noldan farqli va (I), (II) munosabatlarni qanoatlantiradigan bo‘lsin. U vaqtda $\Phi(a) \equiv \Gamma(a)$ ekanini isbotlaymiz.

$\Phi(a) = M(a) \cdot \Gamma(a)$ deb faraz qilamiz; ravshanki, $M(a)$ funksiya ham o‘zining hosilasi bilan birga uzluksiz va noldan farqlidir. Undan tashqari, $\Phi(a)$ va $\Gamma(a)$ funksiyalar (I) va (II) shartlarni qanoatlantirgani uchun $M(a)$ ham quyidagi munosabatlarni qanoatlantiradi:

$$(I') \quad M(a+1) = M(a) \text{ va } (II') \quad M(a)\, M\left(a + \frac{1}{2}\right) = M(2a)$$

(I') dan, $a \to +0$ da $M(a)$ uchun chekli limitning mavjudligi ochiq ko‘rinib turadi. Agar u limit qiymatni $M(0)$ qiymat deb qabul qilsak, u holda $M(a)$ funksiya o‘zining hosilasi bilan birga, hatto $a = 0$ ga qadar qiymatlarda uzluksizdir.

(II’) dan $a = \frac{1}{2}$ qiymatda $M(a) = 1$ kelib chiqadi, demak, hamma $a > 0$ uchun $M(a) > 0$. Bu esa bizga

$$L(a) = logM(a)$$

funksiyani qarashga huquq beradi; $a > O$ da bu funksiya ham o‘zining hosilasi bilan birga uzluksiz bo‘lib, lekin

$$(I'')\ L(a + 1) = L(a) \text{ va } (II'')\ L(a) + L\left(a + \frac{1}{2}\right) = L(2a)$$

shartlarni qanoatlantiradi.

Nihoyat, yana ushbu

$$\Delta(a) = L'(a)$$

uzluksiz funksiyani kiritamiz; bu funksiya

$$(I''')\ \Delta(a + 1) = \Delta(a) \text{ va } (II''')\ \Delta(a) + \Delta\left(a + \frac{1}{2}\right) = 2\Delta(2a)$$

munosabatlarni qanoatlantiradi.

(II''') dan, a nib $\frac{a}{2}$ bilan almashtirib, ushbuni hosil qilamiz:

$$\frac{1}{2}\left\{\Delta\left(\frac{a}{2}\right) + \Delta\left(\frac{a+1}{2}\right)\right\} = \Delta(a).$$

Agar bu yerda yana a ni oldin $\frac{a}{2}$ bilan, sungra $\frac{a+1}{2}$ bilan almashtirsak va topilgan tengliklarni qo‘shsak,

$$\frac{1}{4}\left\{\Delta\left(\frac{a}{4}\right) + \Delta\left(\frac{a+1}{4}\right) + \Delta\left(\frac{a+2}{4}\right) + \Delta\left(\frac{a+3}{4}\right)\right\} = \Delta(a)$$

hosil bo‘ladi. Matematik induksiya metodi bilan ushbu

$$\frac{1}{2^n}\sum_{v=0}^{n-1}\Delta\left(\frac{a+v}{2^n}\right) = \Delta(a)$$

umumiy munosabatni tayinlash yengil.

Lekin a qanday bo‘lsa ham, chap tomondagi yig‘indini

$$\int_0^1 \Delta(x)dx$$

integral uchun integral yig‘indi deb qarash mumkin. Shu sababli

$$\Delta(a) = \lim_{n\to\infty} \frac{1}{2^n} \sum_0^{2^{n-1}} \Delta\left(\frac{a+v}{2^n}\right) = \int_0^1 \Delta(x)dx = L(I) - L(0) = 0$$

[(I") ga asosan]. Bunday holda $L(a) = const$ demak, $M(a) = const$, lekin biz $M\left(\frac{1}{2}\right) = 1$ ekanini ko‘rdik; demak, $M(a) \equiv 1$ va $\Phi(a) = \Gamma(a)$. Shuning isboti talab etilgan edi.

Xulosada, yana shuni ta’kidlab o‘tamizki, differensiallanish talabi muhim ro’lni o‘ynaydi va u talabni chiqarib tashlash mumkin emas. Masalan,

$$L(a) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sin(2^n \pi a)$$

desak, u holda $L(a)$ funksiya (I") va (II") shartlarni qanoatlantiruvchi uzluksiz funksiya bo‘ladi. Biroq, $L(0) = 0$ va $L\left(\frac{1}{4}\right) = \frac{1}{2}$, demak, $L(a)$ — o‘zgarmas sondan iborat emas!

1.3-§ Γ funksiyaning logarifmik hosilasi.

Gamma funksiyaning xossalarini o‘rganishni davom ettirib, uning logarifmik hosilasini, ya’ni

$$\frac{d\log\Gamma(a)}{da} = \frac{\Gamma'(a)}{\Gamma(a)}$$

ifodani tekshiraylik.

7° Bu ifodaning integral shaklidagi turli tasvirlarini (10) formuladan hosil qilish mumkin. Lekin quyidagi mulohazadan foydalanish soddaroq. Haqiqatan,

$$\Gamma(b) - B(a,b) = \Gamma(b) - \frac{\Gamma(a)\Gamma(b)}{\Gamma(a,b)} = \frac{\Gamma(b)b}{\Gamma(a+b)} \cdot \frac{\Gamma(a+b) - \Gamma(a)}{b} =$$

$$= \frac{\Gamma(b+1)}{\Gamma(a+b)} \cdot \frac{\Gamma(a+b) - \Gamma(a)}{b}$$

deylik, agar bu yerda $b \to 0$ da limitga o‘tsak,

$$\frac{\Gamma'(a)}{\Gamma(a)} = \lim_{b\to 0}[\Gamma(b) - B(a,b)].$$

Avvalo quyidagini olamiz [(6) va (2) ga qarang]:

$$\Gamma(b) = \int_0^\infty x^{b-1}e^{-x}\,dx, \quad B(a,b) = \int_0^\infty \frac{x^{b-1}}{(1+x)^{a+b}}\,dx.$$

U vaqtda

$$\frac{\Gamma'(a)}{\Gamma(a)} = \lim_{b\to +0}\int_0^\infty x^{b-1}\left[e^{-x} - \frac{1}{(1+x)^{a+b}}\right]dx.$$

bo‘lib, integral ostida limitga o‘tishni bajarsak, mana bu *Koshi formulasi* topiladi:

$$\frac{\Gamma'(a)}{\Gamma(a)} = \int_0^\infty \left[e^{-x} - \frac{1}{(1+x)^{a+b}}\right]\frac{dx}{x}. \qquad (22)$$

Limitga o‘tishni qonunlashtirish uchun, $x = 0,\ b = 0$ yaqinida

$$\frac{1}{x}\left[e^{-x} - \frac{1}{(1+x)^{a+b}}\right]$$

ifoda x va b ning uzluksiz funksiyasi ekanini va $x^b < 1$ bo‘lishini e’tiborga olamiz. Yetarli katta x na $b \le b_0$ uchun

$$x^{b_0-1}\left[\frac{1}{(1+x)^a} - e^{-x}\right]$$

majoranta bor.

Agar B ning (1) ifodasida avval $x = e^{-t}$ almashtirishni bajarsak,

$$B(a,b) = \int_0^\infty e^{-at}(1-e^{-t})^{b-1}\,dt$$

hosil bo‘lib, u holda quyidagini yozish mumkin:

$$\frac{\Gamma'(a)}{\Gamma(a)} = \lim_{b\to+0}\int_0^\infty [e^{-x}\cdot x^{b-1} - e^{-ax}(1-e^{-x})^{b-1}]\,dx.$$

Bu yerda integral belgisi ostida limitga o‘tsak, boshqa

$$\frac{\Gamma'(a)}{\Gamma(a)} = \int_0^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-ax}}{1-e^{-x}}\right)dx$$

formulaga kelamiz.

Aksincha, integral ostidagi fung'siyadan ko'rsatgichli ifodalarni butunlay chiqarib yuborish mumkin. Shu maqsad bilan (22) da $a = 1$ deymiz:

$$\frac{\Gamma'(1)}{\Gamma(1)} = \Gamma'(1) = \int_0^\infty \left[e^{-x} - \frac{1}{1+x}\right]\frac{dx}{x} = -C,$$

bu yerda C *Eyler o'zgarmasi* deb ataladi. Bu tenglikni (22) dan hadlab ayirsak, mana buni hosil qilamiz:

$$\frac{\Gamma'(a)}{\Gamma(a)} + C = \int_0^\infty \left[\frac{1}{1+x} - \frac{1}{(1+x)^a}\right]\frac{dx}{x}.$$

Nihoyat, $t = \frac{1}{1+x}$ almashtirish bizni *Gauss formulasiga* keltiradi:

$$\frac{\Gamma'(a)}{\Gamma(a)} + C = \int_0^1 \frac{1-t^{a-1}}{1-t}dt. \qquad (24)$$

8° Endi biz a parametr 0 dan $+\infty$ gacha o'sganda, $\Gamma(a)$ funksiyaning qanday o'zgarishi haqida o'zimizga hisobot bera olamiz. U chala formula shuni ko'rsatadiki, ifoda bu vaqtda $-\infty$ dan $+\infty$ gacha o'sadi. Haqiqatan,

$$\int_0^1 \frac{1-t^{a-1}}{1-t}dt < \int_\varepsilon^1 \frac{1-t^{a-1}}{1-t}dt \qquad (\varepsilon > 0)$$

tengsizlikning ($a < 1$) o'ng tomoni $a \to 0$ da

$$-\int_\varepsilon^1 \frac{dt}{t} = log\varepsilon$$

limitga ega bo'lib, bu tengsizlikdan shuni ko'ramizki, masalan, (24) integral $a \to 0$ da $-\infty$ ga intiladi. Shunga o'xshash, $a \to +\infty$ da bu integralning $+\infty$ ga intilishiga ishonch hosil qilish yengil.

