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DIFFERENSIAL TENGLAMALARNING BA’ZI MASALALARINI YECHISHDA OPERATSION HISOBNING TADBIQLARI

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DISSERTATSIYASI

Ish ko‘rilib chiqildi va himoyaga ruxsat berildi.

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KIRISH

Masalaning qo‘yilishi. Differensial operatorlarning umumlashgan fundamental yechimlarini topish, Differensial tenglamalarning klassik bo‘lmagan yechimlari uchun ularning xossalarini taqsimot nuqtai nazari asosida o‘rganish.

Ishning dolzarbligi. Differensial tenglamalarning ba’zi masalalarini yechishda Laplas almashtirishlaridan foydalanib operatsion hisobni tadbiq qilish. Bir jinsli bo‘lmagan differensial tenglarga mos keluvchi differensial operatorlarning fundamental yechimlari fanda muhim ahamiyatga ega.

Ishning maqsadi va vazifalari. Ishning asosiy maqsadi operatsion hisobning usullarini o‘rganish, umumlashgan funksiyalar yordamida differensial operatorlarning fundamental yechimlarini topish, bu yechimlar yordamida bir jinsli bo‘lmagan tenglamalarni yechimini topish.

Ishning ilmiy tadqiqot metodlari. Ishning tadqiqot usullari operatsion hisobning usullarini o’rganish va ularni differensial tenglamalarga tadbiq etish.

Ishning ilmiy ahamiyati. Operatsion hisobning Laplas almashtrishlaridan foydalanib, differensial operatorlarning umumlashgan fundamental yechimlarini topish va ular yordamida bir jinsli bo‘lmagan tenglamalarni yechimlarini topish masalasi ilmiy ahamiyatga egadir.

Ishning amaliy ahamiyati. Ishni amaliy masalalarni echishdagi uni qulayligi nuqtai nazardan muhim ekanligi kelib chiqadi.

Ishning tuzilishi. Bu ish 3-bob va 10-paragrafdan iborat. Birinchi bobning paragraflarida Laplas almashtirishlari o‘rganildi. Ikkinchi bobning paragraflarida umumlashgan funksiyalar, differensial tenglamalarning fundamental yechimlarini topish o‘rganilgan.Uchinchi bobda umumlashgan funksiyalar uchun Laplas almashtrishlari va uning tadbiqlari o‘rganilgan.

II BOB.Laplas almashtirishlari

§1.1. Laplas almashtirishida original va tasvirlar

Matematik fizika tenglamalari uchun qo'yilgan masalalarni yechishda biz Fure qatorlari yoki Fure integrallari tushunchasiga kelamiz. Shu kabi tushunchalardan biri

$$F(s) = \int_a^b e^{-st} f(t)\,dt$$

ko'rinishli Laplas integrali bo'lib qo'yilgan masala yechimini osonlashtirishda muxim o'rin egallaydi. Laplas integralidagi integral xosmas bo'lganligi uchun uni yaqinlashishini aniqlash asosiy hisoblanadi. Bu integralda $f(t)$ funksiyaga original deb aytiladi va u quyidagi shartlarni qanoatlantirishi kerak

1) $f(t) = 0$, agar $t < 0$ bo'lsa bu $f(0) = f(+0)$

2) Shunday σva$M$ o'zgarmaslar mavjudki

$|f(t)| < Me^{\sigma t}$, agar $t > 0$.

3) Har bir chekli $[0,T]$ oraliqda $f(t)$ funksiya chekli sondagi $1-$ tur uzulishga ega bo'lishi mumkin.

$\sigma_0 = \inf_{\sigma>0} \sigma$songa$f(t)$ funksiyaning o'sish ko'rsatkichi deb aytiladi, agar $f(t)$ original bo'lsa, u holda $F(s)$ funksiya tasvir deb aytiladi. Bu tasvir xosmas integral sifatida$ReS \geq \sigma > \sigma_0$yarim tekslikda teks yaqinlashuvchi va

$Rep > \sigma > \sigma_0$tenglikda analitik funksiyadan iborat bo'ladi.

Agar $f(t)$original va$F(s)$ tasvir bo'lsa bu hol $f(t) \leftarrow F(s)$ kabi yoziladi. $f(t)$ni original bo'lishi uchun bu funksiyaning uning o'sish ko'rsatkichi σ_0 sonni topish mumkin.