$\frac{\Gamma'(a)}{\Gamma(a)}$ uzluksiz funksiya ishorasini plyusdan minusga o‘zgarishda, faqat bir marta nolga teng bo‘ladi. Shuningdek ($\Gamma(a) > 0$ bo‘lganidan), $\Gamma'(a)$ hosilaga nisbatan ham shu xulosaga kelish mumkin. Shunday qilib, agar hosilaning ildizini a_0 bilan belgilasak, $0 < a < a_0$ bo‘lganda, Γ funksiya $+\infty$ dan musbat $\Gamma(a_0)$ minimumgacha kamayadi, so‘ngra a parametr $+\infty$ gacha o‘sganda, bu funksiya ham $+\infty$ gacha o‘sadi.

$\Gamma(2) = \Gamma(1) = 1$ bo‘lganidan, *Roll* teoremasiga muvofiq, hosilaning a_0 ildizi 1 bilan 2 orasida yotishi kerak. Aniqroq hisoblash ushbuni beradi:

$$a_0 = 1{,}4616321\ldots., \qquad min\Gamma = \Gamma(a_0) = 0{,}8856032\ldots$$

1-chizmada $\Gamma(a)$ funksiyaning grafigi tasvirlangan (bu yerda bizni $a > 0$ ga javob beruvchi qismigina qiziqtiradi),

1-chizma

Γ funksiya uchun ko‘paytirish teoremasi. 9° Logarifmik hosilaning (24) tasviriga suyanib, Gauss ga qarashli quyidagi ajoyib formulani tayinlaymiz:

$$\Gamma(a)\Gamma\left(a+\frac{1}{n}\right)\dots\Gamma\left(a+\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{na-\frac{1}{2}}}\Gamma(na) \qquad (25)$$

($n -$ istalgan natural son). Bu formula Γ funksiya uchun ko‘paytirish teoremasini ifodalaydi.

(24) da $t = u^n$ deb, ushbuni hosil qilamiz:

$$\frac{\Gamma'(a)}{\Gamma(a)}+C=n\int_0^1\frac{u^{n-1}-u^{na-1}}{1-u^n}du,$$

bundan a ni $a+\frac{v}{n}$ $(v=0,1,\dots,n-1)$ bilan almashtirib, quyidagini topamiz:

$$\frac{\Gamma'\left(a+\frac{v}{n}\right)}{\Gamma\left(a+\frac{v}{n}\right)}+C=n\int_0^1\frac{u^{n-1}-u^{na+v-1}}{1-u^n}du,$$

va v bo‘yicha 0 dan $n-1$ gacha yig‘ib, ushbuga kelamiz:

$$\sum_{v=0}^{n-1}\frac{\Gamma'\left(a+\frac{v}{n}\right)}{\Gamma\left(a+\frac{v}{n}\right)}+nC=n\int_0^1\left[\frac{nu^{n-1}}{1-u^n}-\frac{u^{na-1}}{1-u}\right]du,$$

Bu tenglikni ushbu

$$\frac{\Gamma'(na)}{\Gamma(na)}+C=\int_0^1\frac{1-u^{na-1}}{1-u}du$$

tenglik bilan solishtirib ko‘ramiz. Oxirgi tenglikni n ga ko‘paytirib va uni oldingisidan ayiramiz:

$$\sum_{v=0}^{n-1}\frac{\Gamma'\left(a+\frac{v}{n}\right)}{\Gamma\left(a+\frac{v}{n}\right)}-n\frac{\Gamma'(na)}{\Gamma(na)}=n\int_0^1\left[\frac{nu^{n-1}}{1-u^n}-\frac{1}{1-u}\right]du=$$

$$-nlog\frac{1-u^n}{1-u}\Big|_0^1=-nlogn,$$

demak, quyidagicha yozish mumkin:

$$\frac{d}{da}log\frac{\Gamma(a)\Gamma\left(a+\frac{1}{n}\right)\ldots\Gamma\left(a+\frac{n-1}{n}\right)}{\Gamma(na)}=-nlogn.$$

Bundan, integrallab, quyidagini

$$log\frac{\Gamma(a)\Gamma\left(a+\frac{1}{n}\right)\ldots\Gamma\left(a+\frac{n-1}{n}\right)}{\Gamma(na)}=-anlogn+logC.$$

yoki mana buni hosil qilamiz:

$$\frac{\Gamma(a)\Gamma\left(a+\frac{1}{n}\right)\ldots\Gamma\left(a+\frac{n-1}{n}\right)}{\Gamma(na)}=\frac{C}{n^{na}}.$$

Endi C o‘zgarmasni topish uchun, $a=\frac{\Gamma}{n}$ deymiz. Ravshanki, $C=nE$, bunda E ni 493, 4° da hisoblangan Eyler ko‘paytmasi deb tushunamiz. Uning qiymatini (18) dan olib qo‘ysak, (25) formulaga kelamiz.

Gauss formulasining xususiy holi − oldin mustaqil ravishda chiqarilgan (21) Lejandr formulasidan iboratdir. Haqiqatan, agar (25) da $n=2$ desak,

$$\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)=\frac{\sqrt{2\pi}}{2^{2a-\frac{1}{2}}}\Gamma(2a)$$

formula hosil bo‘ladi, bu (23) formula bilan teng kuchlidir.

Qatorlarga va ko‘paytmalarga ba’zi yoyilmalar. 10° Bu yoyilmalarning manbayi o‘sha (24) formuladir. Integral ostidagi ifodani qatorga yoyaylik:

$$\frac{1-t^{a-1}}{1-t} = (1-t^{a-1})\sum_{v=0}^{\infty} t^v = \sum_{v=0}^{\infty}(t^v - t^{a+v-1}),$$

buning hamma hadlari bir xil ishoraga ega. Hadlab integrallash quyidagini beradi:

$$D log\Gamma(a) + \mathrm{C} = \sum_{м=0}^{\infty}\left(\frac{1}{v+1} - \frac{1}{a+v}\right). \qquad (26)$$

Bu qator $0 < a < a_0$ uchun tekis yaqinlashadi, chunki u

$(a_0 + 1)\sum_1^{\infty}\frac{1}{v^2}$ qator bilan majorlanadi.

Agar qatorni a bo‘yicha hadlab differensiallasak, soddalik jihatdan ajoyib yoyilmani hosil qilamiz:

$$D^2 log\Gamma(a) = \sum_{v=0}^{\infty}\frac{1}{(a+v)^2}. \qquad (27)$$

Bu qator ham $a > 0$ uchun ($\sum_1^{\infty}\frac{1}{v^2}$ qator bilan majorlanib) tekis yaqinlashganidan, hadlab differensiallash qonuniydir.

11° (26) qatorni a bo‘yicha 1 dan $a > 0$ gacha hadlab integrallasak (buni bajarish qonuniydir, chunki qator tekis yaqinlashadi),

$$log\Gamma(a) + C(a-1) = \sum_{v=0}^{\infty}\left(\frac{a-1}{v+1} - log\frac{a+v}{v+1}\right) \qquad (28)$$

hosil bo‘ladi. Bu yerda a ni $a+1$ bilan ($a > -1$ bo‘lganda) almashtirib, yoyilmani quyidagi shaklda yozamiz:

$$log\Gamma(a+1)+Ca=\sum_{n=1}^{\infty}\left(\frac{a}{n}-log\frac{a+n}{n}\right)$$

yoki

$$log\frac{1}{\Gamma(a+1)}=Ca+\sum_{n=1}^{\infty}\left[log\left(1+\frac{a}{n}\right)-\frac{a}{n}\right].$$

Bundan, potensirlab, $\frac{1}{\Gamma(a+1)}$ ning cheksiz ko‘paytmaga yoyilmasini beruvchi *Veyyershtrass* formulasini hosil qilamiz:

$$\frac{1}{\Gamma(a+1)}=e^{Ca}\prod_{n=1}^{\infty}\left(1+\frac{a}{n}\right)e^{-\frac{a}{n}}\qquad(a>-1).\qquad(29)$$

12° (28) ga qaytib, $a=2$ deylik. Bu yerda $log\Gamma(2)=log1=0$ bo‘lganidan:

$$C=\sum_{v=0}^{\infty}\left(\frac{1}{v+1}-log\frac{v+2}{v+1}\right).\qquad(30)$$

Bundan

$$C=\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k}-\log(n+1)\right]$$

ekanligini yo‘lakay sezib, bizga tanish bo‘lgan Eyler o‘zgarmasining ta’rifiga kelamiz.

Nihoyat, (30) ni $a-1$ ga ko‘paytirib, (28) dan hadlab ayirib, C ni chiqarib tashlaymiz:

$$log\Gamma(a)=\sum_{v=0}^{\infty}\left[(a-1)log\frac{v+2}{v+1}-log\frac{a+v}{v+1}\right]=$$

$$= \lim_{n\to\infty} log\left[n^a \frac{1\cdot 2\cdot \ldots \cdot (n-1)}{a(a+1)(a+n-1)}\right].$$

yoki

$$log\Gamma(a) = \lim_{n\to\infty} log\left[n^a \frac{1\cdot 2\cdot \ldots \cdot (n-1)}{a(a+1)(a+n-1)}\right].$$

Bundan, potensirlash natijasida, yuqorida boshqa yo‘l bilan tayinlangan, (7) Eyler-Gauss formulasini qaytadan hosil qilamiz.

II bob. Eyler integrallarining tadbiqlari

2.1-§. Ba’zi aniq integrallarni hisoblash.

Hisoblashlari $\Gamma(a)$ funksiya va uning xossalariga asoslanib bajariladigan ba’zi integrallarni tekshiramiz.

1) Quyidagi

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx$$

formulani a bo‘yicha differensiallab, quyidagiga ega bo‘lamiz:

$$\Gamma'(a) = \int_1^{\infty} x^{a-1} e^{-x} logxdx$$

Bu yerda $a = 1$ deb, $\Gamma'(1) = C$ bo‘lganidan, ushbuni hosil qilamiz:

$$\int_0^{\infty} e^{-x} logxdx = -C.$$

Bunda $x = -logu$ almashtirish bizni

$$\int_0^{\infty} \log(-logu)\, du = -C$$

qiziq integralga olib keladi.

Agar $a = \frac{1}{2}$ ni olib, va $x = t^2$ desak, u holda mana buni topamiz:

$$\int_0^{\infty} e^{-t^2} logtdt = \frac{1}{4}\Gamma'\left(\frac{1}{2}\right) = -\frac{\sqrt{\pi}}{4}(C + 2log2),$$

agar logarifmik qatorni e’tiborga olsak, bu natija (26) yoyilmadan topiladi.

a, bo‘yicha differensiallashni takrorlab,

$$\Gamma''(a) = \int_1^\infty x^{a-1} e^{-x} log^2 x dx$$

tenglikka kelamiz.

$a = 1$ bo‘lganda, u tenglik ushbuni beradi:

$$\int_1^\infty e^{-x} log^2 x dx = \Gamma''(1) = C^2 + \frac{\pi^2}{6}.$$

Agar ma’lum

$$\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$$

qatordan foydalansak, oxirgi natija (27) dan topiladi.

Bu yerda ham $a = \frac{1}{2}$ desak, $x = t^2$ almashtirish yordami bilan yana bunday integralni topamiz:

$$\int_0^\infty e^{-t^2} log^2 t dt = \frac{\sqrt{\pi}}{8}\left[(C + 2log2)^2 + \frac{\pi^2}{2}\right]$$

va h. k.

2)

$$J = \int_0^\infty \frac{sin^p x}{x} dx$$

integral hisoblansin; bu yerda $p -$ toq suratli va toq maxrajli ratsional kasrdir.

Ko‘rsatma. 455, 3° dagi $\int_0^\infty \frac{sinx}{x}$ integralni hisoblashda ishlatilgan metod bilan quyidagini topamiz:

$$J=\int_0^{\frac{\pi}{2}} sin^p t\left\{\frac{1}{t}+\sum_{v=0}^{\infty}(-1)^v\left[\frac{1}{t-v\pi}+\frac{1}{t+v\pi}\right]\right\}dt=\int_0^{\frac{\pi}{2}} sin^{p-1}tdt$$

Javob. $J=\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)}=2^{p-2}\frac{\left[\Gamma\left(\frac{p}{2}\right)\right]^2}{\Gamma(p)}$.

3) Quyidagi integrallar hisoblansin ($b>0$):

$$A=\int_0^{\infty}\frac{cosbx}{x^s}dx, \qquad B=\int_0^{\infty}\frac{sinbx}{x^s}dx.$$

Ma’lumki [(8) ga qarang]:

$$\frac{1}{x^s}=\frac{1}{\Gamma(s)}\int_0^{\infty} z^{s-1}e^{-zs}\,dz,$$

demak,

$$A=\frac{1}{\Gamma(s)}\int_0^{\infty} cosbxdx\int_0^{\infty} z^{s-1}e^{-zs}dz$$

integrallashlarni almashtirib, ushbuni hosil qilamiz:

$$A=\frac{1}{\Gamma(s)}\int_0^{\infty} z^{s-1}dz\int_0^{\infty} e^{-zs}cosbxdx=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{z^s dz}{z^2+b^2}$$

yoki $b^2t=z^2$ desak,

$$A = \frac{b^{s-1}}{2\Gamma(s)} \int_0^{\infty} \frac{t^{\frac{s-1}{2}}}{1+t} dt = \frac{b^{s-1}}{2\Gamma(s)} B\left(\frac{s+1}{2}, \frac{1-s}{2}\right) =$$

$$\frac{b^{s-1}}{2\Gamma(s)} \frac{\pi}{sin\frac{s+1}{2}\pi} = \frac{\pi b^{s-1}}{2\Gamma(s)cos\frac{s\pi}{2}}$$

[(2), (5) ga qarang]. Shunga o‘xshash,

$$B = \frac{\pi b^{s-1}}{2\Gamma(s)sin\frac{s\pi}{2}}$$

Integrallarni almashtirishning asoslanishi $\int_0^{\infty} \frac{sinx}{x} dx$ integralni hisoblashdagidek bajariladi.

4) Ushbu integrallar hisoblansin:

$$\int_0^{\infty} \frac{sinx}{x} logxdx, \quad \int_0^{\infty} \frac{sinx}{x} log^2 xdx$$

3) ga asosan, integral $(0 < s < 2)$

$$J = \int_0^{\infty} \frac{sinx}{x^s} dx = \frac{\pi}{2\Gamma(s)sin\frac{s\pi}{2}}$$

Uni *s* parametr bo‘yicha differensiallab (Lyeybnis qoidasidan foydalanish bilan), mana buni topamiz;

$$\int_0^{\infty} \frac{sinx}{x^s} logxdx =$$

$$= \frac{\pi}{2} \cdot \frac{1}{\left[\Gamma(s) \cdot sin\frac{s\pi}{2}\right]^2} \left\{\Gamma'(s) \cdot sin\frac{s\pi}{2} + \frac{\pi}{2}\Gamma(s)cos\frac{s\pi}{2}\right\}.$$

Leybnis qoidasining qo'llanishiga asos, hosil qilingan integralning s ga nisbatan $x = \infty$ da ($s \geq s_0 > 0$ uchun,), hamda $x = 0$ da ($s \leq s_1$ uchun majoranta $|logx|: x^{s_1 - 1}$) tekis yaqinlashishi bilan qonunlashadi.

Topilgan tenglikni yana bir marta differensiallab (bu yuqoridagidek asoslanadi), quyidagini topamiz:

$$\int_0^\infty \frac{sinx}{x^s} log^2 x dx =$$

$$= \frac{\pi}{\left[\Gamma(s) \cdot sin\frac{s\pi}{2}\right]^3} \left\{\Gamma'(s) \cdot sin\frac{s\pi}{2} + \frac{\pi}{2}\Gamma(s) cos\frac{s\pi}{2}\right\}^2 -$$

$$-\frac{\pi}{2} \cdot \frac{1}{\left[\Gamma(s) \cdot sin\frac{s\pi}{2}\right]^2} \left\{\Gamma''(s) \cdot sin\frac{s\pi}{2} + \pi\frac{\pi}{2}\Gamma'(s) cos\frac{s\pi}{2} - \frac{\pi^2}{4}\Gamma(s) \cdot sin\frac{s\pi}{2}\right\}.$$

Ikkala tenglikda $s = 1$ deb, izlangan integrallarning qiymatlarini topamiz:

$$\int_0^\infty \frac{sinx}{x^s} logx dx = \frac{\pi}{2}\Gamma'(1),$$

$$\int_0^\infty \frac{sinx}{x^s} log^2 x dx = \pi|\Gamma'(1)|^2 - \frac{\pi}{2} \cdot \Gamma''(1) + \frac{\pi^3}{8}.$$

Endi

$$\Gamma'(1) = -C, \qquad \Gamma''(1) = C^2 + \frac{\pi^2}{6}$$

ekanini e'tiborga olib xulosada mana buni hosil qilamiz [1) bilan solishtiring]:

$$\int_0^\infty \frac{sinx}{x^s} logx dx = -\frac{\pi}{2} \cdot C, \quad \int_0^\infty \frac{sinx}{x^s} log^2 x dx = \frac{\pi}{2} \cdot C^2 + \frac{\pi^3}{24}.$$

5) Biz

$$\int_0^{\frac{\pi}{2}} sin^{2a-1}\varphi d\varphi = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(a)}{\Gamma\left(a+\frac{1}{2}\right)} \quad (a > 0)$$

formulaga ega edik.

Buni a bo‘yicha differensiallab [Lyeybnis qoidasini qo‘llanib], quyidagini topamiz:

$$\int_0^{\frac{\pi}{2}} sin^{2a-1}\varphi logsin\varphi d\varphi =$$

$$= \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(a)}{\Gamma\left(a+\frac{1}{2}\right)} \left[\frac{dlog\Gamma(a)}{da} - \frac{dlog\Gamma\left(a+\frac{1}{2}\right)}{da}\right].$$

Agar (24) Gauss formulasidan foydalanilsa, u holda qavslar ichidagi ifoda $\int_0^1 \frac{t^{a-\frac{1}{2}}-t^{a-1}}{1-t} dt$ shaklida yoziladi. Endi $2a - 1 = 2n$ deymiz (bu yerda $n -$ istalgan natural son yoki nol) va $t = u^2$ almashtirishni bajaramiz. U holda

$$\int_0^{\frac{\pi}{2}} sin^{2n}\varphi logsin\varphi d\varphi = -\frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \int_0^1 \frac{u^{2n}}{1+u} du$$

hosil bo‘ladi.

Bu formula $n = 0$ da bizga ma’lum bo‘lgan

$$\int_0^{\frac{\pi}{2}} logsin\varphi d\varphi = -\frac{\pi}{2} log2$$

natijani beradi. Biz $n \geq 1$ da yangi

$$\int_0^{\frac{\pi}{2}} sin^{2n}\varphi logsin\varphi d\varphi = \frac{\pi}{2}\frac{(2n-1)!!}{2n!!}\left(1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{2n}-log2\right)$$

integralni hosil qilamiz.

6)Quyidagi integrallar hisoblansin (a $> 0,\ p > 0$):

$$u = \int_0^{\infty} e^{-ax}x^{p-1}cosbxdx, \qquad u = \int_0^{\infty} e^{-ax}x^{p-1}sinbxdx.$$

U yerdagidek, b ning $w = u + vi$ funksiyasi uchun

$$\frac{dw}{db} = -\frac{p}{a^2+b^2}(b-ai)w$$

differensial tenglama hosil bo‘ladi, uni mana bu

$$\frac{dw}{db} = pw \cdot \frac{i}{a-bi}$$

shaklda yozish mumkin.

Bu tenglamaga asoslanib

$$w(a-bi)^p = c = const$$

ekanini tekshirish oson. Bu yerda $b = 0$ desak, $c = \Gamma(p)$ bo‘ladi SHunday qilib:

$$w = \frac{\Gamma(p)}{(a-bi)^p} = \frac{\Gamma(p)}{(a^2+b^2)^p}(a+bi)^p =$$

$$= \frac{\Gamma(p)}{(a^2+b^2)^{p/2}}\left\{cosparctg\frac{b}{a} + isinparctg\frac{b}{a}\right\}.$$

Nihoyat, haqiqiy va mavhum qismlarini ayrim-ayrim tenglashtirib, ushbuni topamiz:

$$u = \frac{\Gamma(p)}{(a^2 + b^2)^{p/2}} cosp\theta, \;\; v = \frac{\Gamma(p)}{(a^2 + b^2)^{p/2}} sinp\theta,$$

bunda qisqalik uchun $\theta = arctg\frac{b}{a}$ faraz qilingan.