Misol.$f(t) = a_n t^n + \ldots + a_1 t + a_0$bo'lsa uning o'sish ko'rsatkichini topamiz.

$\sigma > 0$bo'lganda

$$\lim_{t\to+\infty} \frac{a_n t^n + \ldots + a_1 t + a_0}{e^{\sigma t}} = 0$$

bo'lgani uchun

$$|a_n t^n + \ldots + a_1 t + a_0| < M(\sigma)e^{\sigma t}, \quad t > 0$$

bo‘ladi. Shuning uchun

$$\sigma_0 = \inf_{\sigma > 0} \sigma = 0$$

bo‘ladi.

Shuning uchun har bir ko‘phadningo‘sish darajasi nolga teng bo‘ladi.Davriy fizik hodisalarda $f(t)$ funksiyani garmonik funksiyalar yig‘indisi ko‘rinishda yozish Laplas integraliga olib keladi. Davri 2π ga teng bo‘lgan $f(t)$ funksiyaning Fure qatori

$$f(t) = a_0 + 2\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \qquad (1.1.1)$$

ko‘rinishda bo‘lib, uning Fure koeffitsentlari a_n va b_n lar quyidagi formulalar orqali topiladi.

$$\begin{cases} a_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\cos nt\,dt \\ b_n == \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\sin nt\,dt \end{cases} \qquad (n = 0,1,2,\ldots) \qquad (1.1.2)$$

Agar $f(t)$ funksiya har bir chekli intervalda chekli sondagi ekstremumlarga ega va chekli sondagi birinchi turdagi uzilish nuqlasiga ega bo‘lsa (Dirixle sharti) u holda (1.1.1) qator $f(t)$ funksiyaning uzluksizlik nuqtalarida funksiya qiymatlariga yaqinlashadi va uzilish nuqtalarida

$$\frac{f(t-0) + f(t+0)}{2}$$

shartni qanoatlantiradi.

Quyidagi belgilashlarni kiritamiz.

$$a_n = r_n \cos\varphi_n, b_n = r_n \sin\varphi_n \qquad (1.1.3)$$

bundan

$$r_n = \sqrt{a_n^2 + b_n^2},\ tg\varphi_n = \frac{b_n}{a_n}$$

natijada (1.1.1) formula

$$f(t) = r_0 + 2\sum_{n=1}^{\infty} r_n \cos(nt - \varphi_n) \quad (1.1.4)$$

ko‘rinishni oladi. $f(t)$ funksiya garmonik to‘lqinlar superpozissiyasi ko‘rinishida yozildi.

Fizikada uchraydigan funksiyalar biror differensial tenglamani qanoatlantirganligi uchun hisoblashlarni bajarishda (1.1.4) ning hosilasini olishga to‘gri kelali.Hosilani hisoblashni osonlashtirish maqsadida Fure qatorini kompleks ko‘rinishga keltirish maqsadga muvofiqdir. Kompleks ko‘rinishli Fure qatori

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{int} \quad (1.1.5)$$

ko‘rinishda bo‘lib, uning Fure koeffitsenti

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int} dt \quad (1.1.6)$$

formula orqali hisoblanadi. Yuqorida yozilgan (1.1.1) va (1.1.6) qatorlar haqiqi $f(t)$ funksiya uchun bir xil ekanligini ko‘rish qiyin emas. Haqiqatdan ham, bu holda C_0 va $C_{-n} = \overline{C_n}$ bo‘lgani uchun

$$f(t) = C_0 + \sum_{n=1}^{\infty} \left(C_n e^{int} + \overline{C_n} e^{-int}\right) = C_0 + 2\sum_{n=1}^{\infty} \mathrm{Re}\left(C_n e^{-int}\right)$$

agar biz

$$C_0 = a_0, \quad C_n = a_n - jb_n \qquad (n = 1,2,\ldots) \quad (1.1.7)$$

deb olsak oxirgi formuladan

$$f(t) = a_0 + 2\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

(1.1.6) formulaga asosan

$$a_n = \mathrm{Re}\, C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \cos nt\, dt$$

$$b_n = -Im C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

Shuning uchun (1.1.7) shartlarda (1.1.5) va (1.1.6) formulalar (1.1.1) va (1.1.2) ga teng kuchli ekan. Shunday qilib biz quyidagi xulosaga kelamiz. $(-\pi,\pi)$ oraliqdan tashqarida davriy davom ettirilgan $f(t)$ funksiyaning e^{int} $(n = 0, \pm 1, \pm 2, ..)$ ga nisbatan C_n spektri (1.1.6) integral yordamida topiladi. $f(t)$ ning spektri C_n bilish natijasida (1.1.5) formula orqali $f(t)$ ning o‘zini topish mumkin. Buning natijasida $f(t)$ orqali berilgan fizik hodisaning harakterini aniqlash mumkin.Endi kompleks ko‘rinishli (1.1.5) Fure qatoriga mos (1.1.4) ko‘rinishli qatorni keltiramiz. Buning uchun (1.1.7) va (1.1.3) tengliklardan foydalanamiz. Bu tengliklardan

$$C_0 = r_0,\ \varphi_0$$

$$C_n = r_n(\cos\varphi - i\sin\varphi_n) = r_n e^{-i\varphi_n} \quad (n = 1,2,..)$$

$$C_{-n} = \overline{C_n} = r_n e^{i\varphi_n}$$

bo‘lganligidan $r_{-n} = r_n,\ \ \varphi_{-n} = -\varphi_n$ bo‘lganliuchun $C_{-n} = r_{-n}e^{i\varphi(-n)}$ bo‘ladi.bulard an

$$f(t) = \sum_{n=\infty}^{\infty} r_n e^{i(nt-\varphi_n)} \qquad (1.1.8)$$

ko‘rinishliqatorhosilqilamiz.(1.1.5) qatordagi $C_n e^{int}$ had $C_n = r_n e^{i\varphi(n)}$ bo‘lgandakomplekstebranishnianiqlaydi, butebranishningchastotasi n, amplitudasi r_n vaboshlang‘ichfazosi φ_n daniborat.

$$\begin{aligned} &Demak,\ amplituda = |C_n| \\ &boshlang\acute{}ich\ \ faza = \arg C_n \end{aligned} \qquad (1.1.9)$$

Komplekstebranishlarning (1.1.5) ko‘rinishdagi C_n ketma–ketlikningqiymatlarinispektr $f(t)$ ningspektridebaytishmumkin.Buxulosalardank eyin (1.1.5) dagi $f(t)$ funksiyakompleksfunksiyabo‘ladi.

Agarqaralayotganfizikhodisadavriybo‘lmasa, uholda $f(t)$ funksiya t ning $(-\infty < t < \infty)$ oraliqdagibirorfunksiyasiuchunma'lums hartlarbajarilgandaFureqatorio‘rnigaquyidagiFureintegralimosqo‘yiladi.

$$f(t) = \int_{-\infty}^{\infty} F(y)\, e^{iyt} dy \qquad (1.1.10)$$

bunda $F(y)$ funksiya $f(t)$ funksiya bilan

$$F(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{iyt} dt \qquad (1.1.11)$$

formula orqali bog‘langan bo‘ladi. Bu integral tasvir o‘rinli bo‘lishi uchun $\int_{-\infty}^{\infty} |f(t)| dt$ integralning yaqinlashuvchi bo‘lishi kerak.Bu shart bajarilganda (1.1.10) integral yaqinlashuvchi bo‘ladi. Bu qator $f(t)$ ning uzilish nuqtasida ham yaqinlashuvchi va u Fure qatoridagidek $\frac{f(t-0) + f(t+0)}{2}$ qiymatga intiladi.(1.1.10) va (1.1.11) formulalar huddi (1.1.5) va (1.1.6) formulalardek hosil qilinadi. Bunda C_n koeffitsent o‘rniga $F(y)$, n o‘rniga y mos qo‘yiladi. Lekin $F(y)$ ni $f(t)$ spektri sifatida qarash mumkin emas. Chunki bunda har bir olingan tebranish $F(y)dy$ cheksiz kichik bo‘lgani uchun chekli amplitudaga ega emas. Shuning uchun (1.1.10) integral tasvirga fizik ma'no berish uchun $F(y)$ ga massa zichlik deb qaraymiz. Oy o‘qi bo‘yicha biror massa taqsimlangan bo‘lib, uning zichligi $F(y)$ bo‘isin. Bu holda dy chiziqli eliment cheksiz kichik $F(y)dy$ massaga ega, chekli massaga chekli oraliqga ega bo‘ladi. Bunday massa taqsimlanishini

$$\int_{-\infty}^{y} F(\eta)\, d\eta = G(y) \qquad (1.1.12)$$

ko‘rinishni olish mumkin. $G(y)$ funksiyani massa taqsimlanishining funksiyasi yoki taqsimlanish deb ataymiz. (1.1.12) formuladan

$$\frac{dG(y)}{dy} = F(y) \text{ yoki } F(y)dy = dG(y) \qquad (1.1.13)$$

tengliklarni yozish qiyin emas.