$\sqrt{a^2 + b^2}$ ni $\frac{b}{sin\theta}$ yoki $\frac{a}{cos\theta}$ bilan almashtirib, natijani bunday yozish mumkin:

$$u = \frac{\Gamma(p)}{b^p} sin^p\theta cosp\theta = \frac{\Gamma(p)}{a^p} cos^p\theta cosp\theta,$$

$$v = \frac{\Gamma(p)}{b^p} sin^p\theta sinp\theta = \frac{\Gamma(p)}{b^p} cos^p\theta sinp\theta.$$

Bu yerdan $p = 1 - s$ deb va a ni 0 ga intiltirib (>0 da

burchak $\theta = arctg\frac{b}{a}$ bo‘lib, u vaqtda $\frac{\pi}{2}$ intiladi), 3) masaladagi A va B integrallarni topish taklif etiladi.

u va v integrallarni p bo‘yicha differensiallab, qator yaigi integrallar hosil qilish mumkin, buni o‘quvchiga tavsiya etamiz.

7) u va v uchun topilgan qiymatlar bizga boshqa qiziq integrallarni hisoblashga imkon beradi. Ushbu

$$\frac{\Gamma(p)}{a^p} cos^p\theta \cdot cosp\theta = \int_0^\infty e^{-ax} x^{p-1} cosbxdx$$

tenglikning ikkala tomonini quyidagiga ko‘paytamiz ($0 < q < p$ va $q < 1$ hisoblab):

$$a^q \cdot tg^{q-1}\theta \cdot \frac{d\theta}{cos^2\theta} = b^{q-1}db$$

va chap tomonda θ bo‘yicha 0 dan $\frac{\pi}{2}$ gacha, o‘ng tomonda esa b bo‘yicha 0 dan ∞ gacha integrallaymiz. Natijada

$$J_1 = \int_0^{\frac{\pi}{2}} cos^{p-q-1}\theta \cdot sin^{q-1}\theta \cdot cosp\theta d\theta =$$

$$\frac{a^{p-q}}{\Gamma(p)} \int_0^{\infty} b^{q-1} db \int_0^{\infty} e^{-ax} x^{p-1} cospxdx$$

hosil bo‘ladi. Agar o‘ng tomondagi integrallarni almashtirsak, u bizni birdaniga J_1 integralni hisoblashga olib keladi:

$$J_1 = \frac{a^{p-q}}{\Gamma(p)} \int_0^{\infty} e^{-ax} x^{p-1} cospxdx \int_0^{\infty} \frac{cosbx}{b^{1-q}} db.$$

3) dan ichki intogralning qiymati $\Gamma(q) cos\frac{q\pi}{2} x^{-q}$ bo‘lishini aniqlash yengil, demak:

$$J_1 = \frac{a^{p-q}\Gamma(q) cos\frac{q\pi}{2}}{\Gamma(p)} \int_0^{\infty} e^{-ax} x^{p-q-1} dx,$$

va

$$J_1 = \int_0^{\frac{\pi}{2}} cos^{p-q-1}\theta \cdot sin^{q-1}\theta \cdot cosp\theta d\theta = \frac{\Gamma(q)\Gamma(p-q)}{\Gamma(p)} cos\frac{q\pi}{2}.$$

Shunga o‘xshash,

$$J_2 = \int_0^{\frac{\pi}{2}} cos^{p-q-1}\theta \cdot sin^{q-1} p\theta d\theta = \frac{\Gamma(q)\Gamma(p-q)}{\Gamma(p)} cos\frac{q\pi}{2}.$$

Endi, integrallarni almashtirishning qanday asoslanishini ko‘rsatamiz (busiz, albatta, natija tayinlangan deb hisoblanishi mumkin emas). Quyidagi

$$\int_{0}^{\infty} x^{p-1}e^{-ax}b^{q-1}cosbxdx$$

integral $0 < b_0 \leq b \leq B < +\infty$ uchun tekis yaqinlashganligi sababli:

$$\int_{b_0}^{B} b^{q-1}db \int_{0}^{\infty} x^{p-1}e^{-ax}\, cosbxdx = \int_{0}^{\infty} e^{-ax}x^{p-1}dx \int_{b_0}^{B} b^{q-1}\, cosbxdb =$$

$$\int_{0}^{\infty} e^{-ax}x^{p-q-1}dx \int_{b_0x}^{Bx} u^{q-1}\, cosudu.$$

Endi $\int_0^\infty u^{q-1}\, cosudu$ integralning mavjudligidan $b_0 \to 0$ va $B \to +\infty$ da ichki integral chegaralangan holda qolib, unga yaqinlashadi:

$$\left| \int_{b_0x}^{Bx} u^{q-1} \cdot cosudu \right| \leq L$$

demak, integral ostidagi butun ifoda $L \cdot e^{-ax}x^{p-q-1}$ funksiya bilan majorlanadi, shu sababli$b_0 \to 0$ va $B \to +\infty$ da integral ostida limitga o‘tish mumkin va h. k.

8) Bunday faraz qilamiz:

$$\psi(t) = Dlog\Gamma(t) = \int_{0}^{1} \frac{1 - x^{t-1}}{1 - x} dx - C$$

[(24) ga qarang]. U holda

$$\int_{0}^{1} \frac{x^p - x^q}{1 - x} dx = \psi(q + 1) - \psi(p + 1), \qquad (p + 1 > 0,\ \ q + 1 > 0 \text{ uchun})$$

Buni nazarda tutib,

$$J = \int_0^1 \frac{(1 - x^\alpha)(1 - x^\beta)}{(1 - x)logx} dx$$

$$\alpha > -1, \beta > -1, \alpha + \beta > -1$$

integralni ko‘zdan kechiramiz. Buning α bo‘yicha hosilasi:

$$\frac{dJ}{d\alpha} = -\int_0^1 \frac{x^\alpha(1 - x^\beta)}{1 - x} dx = \psi(\alpha + 1) - \psi(\alpha + \beta + 1) =$$

$$= \frac{d}{d\alpha} log \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + 1)}.$$

SHuning uchun:

$$J = log \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + 1)} + C.$$

Lekin, $\alpha = 0$ da $J = 0$ bo‘lganidan $C = log\Gamma(\beta + 1)$ bo‘lishi zarur, demak:

$$J = log \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} + C.$$

Kuyidagi integrallar shuning singari topiladi:

$$K = \int_0^1 \frac{x^\alpha(1 - x^\beta)(1 - x^\gamma)}{(1 - x)logx} =$$

$$= log \frac{\Gamma(\alpha + \gamma + 1)\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha + 1)\Gamma(\alpha + \beta + \gamma + 1)},$$

$$(\alpha > -1, \qquad \alpha + \beta > -1, \qquad \alpha + \gamma > -1, \qquad \alpha + \beta + \gamma > -1)$$

$$L=\int_0^1 \frac{(1-x^{\alpha})\left(1-x^{\beta}\right)(1-x^{\gamma})}{(1-x)logx}dx=$$

$$=log\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)},\ \ \alpha>-1 \text{ va x.k}$$

va shunga o‘xshash.

Agar K integralda $\gamma=\frac{1}{2},\alpha=\frac{a}{2}-1,\beta=\frac{b-a}{2}$ deb olsak, va demak, $z=t^2$ almashtirishni bajarsak,

$$\int_0^1 \frac{t^{a-1}-t^{b-1}}{(1+t)logt}dt=log\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\qquad (a,b>0).$$

integralga kelamiz. Bundan $b=1-a$ uchun ajoyib integral hosil qilinadi:

$$\int_0^1 \frac{t^{a-1}-t^{-a}}{(1+t)logt}dt=logtg\frac{a\pi}{2}\quad (0<a<1).$$

Γ funksiyani kiritish bilan, integrallarni chekli formulalar orqali tasvirlash imkoniyatining qanchalik kengayganligini ko‘rsatish uchun yuqorida keltirilgan misollar yetarlidir. Hatto, chekli formula o’z ichiga elementar funksiyalardan boshqa funksiyalarni olmagan hollarda ham, bunday formulani topish (hech bo‘lmaganda, hisoblashlarni bajarish) ishi Γ funksiyadan foydalanish bilan ko‘pincha osonlashadi.

1) Quyidagi integral topilsin:

$$\int_0^1 x^{p-1}(1-x^m)^{q-1}dx\ \ (p,q,m>0).$$

Buni $x^m=y$ almashtirish bilan birinchi tur Eyler integralini topamiz.

$$\frac{1}{m}B\left(\frac{p}{m},q\right)=\frac{1}{m}\frac{\Gamma\left(\frac{p}{m}\right)\Gamma(q)}{\Gamma\left(\frac{p}{m}+q\right)}.$$

Bu formula bilan,

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}}\cdot\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}=\frac{\pi}{4}$$

ni hosil qilamiz.

2)Ushbu

$$\int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{[\propto x+\beta(1-x)+\gamma]^{p+q}}dx \quad (\propto,\beta,\gamma,p,q>0)$$

integral hisoblansin.

Buvda

$$\frac{(\alpha+\beta)x}{\propto x+\beta(1-x)+\gamma}=t, \qquad \frac{(\alpha+\beta)(1-x)}{\propto x+\beta(1-x)+\gamma}=1-t$$

almashtirishning yordami bilan, integral

$$\frac{1}{(\alpha+\gamma)^p(\beta+\gamma)^q}\int_0^1 t^{p-1}(1-t)^{q-1}dt=\frac{B(p,q)}{(\alpha+\gamma)^p(\beta+\gamma)^q}$$

shaklga keltiriladi.

3)Quyidagi integrallar topilsin:

$$a)\ \int_0^1 \frac{x^{a-1}(1-x)^{b-1}}{(x+p)^{a+b}}dx,\ \ b)\ \int_{-1}^{+1} \frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}}\,dx.$$

(a) Almashtirish $y = (1+p)\frac{x}{x+p}$. Almashtirish (b) $u = \frac{1}{2} \cdot \frac{(1+x)^2}{1+x^2}$ qilib quyidagi natijalarga kelamiz.

$$(a)\ \frac{1}{(1+p)^a p^b} B(a,b);\quad (b)\ 2^{m+n-2} B(m,n).$$

$n = 1 - m$ va $2m - 1 = cos2\alpha$ deb $x = tg\varphi$ almashtirishni bajarsak,

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{cos\varphi + sin\varphi}{cos\varphi - sin\varphi}\right)^{cos2\alpha} d\varphi = \frac{\pi}{2sin(\pi cos^2\alpha)}$$

topiladi.

4)Quyidagi integrallar hisoblansin:

$$(a)\ \int_0^{\frac{\pi}{2}} sin^{a-1}\varphi d\varphi = \int_0^{\frac{\pi}{2}} cos^{a-1}\varphi d\varphi \quad (a > 0);$$

$$(b)\ \int_0^{\frac{\pi}{2}} tg^c\varphi d\varphi \quad (|c| < 1);\quad (v)\ \int_0^{\frac{\pi}{2}} sin^{a-1}\varphi cos^{b-1}\varphi d\varphi \quad (a, b > 0).$$

Ye ch i l i sh i. Oxirgi (v) misoldan boshlaylik. Unda $x = sin\varphi$ deb, uni

$$\int_0^1 x^{a-1}(1-x^2)^{\frac{b}{2}-1}\, dx$$

integralg'a keltiramiz, demak, 1) masaladan foydalanib, quyidagiga ega bo‘lamiz:

$$\int_0^{\frac{\pi}{2}} sin^{a-1}\varphi cos^{b-1}\varphi d\varphi = \frac{1}{2} B\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}.$$

(a) $b = 1$ bo‘lgan xususiy holda quyidagini hosil qilamiz:

$$\int_0^{\frac{\pi}{2}} sin^{a-1}\varphi d\varphi = \frac{\sqrt{\pi}}{2} = \frac{1}{2}\frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)}.$$

Lejandr formulasi yordami bilan, bu formula

$$\int_0^{\frac{\pi}{2}} sin^{a-1}\varphi d\varphi = 2^{a-1}\frac{\left[\Gamma\left(\frac{a}{2}\right)\right]^2}{\Gamma(a)} = 2^{a-1}B\left(\frac{a}{2},\frac{b}{2}\right)$$

shaklga keltirilishi mumkin.

(b) (v) da $a = 1 + c$ va $b = 1 - c$ desak, bunda $|c| < 1$ to‘ldirish formulasidan foydalanib, ushbuni topamiz:

$$\int_0^{\frac{\pi}{2}} tg^c\varphi d\varphi = \frac{1}{2}\Gamma\left(\frac{1+c}{2}\right)\left(\frac{1-c}{2}\right) = \frac{\pi}{2cos\frac{c\pi}{2}}$$

5)Ushbu

$$r^4 = sin^3\theta cos\theta$$

egri chiziq bilan chegaralangan shaklning R yuzi topilsin.

Yechilishi. Egri chiziq I va III choraklarda ikkita halqaga ega; shu sababli bulardan bittasining yuzini ikkiga ko‘paytirshi kifoya. Yuzning qutb koordmmatalardagi for mulasi bo‘yicha:

$$P = 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} sin^{8/2}\theta cos^{1/2}\theta d\theta = \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\Gamma(2)} =$$

$$= \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi\sqrt{2}}{8}.$$

6) (a) Ushbu

$$r^m = a^m cosm\theta$$

egri chiziqning (m — natura son) bitta o‘rami bilan cheklangan R yuz va (b) shu o‘ramniig S uzunligi hisoblansin.

Ye ch i l i sh i.

$$(a) \quad P = 2 \cdot \frac{a^2}{2} \int_0^{\frac{\pi}{2m}} cos^{\frac{2}{m}} m\theta d\theta = \frac{a^2}{m} \int_0^{\frac{\pi}{2}} cos^{\frac{2}{m}} \varphi d\varphi =$$

$$= \frac{a^2}{m} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{1}{m} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{m} + 1\right)} = \frac{\pi a^2}{\sqrt[m]{4}} \cdot \frac{\Gamma\left(\frac{2}{m}\right)}{\left[\Gamma\left(\frac{1}{m}\right)\right]^2}.$$

(b)Yoy uzunligining qutb koordinatlardagi formulasi bo‘yicha quyidagini

topamiz:

$$S = 2a \int_0^{\frac{\pi}{2m}} cos^{\frac{1}{m}-1} m\theta \, d\theta = \frac{2a}{m} \int_0^{\frac{\pi}{2}} cos^{\frac{1}{m}-1} \varphi \, d\varphi =$$

$$\frac{a}{m} \cdot 2^{\frac{1}{m}-1} \cdot \frac{\left[\Gamma\left(\frac{1}{2m}\right)\right]^2}{\Gamma\left(\frac{1}{m}\right)}.$$

7) Mustaqil yechish uchun misollar:

$$(a) \int_0^{\pi} \frac{d\theta}{\sqrt{3 - cos\theta}}$$

$$(b) \int_0^{\pi} \left(\frac{sin\varphi}{1 + kcos\varphi}\right)^{a-1} \frac{d\varphi}{1 + kcos\varphi} \quad (a > 0, 0 < k < 1).$$

(a) Almashtirish: $cos\theta = 1 - 2\sqrt{x}$,

Javob. $\frac{1}{4\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$.

(b) Almashtirish $tg\,\frac{\theta}{2} = \sqrt{\frac{1-k}{1+k}}\,tg\,\frac{\varphi}{2}$.

Javob. $\frac{2^{a-1}}{(1-k^2)^{a/2}} \cdot \frac{\left[\Gamma\left(\frac{a}{2}\right)\right]^2}{\Gamma(a)}$.

2.2-§ Stirling formulasi.

Endi $log\Gamma(a)$ uchun qulay taqribiy hisoblash formulalarini chiqarishga va bu logarifmni (va Γ funksiyaning o‘znni) hisoblash masalasiga kelamiz.

Bizga, Γ funksiyaning logarifmik hosilasi

$$Dlog\Gamma(a) = \int_0^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-ax}}{1-e^{-x}}\right) dx$$

uchun chiqarilgan (23) formula boshlang‘ich punkt bo‘lib xizmat qiladi.

Integral ostidagi ifoda $x \geq 0$ va $a > 0$ uchun ikkala argument x va a ning uzluksiz funksiyasini tasvirlaganidan ($x = 0$ da qatorga yoyish bilan bunga ishonish mumkin) va $x = \infty$ da esa, integral a ga nisbatan $a \geq a_0 > 0$ uchun $\left(\frac{e^{-x}}{x} - \frac{e^{-a_0 x}}{1-e^{-x}} \text{ majoranta}\right)$ tekis yaqinlashganidan, integral ostida a bo‘yicha 1 dan a gacha integrallash mumkin:

$$log\Gamma(a) = \int_0^\infty \left[(a-1)e^{-x} - \frac{e^{-x} - e^{-ax}}{1-e^{-x}}\right] \frac{dx}{x} \quad (a > 0).$$

Integrallash o‘zgaruvchisining ishorasini o‘zgartirib, $[-\infty, 0]$ oraliqqa o‘tamiz:

$$log\Gamma(a) = \int_{-\infty}^0 \left[\frac{e^{ax} - e^x}{e^x - 1} - (a-1)e^x\right] \frac{dx}{x}. \qquad (33)$$

Bu integral ham $0 < a_0 \leq a \leq A < +\infty$ uchun $x = -\infty$ da tekis yaqinlashadi; yana a bo‘yicha a dan $a+1$ gacha integral belgisi ostida integraldaymiz;

$$R(a)=\int_{a}^{a+1} log\Gamma(a)da=\int_{-\infty}^{0}\left[\frac{e^{ax}}{x}-\frac{e^{x}}{e^{x}-1}-\left(a-\frac{1}{2}\right)e^{x}\right]\frac{dx}{x}. \qquad (34)$$

Biz (33) ifodani soddalashtirish uchun, topilgan integraldan va shuningdek, *Frullanining*

$$\frac{I}{2}loga=\int_{0}^{\infty}\frac{e^{-x}-e^{-ax}}{2}\cdot\frac{dx}{x}=\int_{-\infty}^{0}\frac{e^{ax}-e^{x}}{2}\cdot\frac{dx}{x} \qquad (35)$$

elementar integralidan foydalanamiz (33) dan (34) ni ayirib va (35) ni qo‘shib, ushbuni hosil qilamiz:

$$log\Gamma(a)-R(a)+\frac{I}{2}loga=\int_{-\infty}^{0}\left[\frac{1}{e^{x}-1}-\frac{1}{x}+\frac{1}{2}\right]\frac{e^{ax}dx}{x},$$

o‘ng‘aylik uchun:

$$\omega(a)=\int_{-\infty}^{0}\left[\frac{1}{e^{x}-1}-\frac{1}{x}+\frac{1}{2}\right]\frac{e^{ax}dx}{x} \qquad (36)$$

faraz etib va $R(a)$ o‘rniga bizga ma’lum *Raabe* integralining (20) ifodasini qo‘ysak,

$$log\Gamma(a)=log\sqrt{2\pi}+\left(a-\frac{1}{2}\right)loga-a+\omega(a) \qquad (37)$$

hosil bo‘ladi.

Giperbolik kotangensning sodda kasrlar yoyilmasiga asosan:

$$cthx=\frac{1}{x}+\sum_{k=1}^{\infty}\frac{2x}{x^{2}+4k^{2}\pi^{2}}$$

va bu yoyilma $x \neq 0$ qiymatlar uchun haqiqiy edi. Bu yerda x ni $\frac{x}{2}$ bilan almashtirib, uni

$$\frac{x}{e^x - 1} + \frac{x}{2} = 1 + \sum_{k=1}^{\infty} \frac{2x^2}{x^2 + 4k^2\pi^2}$$

shaklga keltirish yoki, nihoyat, quyidagini hosil qilish mumkin:

$$f(x) - \frac{1}{x}\left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right) = 2\sum_{k=1}^{\infty} \frac{1}{x^2 + 4k^2\pi^2}$$

$f(x)$ funksiya (36) ifoda integral ostidagi funksiya ekanini ko‘ramiz.