Taqsimlanish funksiyasini chekli oraliqda taqsimlangan massa deb qarash mumkin.Agar biz $f(t)$ ning spektrini massa taqsimlanish sifatida qarasak $G(y)$ funksiyani spektral taqsimlanish deb atash mumkin. $F(y)$ funksiyani esa spektral zichlik deb ataymiz. (l.1.l0) formulaga $f(t)$ funksiyaning spektral tasviri deb atash mumkin.Yoki (1.1.10) spektral taqsimlanish $G(y)$ funksiya orqali tasvirlash mumkin.

$$f(t)=\int_{-\infty}^{\infty} e^{iyt}dG(y) \qquad (1.1.14)$$

agar biz $r(y)=|F(y)|, -\varphi(y)=arg F(y)$ belgilashlar kiritsak $F(y)=r(y)e^{i\varphi(y)}$ ko‘rinishda yozish mumkin. Bularga asosan (1.10) o‘rinli

$$f(t)=\int_{-\infty}^{\infty} r(y)e^{i(yt-\varphi(y))}dy \qquad (1.1.15)$$

formulao‘rinli bo‘ladi. Demak, y chastotaga ega bo‘lgan tebranish

$$r(y)=|F(y)|dy$$

amplitudaga ega va uning boshlang‘ich fazosi $-\varphi(y)=arg F(y)$. Shuning uchun $r(y)=|F(y)|$ kattalikni amplituda zichligi deb ataymiz.

(1.1.1) da t ni $(-\infty,\infty)$ oraliqda deb qaragan edik. Lekin ko‘pchilik amaliy masalalarni yechishda biz $0\le t<\infty$ oraliqda qaraladigan hodisalar bilan ish ko‘ramiz. Agar biz $f(t)$ funksiyani $[0,\infty)$ dan $(-\infty,\infty)$ oraliqqa nol qilib davom ettirsak (1.1.11) formulada integralning quyi chegarasini nolga teng deb olish mumkin, yani

$$F=(y)=\frac{1}{2\pi}\int_{0}^{\infty} e^{-iyt}f(t)dt \qquad (1.1.16)$$

U holda $f(t)$ ning (1.1.10) ko‘rinishli spektral tasviri

$$\int_{-\infty}^{\infty} e^{iyt}F(y)dy=\begin{cases} f(t), & агар\ t>0, \\ 0, & агар\ t<0 \end{cases} \qquad (1.1.17)$$

natijada biz (1.1.16) integrating uzoqlashuvchi bo‘lgan holini chetlagan bo‘lamiz. $f(t)$funksiyao‘rniga $e^{-xt}f(t)$ parametirga bog‘liq so‘nuvchi funksiyani qaraymiz. Bu funksiyaning spektiral zichligini $F_x(y)$ deb belgilaymiz.

$$F_x(y)=\frac{1}{2\pi}\int_0^{\infty} e^{-iyt}\left[e^{-xt} f(t)\right]dt \qquad (1.1.18)$$

Harqanday chegaralangan $f(t)$ funksiyalar uchun (1.1.18) integral yaqinlashuvchi bo‘ladi.(1.1.18) integral $e^{\alpha t}$ $(\alpha > 0,\ x>\alpha)$ ko‘rinishlio‘suvchi funksiyalar uchun ham yaqinlashuvchi bo‘ladi.Shunday qilib $f(t)$ funksiya o‘rniga $e^{-xt} f(t)$ funksiyani kiritish natijasida biz (1.1.16) integralni yaqinlashuvchi qilishimiz mumkin.Bunday funksiyalar uchun (1.1.10) integral tasvir quyidagi ko‘rinishni oladi.

$$\int_{-\infty}^{\infty} e^{iyt} F_x(y)dy=\begin{cases} e^{-xt} f(t), & агар \quad t>0 \\ 0, & агар \quad t<0 \end{cases} \qquad (1.1.19)$$

(1.1.18) va (1.1.19) larni boshqacha yozish mumkin.

$$\frac{1}{2\pi}\int_0^{\infty} e^{-(x+iy)t} f(t)dt = F_x(y) \qquad (1.1.20)$$

$$\int_{-\infty}^{\infty} e^{(x+iy)t} F_x(y)dy=\begin{cases} f(t), & агар \quad t>0 \\ 0, & агар \quad t<0 \end{cases} \qquad (1.1.21)$$

agar biz $x+iy$ ni s orqali ifodalasak kompleks o‘zgaruvchili $s=x+iyF_x(y)$ funksiya hosil qilamizki

$$F_x(y)=F(x+iy)=F(s)$$

bo‘ladi.

(1.1.21) formulada $-\infty < y < \infty$ bo‘lib x parameter bo‘lgani uchun $S=x+iy$ o‘zgaruvchi $x-i\infty$ dan $x+i\infty$ gacha o‘zgaradi.Demak, s ning o‘zgarishi kompleks tekislikning vertikal x o‘qida sodir bo‘lar ekan. Shuning uchun $f(t)$ funksiyani $2\pi f(t)$ bilan almashtirib, $ds = idy$ ekanligini hisobga olganda (1.1.20) va (1.1.21) formulalarni quyidagicha yozamiz.

$$\int_0^{\infty} e^{-st} f(t)dt = F(s) \qquad (1.1.22)$$

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{-is}F(s)ds=\begin{cases} f(t), & агар \quad t>0 \\ 0, & агар \quad t<0 \end{cases} \qquad (1.1.23)$$

(1.1.22) ning chap tomoni Laplas integralidan iborat.(1.1.23) esa Laplas almashtirishideyiladi.

Shunday qilib (1.1.22) va (1.1.23) forrnulalarga quyidagicha fizik ma'no berish mumkin bo‘ladi.

Agar $F(s)$ kompleks o‘zgaruvchili $(s = x + iy)$ Laplas integralidan iborat bo‘lsa, u holda $F(x+iy)$ funksiya $e^{-xt} f(t)$ so‘nuvchi funksiyaning spektral zichligi bo‘ladi va bunday o‘zgaruvchili doiraviy chastotadan iborat. $f(t)$ funksiyani $F(s)$ spektral zichlik orqali (1.1.23) formula orqali topish mumkin. Agar biz (1.1.23) formulani spektral lasvir deb qaraydigan bo‘lsak uni quyidagicha yozish mumkin.

$$\int_{-\infty}^{\infty} e^{iyt} F(x+iy)\, ds = \begin{cases} 2\pi e^{-xt} f(t), & агар \quad t > 0 \\ 0, & агар \quad t < 0 \end{cases} \qquad (1.1.24)$$

(1.1.24) $2\pi e^{-xt} f(t)$ funksiyaning t ning manfiy qiymatlarida nol qilib davom etlirilgan spektral lasvirdan iborat.(1.1.24) dagi e^{-xt} chap tomonga o‘tkazamiz.

$$\int_{-\infty}^{\infty} e^{xt} e^{iyt} F(x+iy)\, dy = 2\pi f(t) \qquad (1.1.25)$$

§1.2. Laplas integralidan kelib chiqadigan funksiya xossalari

Agar x parametrning x_0 qiymatida $e^{-x_0 t} f(t)$ so‘nuvchi funksiyaning spektral zichlik mavjud bo‘lsa, u holda $x > x_0$ bo‘lgan kuchli so‘nishlar uchun ham bu spektral zichlik mavjud bo‘ladi. Haqiqatdan ham (1.1.22) integral $s = x_0 + iy$ da yaqinlashuvchi bo‘lsa, u holda bu integral $x > x_0$ bo‘lgandagi $s = x + iy$ lar uchun ham yaqinlashuvchi bo‘ladi. Bulardan (1.1.22) integralning yaqinlashish sohasi $\mathrm{Re}\, s > \beta$ bo‘ladi, bu sohaga yaqinlashish yarim tekisligi deb aytamiz.