Istalgan musbat m sonni olib, qatorning har bir hadini unga aynan teng bo‘lgan yig‘indi bilan almashtiramiz;

$$\frac{1}{x^2 + 4k^2\pi^2} = \frac{1}{4k^2\pi^2} - \frac{x^2}{(4k^2\pi^2)^2} + \frac{x^4}{(4k^2\pi^2)^3} - \dots + (-1)^{m-1}\frac{x^{2m-2}}{(4k^2\pi^2)^m} +$$

$$+(-1)^m \frac{x^{2m}}{(4k^2\pi^2)^{m+1}} \cdot \frac{1}{1 + \frac{x^2}{4k^2\pi^2}}.$$

Endi

$$(-1)^{n-1}\frac{x^{2n-2}}{(4k^2\pi^2)^n} \qquad (1 \le n \le m)$$

shaklidagi qo‘shiluvchilarni ($k = 1, 2, \ldots$ da) alohida yig‘amiz. Odatdagicha,

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = s_{2n}$$

faraz qilib,

$$(-1)^{n-1}\frac{1}{(2\pi)^{2n}} \cdot s_{2n} \cdot x^{2n-2}$$

natijani hosil qilamiz, agar *Bernullining* n-sonini kiritsak:

$$B_n = \frac{2 \cdot (2n)!}{(2\pi)^{2n}} \cdot s_{2n} \qquad (38)$$

bu natija

$$(-1)^{n-1} \frac{B_n}{2 \cdot (2n)!} \cdot x^{2n-2}$$

shaklni oladi; $\frac{1}{1+\frac{x^2}{4k^2\pi^2}}$ ga ko‘paytirilgan keyingi qo‘shiluvchilar esa musbat to‘g‘ri kasrlarni tasvirlaydi, ularni yig‘ib

$$(-1)^m \cdot \breve{\theta} \cdot \frac{B_{m+1}}{2 \cdot (2m+2)!} \cdot x^{2m}$$

hadga kelamiz, bu yerda $\breve{\theta}$ ham musbat to‘g‘ri kasrdir.

Oxirda $f(x)$ uchun quyidagi ifodani topamiz:

$$f(x) = \frac{B_1}{2!} - \frac{B_2}{4!}x^2 + \frac{B_3}{6!}x^4 - \cdots + (-1)^{m-1}\frac{B_m}{2m!}x^{2m-2} +$$

$$+(-1)^m \cdot \breve{\theta} \cdot \frac{B_{m+1}}{(2m+2)!} \cdot x^{2m} \qquad \left(0 < \breve{\theta} < 1\right).$$

Buni (36) ga qo‘yib, hadlab integrallaymiz. Endi

$$\int_{-\infty}^{0} e^{ax} x^{2n} dx = \int_{0}^{\infty} e^{-ax} x^{2n}\, dx = \frac{2n!}{a^{2n+1}}$$

va

$$\int_{-\infty}^{0} e^{ax} \cdot \breve{\theta} \cdot x^{2m} dx = \theta \int_{-\infty}^{0} e^{ax} x^{2m}\, dx = \theta \cdot \frac{2m!}{a^{2m+1}} \qquad (0 < \theta < 1)$$

bo‘lganidan, quyidagini topamiz:

$$\omega(a) = \frac{B_1}{1 \cdot 2} \cdot \frac{1}{a} - \frac{B_2}{3 \cdot 4} \cdot \frac{1}{a^3} + \frac{B_3}{5 \cdot 6} \cdot \frac{1}{a^5} - \cdots +$$

$$+(-1)^{m-1} \frac{B_m}{(2m-1)2m} \frac{1}{a^{2m-1}} +$$

$$+(-1)^m \theta \cdot \frac{B_{m+1}}{(2m+1)(2m+2)} \frac{1}{a^{2m+1}} \qquad (0 < \theta < 1).$$

Nihoyat, agar (37) da $\omega(a)$ o‘rniga topilgan ifodani qo‘ysak. *Stirling* (J. Stirling) nomi bilan ataluvchi formulaga kelamiz,

$$log\Gamma(a) = log\sqrt{2\pi} + \left(a - \frac{1}{2}\right) loga - a + \frac{B_1}{1 \cdot 2} \cdot \frac{1}{a} -$$

$$- \frac{B_2}{3 \cdot 4} \cdot \frac{1}{a^3} + \cdots + (-1)^{m-1} \frac{B_m}{(2m-1)2m} \cdot \frac{1}{a^{2m-1}} +$$

$$+(-1)^m \theta \cdot \frac{B_{m+1}}{(2m+1)(2m+2)} \frac{1}{a^{2m+1}} \qquad (0 < \theta < 1)$$

$m = 0$ bo‘lgan zng sodda holda formula *!*

$$log\Gamma(a) = log\sqrt{2\pi} + \left(a - \frac{1}{2}\right) loga - a + \frac{\theta}{12a} \qquad (0 < \theta < 1).$$

shaklga keladi, chunki $B_1 = \frac{1}{6}$. Bu yerda a ni n natural songa teng deb (va ikkala tomonga $logn$ ni qo‘shib), bizga tanish formulani hosil qilamiz:

$$logn! = log\sqrt{2\pi} + \left(n - \frac{1}{2}\right) logn - n + \frac{\theta}{12n} \qquad (0 < \theta < 1).$$

Agar (θ ko‘paytuvchini o‘z ichiga olgan) qo‘shimcha hadni tashlab, (39) formuladagi qatorning hadlarini cheksiz davom ettirsak, u holda Stirling qatori kelib chiqadi Bu qator uzoqlashuvchi bo‘ladi. Haqiqatan (38) ga asosan, Stirling qatorining umumiy hadi $n \to +\infty$ da:

$$\frac{B_n}{(2n-1)2n} \cdot \frac{1}{a^{2n-1}} = \frac{1}{\pi} \cdot \frac{(2n-2)!}{(2\pi a)^{2n-1}} s_{2n} \to \infty.$$

Biroq, bu qatorning (39) formula bo‘yicha berilgan parchasidan taqribiy hisoblashlarda muvaffaqiyatli foydalanish mumkin. Qo‘shimcha hadning shaklidan ko‘rinadiki, xato absolyut qiymat bo‘yicha tashlangan hadlarning birinchisidan kichikdir. Qatorning hadlari avval tezlik bilan kamayadi, so‘ngra o‘sadi — hatto cheksizlikkacha; shu sababli qatorni absolyut qiymat jihatdan eng kichik bo‘lgan hadida uzish foydaliroqdir.

Albatta, o‘zgarmas a da bu yo‘l bilan ixtiyoriy berilgan aniqlikdagi taqribiy qiymatni topish mumkin emas. Biroq, a yetarli katta qilib olinganda bunga erishish mumkin.

O‘quvchiga chuqurroq tushunishga imkon tug‘dirish uchun, yaqin punktlarda bunga o‘xshash uzoqlashuvchi qatorlar haqidagi masalani umumiy nuqtai nazardan tekshiramiz.

2.3-§Asimptotik qatorlar.

Agar $x \geq x_0$ da aniqlangan biror $F(x)$ funksiyani x ning katta qiymatlarida tekshirishga to‘g‘ri kelsa, u holda $x \to \infty$ shartida

$$\rho(x) = F(x) - S(x)$$

ayirma nolga intiladigan, tuzilishi sodda bo‘lgan $S(x)$ funksiyani topish ahamiyatga egadir. U vaqtda yetarli katta x uchun istalgan darajadagi aniqlikda $F(x) = S(x)$ deyish mumkin. Ravshanki, $x \to \infty$ da ($\frac{1}{x}$ ga nisbatan) cheksiz kichik $\rho(x)$ ayirmaning tartibi qanchalik yuqori bo‘lsa, taqribiy formula shunchalik qiymatli bo‘ladi.

$F(x)$ funksiya x ning ($x \geq x_0$ uchun) manfiy darajalari bo‘yicha yoyiladigan, ya’ni

$$F(x) = A_0 + \frac{A_1}{x} + \cdots + \frac{A_n}{x^n} + \frac{A_{n+1}}{x^{n+1}} + \cdots$$

bo‘lsa, bu yoyilmani n −hadda uzib, ya’ni

$$S_n(x) = A_0 + \frac{A_1}{x} + \cdots + \frac{A_n}{x^n}, \qquad \rho(x) = \frac{A_{n+1}}{x^{n+1}} + \cdots$$

faraz qilib, quyidagiga ega bo‘lar edik:

$\lim_{x\to\infty} x^n \rho_n(x) = 0$ yoki $\rho_n(x) = o\left(\frac{1}{x^n}\right)$, (40)

demak, $\rho_n(x)$ ayirma n dan yuqori tartibli cheksiz kichik miqdor bo‘lar edi.

Lekin bunday yoyilmaga ega bo‘lmagan $F(x)$ funksiya uchun ham, ba’zan shunday

$$A_0 + \frac{A_1}{x} + \cdots + \frac{A_n}{x^n} + \frac{A_{n+1}}{x^{n+1}} + \cdots$$

(ehtimol uzoqlashuvchi) qatorni tuzish mumkinki, $F(x)$ funksiya va bu qatorning $n = S_n(x)$ parchasi orasidagi $\rho_n(x)$ ayirma ixtiyoriy, lekin o‘zgarmas n da (40) talabni qanoatllntiradi. Bunday shartlarda, qator $F(x)$ funksiyaning asimptotik yoyilmasini (yoki tasvirini) beradi deyiladi va bu fakt bunday yoziladi:

$$F(x) \sim \sum_{k=0}^{\infty} \frac{A_k}{x^k}.$$

Buning ma’nosi: yozilgan qator sifati (40) tenglik bilan xarakterlanadigan taqribiy

$$F(x) = A_0 + \frac{A_1}{x} + \cdots + \frac{A_n}{x^n}$$

formulalarning ($n = 0, 1, 2, \ldots$) manbai bo‘ladi.

Asimptotik yoyilmalarga doyr ikkita sodda misol keltiramiz:

1) Faraz qilaylik ($x > 0$ uchun):

$$F(x) = \int_x^{\infty} e^{x-t} \frac{dt}{t}$$

bo‘lsin. Takroriy bo‘laklab integrallash bilan quyidag‘ini hosil qilish yengil:

$$F(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \cdots + (-1)^{n-1} \frac{(n-1)!}{x^n} + \rho_n(x),$$

bu yerda

$$\rho_n(x) = (-1)^n \cdot n! \int_x^{\infty} \frac{e^{x-t}}{t^{n+1}} dt.$$

Lekin

$$\int_x^{\infty} \frac{e^{x-t}}{t^{n+1}} dt = -\frac{e^{x-t}}{t^{n+1}} \Big|_{t=x}^{t=\infty} - (n+1) \int_x^{\infty} \frac{e^{x-t}}{t^{n+1}} dt < \frac{1}{x^{n+1}}$$

bo‘lganidan,

$$|\rho_n(x)| < \frac{n!}{x^{n+1}}$$

bo‘lib, (40) shart bajariladi. Shunday qilib, $F(x)$ funksiya uchun:

$$F(x) \sim \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \dots + (-1)^{n-1} \frac{(n-1)!}{x^n} + \cdots$$

asimptotik qatorni topamiz.

Bu qatorning uzoqlashuvchi ekaniga ishonish yengil, chunki $n \to \infty$ da uning umumiy hadi ∞ ga intiladi.

2)Endi ($x > 0$ uchun)

$$F(x) = \sum_{k=1}^{\infty} \frac{c^k}{x+k}$$

deylik, bu yerda $0 < c < 1$ (qator uzoqlashuvchi).

$k < x$ bo‘lganda

$$\frac{1}{x+k} = \frac{1}{x} - \frac{k}{x^2} + \frac{k^2}{x^2} - \frac{k^3}{x^3} + \cdots;$$

agar $k > x$ bo‘lsa, u holda bu qator uzoqlashadi.

Lekin, $F(x)$ funksiyani aniqlovchi qatorga bu yoyilmani formal ravishda qo‘yib, o‘xshash hadlarni yig‘ishtirsak,

$$\frac{A_1}{x} + \frac{A_2}{x^2} + \dots + \frac{A_n}{x^n} + \cdots$$

qator hosil bo‘ladi, bu yerda

$$A_n = (-1)^{n-1}\sum_{k=1}^{\infty} k^{n-1}c^k$$

A_n koeffitsientlarni aniqlovchi qatorlarning hammasi yaqinlashuvchi ekaniga ishonish yengil. Lekin oldingi qator oshkora uzoqlashuvchidir, chunki

$$|A_n| \geq n^{n-1}c^n \quad va \quad \left|\frac{A_n}{x^n}\right| \geq \frac{n^{n-1}c^n}{x^n},$$

oxirgi ifoda esa $n \to \infty$ da ∞ ga intiladi.

Biroq, topilgan qator har holda $F(x)$ funksiyaning asimptotik tasvirini berishini aniqlaymiz.

Uning n −parchasi quyidagidan iborat:

$$S_n(x) = \sum_{v=1}^{n}\frac{A^v}{x^v}\sum_{k=1}^{\infty} c^k \sum_{v=1}^{n}\frac{(-1)^{v-1}k^{v-1}}{x^v} =$$

$$= \sum_{k=1}^{\infty}\left[1 + (-1)^{n+1}\frac{k^n}{x^n}\right]\frac{c^k}{x+k},$$

demak,

$$|\rho_n(x)| = \frac{1}{x^n}\sum_{k=1}^{\infty} k^n \frac{c^k}{x+k} \leq \frac{|A_{n+1}|}{x^{n+1}}$$

va (40) shart yana bajariladi.

Stirling qatoriga qaytib:

$$log\Gamma(a) - \left(a - \frac{1}{2}\right) loga - a \sim log\sqrt{2\pi} + \frac{B_1}{1\cdot 2}\cdot\frac{1}{a} -$$

$$-\frac{B_2}{3\cdot 4}\cdot\frac{1}{a^3}+\cdots+(-1)^{n-1}\frac{B_n}{(2n-1)2n}\cdot\frac{1}{a^{2n-1}}+\cdots, \qquad (41)$$

bu qator chap tomondagi funksiyaning asimptotik tasvirini beradi deyishimiz mumkin. Haqiqatan, (39) formuladan

$$\rho_{2n-1}(a)=\rho_{2n}(a)=(-1)^n\theta\cdot\frac{B_{n+1}}{(2n+1)(2n+2)}\cdot\frac{1}{a^{2n+1}}$$

ekanligi ravshan, bundan

$$\rho_{2n}(a)=o\left(\frac{1}{2^{2n}}\right).$$

Bunga o‘xshash asimptotik qatorlar differensial tenglamalar nazariyasida va osmon mexanikasida muhim rolni o‘ynaydi.

Asimptotik qatorlar ustida amallar. Eng avval $F(x)$ funksiya asimptotik yoyilmaga yo‘l qo‘ysa, u yoyilma yagona ekanini eslatib o‘tamiz. Haqiqatan, agar

$$F(x) \sim \sum_{0}^{\infty} A_n x^{-n} \quad va \quad F(x) \sim \sum_{0}^{\infty} B_n x^{-n}$$

bo‘lsa, har bir n uchun

$$\lim_{n\to\infty}\left[(A_0 - B_0) + \frac{A_1 - B_1}{x} + \cdots + \frac{A_n - B_n}{x^n}\right] x^n = 0$$

munosabat bajarilar edi, bu esa faqat

$$A_0 = B_0, A_1 = B_1, \dots A_n = B_n$$

shartdagina mumkindir.

Agar

$$F(x) \sim \sum_{0}^{\infty} A_k x^{-k} \quad va \quad F(x) \sim \sum_{0}^{\infty} B_k x^{-k} \qquad (42)$$

bo‘lsa, u holda, ravshanki,

$$F(x) \pm G(x) \sim \sum_{0}^{\infty} (A_k + B_k)\, x^{-k}.$$

Endi, $F(x) \cdot G(x)$ funksiyaning asimptotik yoyilmasini formal ravishda (42) yoyilmalarni ko‘paytirish bilan topish mumkin ekanligini ko‘rsatamiz.

Istalgan n da ushbuga egamiz:

$$F(x) = A_0 + \frac{A_1}{x} + \cdots + \frac{A_n}{x^n} + o\left(\frac{1}{x^n}\right)$$

va

$$G(x) = B_0 + \frac{B_1}{x} + \cdots + \frac{B}{x^n} + o\left(\frac{1}{x^n}\right)$$

Bularni ko‘paytirib, ushbuni hosil qilamiz:

$$F(x) \cdot G(x) = C_0 + \frac{C_1}{x} + \cdots + \frac{B_n}{x^n} + o\left(\frac{1}{x^n}\right)$$

bu yerda

$$C_m = \sum_{k=0}^{m} A_k B_{m-k}.$$

Bu esa ushbu

$$F(x) \cdot G(x) \sim \sum_{m=0}^{\infty} C_m x^{-m}$$

tasdiq bilan teng kuchli bo‘lib, shuning isboti talab etilgan edi. So‘ngra, agar $F(x)$ funksiya x^2 haddan boshlanuvchi asimptotik

$$F(x) \sim \sum_{k=2}^{\infty} \frac{A_k}{x^k}$$

yoyilmaga ega bo‘lsa, u holda bu yoyilmani x dan ∞ gacha formal ravishda integrallash mumkin, u vaqtda mana bu asimptotik yoyilma:

$$\int_x^{\infty} F(x)\, dx \sim \sum_{k=2}^{\infty} \int_x^{\infty} \frac{A_k}{x^k} dx$$

yoki

$$\int_x^{\infty} F(x)\, dx \sim \sum_{k=2}^{\infty} \frac{A_k}{(k-1)x^{k-1}} \qquad (43)$$

bo‘dadi.

Haqiqatan,

$$S_n(x) = \sum_{k=2}^{\infty} \frac{A_k}{x^k}, \qquad \rho_n(x) = F(x) - S_n(x)$$

deb, yetarli katta x uchun quyidagiga ega bo‘lamiz:

$$|\rho_n(x)| \cdot x^n < \varepsilon \tag{44}$$

bu yerda $\varepsilon > 0$ oldindan berilgan ixtiyoriy son.

U vaqtda

$$\int_x^{\infty} F(x)\,dx = \int_x^{\infty} S_n(x)\,dx + \int_x^{\infty} \rho_n(x)\,dx =$$

$$= \sum_{k=2}^{\infty} \frac{A_k}{(k-1)x^{k-1}} + \int_x^{\infty} \rho_n(x)\,dx.$$

Biroq (44) ga asosan, yetarli katta x uchun

$$\left| \int_x^{\infty} \rho_n(x)\,dx \right| \le \int_x^{\infty} |\rho_n(x)|\,dx < \varepsilon \int_x^{\infty} \frac{dx}{x^n} = \frac{\varepsilon}{(n-1)x^{n-1}},$$

demak

$$\lim_{x\to\infty} x^{n-1} \cdot \int_x^{\infty} \rho_n(x)\,dx = 0,$$

shu bilan (43) isbotlanadi.

Asimptotik yoyilmalarni formal differensiallash, umuman aytganda, qonunsiz ekanini eslatib o‘tish qiziqdir. Misol uchun $F(x) = e^{-x} sine^{x}$ funksiyani tekshiraylik. Istalgan n da

$$\lim_{x\to\infty} F(x)x^n = 0$$

bo‘lganidan, $F(x)\sim 0$, ya’ni $F(x)$ funksiyaning asimptotik yoyilmasi nollardan iboratdir. Biroq $F(x) = e^{-x}sine^x + cose^x$ hosila uchun bunday yoyilma umuman mumkin emas, chunki hatto $\lim_{x\to\infty} F'(x)$ limit mavjud emas.

Nihoyat, $F(x)$ funksiya ozod hadsiz asimptotik yoyilmaga ega ya’ni:

$$F(x)\sim \sum_{k=1}^{\infty} A_k x^{-k}$$

deb faraz etib, uni formal ravishda potensirlash mumkinligini isbotlaymiz. Bu bilan, biz aytmoqchimizki, agar

$$e^{F(x)} = 1 + \sum_{m=1}^{\infty} \frac{1}{m!}[F(x)]^m$$

qatorda $F(x)$ ni uning yoyilmasi bilan almashtirib va formal ravishda darajaga ko‘tarib, o‘xshash hadlarni yig‘sak, $e^{F(x)}$ funksiyaning asimptotik yoyilmasini hosil qilish mumkin.

Shu maqsad bilan (istalgan n uchun) $x \to \infty$ da

$$x^n\left[e^{F(x)} - e^{S_n(x)}\right] = x^n e^{S_n(x)}\left[e^{\rho_n(x)} - 1\right] =$$

$$= e^{S_n(x)} \cdot \frac{e^{\rho_n(x)} - 1}{\rho_n(x)} \cdot x^n \quad \rho_n(x) \to 0$$

ekanini eslatib o‘tamiz, demak,

$$e^{F(x)} = e^{S_n(x)} + o\left(\frac{1}{x^n}\right)$$

Ikkinchi tomondan

$$e^{S_n(x)} = 1 + \sum_{k=1}^{\infty} \frac{|S_n(x)|^k}{k!} = 1 + \frac{A_1}{x} + \left[\frac{A_1^2}{2!} + \frac{A_2}{1!}\right]\frac{1}{x^2} +$$

$$\left[\frac{A_1^3}{3!} + \frac{2A_1A_2}{2!} + \frac{A_3}{1!}\right]\frac{1}{x^3} + \cdots$$

$$\ldots + \left[\frac{A_1^n}{n!} + \frac{(n-1)A_1A_2 + \cdots}{(n-1)!} + \cdots + \frac{A_n}{1!}\right]\frac{1}{x^n} + o\left(\frac{1}{x^n}\right).$$

Bu yerdan

$$e^{F(x)} \sim 1 + \frac{A_1}{x} + \cdots + \left[\frac{A_1^n}{n!} + \cdots + \frac{A_n}{1!}\right]\frac{1}{x^n} + \cdots$$

ekanligi kelib chiqadi; shuni isbotlash talab etilgan edi.

Agar topilgan natijani (41) Stirling qatoriga tatbiq etsak, u holda B_k koeffitsientlar o‘rniga ularning son qiymatlarini qo‘yib, $\Gamma(a)$ funksiyaning o‘zi uchun quyidagi yoyilmani hosil qilamiz:

$$\frac{\Gamma(a)}{\sqrt{2\pi} \cdot e^{-a} \cdot a^{a-\frac{1}{2}}} \sim 1 + \frac{1}{12a} + \frac{1}{288a^2} - \frac{139}{51840a^3} - \frac{571}{2488320a^4} + \cdots$$

2.4-§ Bazi masalalar

1) Eyler o'zgarmasini hisoblash. (37) formulaga qaytib, uni a bo'yicha differensiallaymiz:

$$D log\Gamma(a) = log a - \frac{1}{2a} + \omega'(a),$$

bu yerda

$$\omega'(a) = \int_{-\infty}^{0} x e^{ax} f(x)\, dx.$$

Avvalgi hisoblashlarni takrorlab, mana buni hosil qilamiz:

$$\omega'(a) = -\frac{B_1}{2} \cdot \frac{1}{a^2} + \frac{B_2}{4} \cdot \frac{1}{a^4} - \cdots - (-1)^m \frac{B_m}{2m} \cdot \frac{1}{a^{2m}} +$$

$$+(-1)^{m+1}\theta' \cdot \frac{B_{m+1}}{2m+2} \frac{1}{a^{2m+2}} \qquad (0 < \theta' < 1). \qquad (45)$$

Bundan asimptotik yoyilmaga kelamiz:

$$D log\Gamma(a) - log a + \frac{1}{2a} \sim -\frac{B_1}{2} \cdot \frac{1}{a^2} + \frac{B_2}{4} \cdot \frac{1}{a^4} - \cdots$$

$$\ldots + (-1)^n \frac{B_n}{2n} \cdot \frac{1}{a^{2n}} + \cdots$$

Formal jihatdan buni *Stirling* qatorini hadlab differensiallash bilan topish mumkin.

Topilgan (45) formuladan *Eylerning* C o'zgarmasini hisoblash uchun qulay usulni chiqarish mumkin.

Gaussning (24) formulasida a ni natural k song'a teng deb, quyidagini topamiz:

$$C=\int_0^1 \frac{1-t^{k-1}}{1-t}dt-Dlog\Gamma(k).$$

Lekin

$$\frac{1-t^{k-1}}{1-t}=1+t+\cdots+t^{k-2},$$

demak,

$$\int_0^1 \frac{1-t^{k-1}}{1-t}dt=1+\frac{1}{2}+\cdots+\frac{1}{k-1}.$$

(45) formuladan foydalannb, $a=k$ da ushbuni hosil qilamiz:

$$C=1+\frac{1}{2}+\cdots+\frac{1}{k-1}-logk+\frac{1}{2k}+\frac{1}{12k^2}-\frac{1}{120k^4}+$$

$$+\frac{1}{252k^6}-\frac{1}{240k^8}+\cdots+(-1)^n\frac{B_n}{2n}\cdot\frac{1}{k^{2n}}+(-1)^{n+1}\theta'\cdot\frac{B_{n+1}}{2n+2}\frac{1}{k^{2n+2}}$$

$$(0<\theta'<1)$$

Bu formula bo‘yicha, $k=10$ deb va k^{12} li hadgacha hisoblab, *Eyler* C ning qiymatidi 15 –raqamgacha topgan:

$$C=\ 0{,}577\ 215\ 664\ 901\ 532\ldots$$

2) Γ funksiyaning o‘nli logarifmlari jadvalini tuzish. Bunday jadvalni tuzish uchun qisqa yo‘l ko‘rsatamiz.

(26) formulaga qaytamiz, unda a ni $a+1$ ga almashtirib, uni

$$\frac{dlog\Gamma(a+1)}{da} = -C + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{a+k}\right)$$

shaklda yozamiz. Ketma-ket differensiallash bilan n −hosila uchun

$$\frac{d^n log\Gamma(a+1)}{da^n} = (-1)^n(n-1)!\sum_{k=1}^{\infty}\frac{1}{(a+k)^n}$$

formulaga kelamiz (hosil qilinadigan qatorlarning tekis yaqinlaigishi hadlab integrallashning qonuniy ekanini ko‘rsatadi).

Shunday qilib, Teylor qatorining koeffitsientlarini topamiz:

$$\frac{1}{n!}\left[\frac{d^n \log\Gamma(a+1)}{da^n}\right]_{a=0} = (-1)^n\frac{S_n}{n}$$

bunda

$$S_n = \sum_{k=1}^{\infty}\frac{1}{k^n}.$$

U vaqtda $|a| < 1$ uchun quyidagiga ega bo‘lamiz:

$$log\Gamma(a+1) = -Ca + \frac{1}{2}s_2a^2 + \frac{1}{3}s_3a^3 + \frac{1}{4}s_4a^4 - \cdots$$

s_k sonlari (ayniqsa, katta k uchun) 1 ga yaqin turganidan, $|a| < 1$ da ham o‘rinli bo‘lgan

$$log(a+1) = a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \frac{1}{4}a^4 + \cdots$$

yoyilmani yuqoridagi yoyilmaga hadlab qo‘shish foydalidir, bu esa bizga quyidagini beradi:

$log\Gamma(a+1) =$

$$= -log(a+1) + (1-C)a + \frac{1}{2}(s_2 - 1)a^2 + \frac{1}{3}(s_3 - 1)a^3 + \cdots$$

Buni M modulga ko‘paytirib va

$$M(1-C) = C_1, \ \frac{1}{2}M(s_2 - 1) = C_2, \ \frac{1}{3}M(s_3 - 1) = C_3, ...$$

deb olib, ushbuni hosil qilamiz:

$$log_{10}\Gamma(a+1) = -log_{10}(a+1) + C_1 a + C_2 a^2 - C_3 a^3 + C_4 a^4 - \cdots \quad (46)$$

Bunda a ni $-a$ ga almashtiramiz va hosil bo‘lgan

$$log_{10}\Gamma(1-a) = -log_{10}(1-a) - C_1 a + C_2 a^2 + C_3 a^3 + C_4 a^4 + \cdots$$

yoyilmani avvalgidan ayiramiz. To‘ldirish formulasi bo‘yicha:

$$\Gamma(1-a)\Gamma(1+a) = \frac{a\pi}{sina\pi}$$

va

$$log_{10}\Gamma(1-a) = -log_{10}(1+a) + log_{10}\frac{a\pi}{sina\pi}$$

bo‘lganidan, mana buni topamiz:

$$log_{10}\Gamma(a+1) = \frac{1}{2}log_{10}\frac{a\pi}{sina\pi} - \frac{1}{2}log_{10}\frac{1+a}{1-a} +$$

$$-C_1 a - C_3 a^3 - C_5 a^5 - \cdots \qquad (47)$$

Lejandr C_n koeffitsientlarning qiymatlarini ($n \leq 15$ shartda) va ularning logarifmlarini berdi, so‘ngra (46) va (47) formulalardan foydalanib, va a ga 1 dan

2 gacha 0,001 oralatib qiymatlar berib, $\Gamma(a)$ ning logarifmlarini avval 7 ta va keyin 12 ta o‘nli raqamlar bilai hisobladi.

Xulosa

“Gamma” funksiyani o‘rganishni shu bilan tamomlab, shuni ko‘ramizki, bu funksiyaning a parametrga bog‘liq integral shaklida tasvirlanishidan foydalanib, biz uning chuqur xossalarinigina emas, balki uni hisoblashni ham o‘rgandik. Elementar funksiyalarni qay darajada o‘zlashtirgan bo‘lsak, bu yangi funksiyani ham shu darajada o‘zlashtirdik.

Umuman aytganda integrallarni (“Gamma” va “Betta” funksiyalari) oily matematikaning barcha bo’limlaridan turli masalalarni yechishda qo’llaniladi.

Ushbu “Bitiruv Malakaviy Ishda” da biz bu funksiyalarning ta’riflari, xossalari va ular bilan bog’liq funksiyalarni ko’rib o’tdik. Fikrimizcha biz ko’rib chiqqan masalalar bu funksiyalarni o’rganishimiz va tadbiq etishimizda muhim ahamiyatga ega. Men bu mavzuni o’rganishda bayon qilishim davomida 4 yil ichida o’qigan fanlarimni qaytarib chiqdim deb hisoblasam bo’ladi. Keyingi faoliyatlarimda bu mavzuni yana ham chuqurroq o’rganib, yangi tadbiqlarni topaman degan umiddaman.

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