Ko‘pchilik amaliy masalalarni yechishda $f(t)$ funksiya haqiqiy bo‘ladi.Bu hol uchun

$$F(\bar{s}) = \int_0^\infty e^{-\bar{s}t} f(t)dt = \overline{\int_0^\infty e^{-st} f(t)dt} = \overline{F(s)}$$

bundao’zgaruvchining haqiqiy o‘qqa nisbatan oyna aksidagi qiymatiga $F(s)$ funksiyaning bu o‘qdagi oyna aksidagi qiymatiga teng bo‘ladi. $F(s)$ funksiyaning asosiy xossalari y bitta yarim tekislikda aniqlangan bo‘ladi va bu yarim tekislikda y, bitta yarim tekislikda aniqlangan bo‘ladi va bu yarim tekislikda y analitik funksiyadan iborat, shuning uchun uni kompleks funksiya sifatida istalgancha differensiallash mumkin. $F(s)$ ning bu hosilalarini integral ostida hisoblanadi, yani

$$F(s)\ F^{(n)}(s) = (-1)^n \int_0^\infty e^{-st} t^n f(t)dt \tag{1.2.1}$$

shunday qilib, $F(s)$ funksiya uchun kompleks o‘zgaruvchili funksiyalarning asosiy usullarini qo‘llash mumkin.Bu asosda hosil qilingan $F(s)$ funksiya xossalaridan $f(t)$ ning muhim xossalarini ham hosil qilish mumkin.Bu muhim xossalarni keltirishdan oldin ba'zi $f(t)$ funksiyalar uchun $F(s)$ funksiyalarni topamiz.

Laplas integralini hisoblashda $f(t)$ ning qiymatini $t>0$ uchun aniqlash kifoya, $t<0$ uchun uining qiymatlarini qanday ekanligi bilish shart emas, shuning uchun $t<0$ lar uchun biz $f(t)=0$ deb olishimiz zarur.

Misol 1.2.1. $f(t)=1$ bo‘lsin, t< 0 bo‘lganda bu funksiyani 0 ga teng deb, biz funksiyani birlik sakrashga ega bo‘lgan funksiya deb olamiz.Shunday qilib,

$$u(t)=\begin{cases}1, & агар\ t>0\\ 0, & агар\ t<0\end{cases} \qquad (1.2.2.)$$

$t=0$ nuqtada $u(t)$ funksiya aniqlanrnagan,bu esa muhim emas. Ba'zi hollarda $u(0)=*$ yoki $u(0)=1$, $u(0)=\frac{1}{2}$ - deb olinadi. Bu funksiyaning spektiral zichligini topamiz

$$F(s)=\int_0^{\infty} e^{-st}u(t)dt=\int_0^{\infty} e^{-st}dt=-\frac{1}{s}e^{-t}I_0^{\infty}=\frac{1}{s} \qquad (1.2.3)$$

bu tenglik bajarilishi uchun $e^{-st}\to 0$, agar, ya'ni $\operatorname{Re}s>0$ bo‘lishi kerak.Demak, $2\pi e^{-xt}u(t)$ funksiyaning spektral zichligi $F(s)=\frac{1}{s}=\frac{1}{x+iy}$ ko‘rinishda bo‘ladi.Endi (1.2.3) dan Laplas almashtirishi (1.1.23) formulani qo‘llaymiz.

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{ts}\frac{1}{s}ds=u(t)=\begin{cases}1, & агар\ t>0\\ 0, & агар\ t<0\end{cases} \qquad (1.2.4)$$

yoki spektral tasvir ko‘rinishda yozadigan bo‘lsak

$$\int_{-\infty}^{\infty} e^{iyt}\frac{1}{x+iy}dy=2\pi e^{-xt}u(t)\ (x>0) \qquad (1.2.5)$$

formula hosil bo‘ ladi.

Misol 1.2.2. Birlik sakrash $t=a>0$ da bo‘lsin.Bu holda funksiya $u(t-a)$ ko‘rinishda yoziladi. Bu funksiya spektral zichligi

$$F(s) = \int_0^\infty e^{-st} u(t-a)\,dt = \int_a^\infty e^{-st} dt = -\frac{e^{-st}}{s} I_a^\infty = \frac{e^{-at}}{s}, \ \operatorname{Re} x > 0 \quad (1.2.6)$$

demak, vaqt funksiyasini a - ga intilishi spektral zichlikni a^{-as} - ga ko‘paytirish natajasida hosil bo‘larekan.

Misol 1.2.3. $f(t) = a^{\alpha t}$ (α - ixtiyoriy kompleks son)

$$f(t) = \begin{cases} e^{\alpha t}, & t > 0 \\ 0\ , & t < 1 \end{cases}$$

bunday funksiyani $u(t)e^{\alpha t}$ ko‘rinishda yozish mumkin. Agar $\alpha = \delta + i w$ bo‘lsa $w \neq 0$ va $\delta = 0$ bo‘lganda bu funksiya kompleks tebranishni bildiradi, agar $w \neq 0$ va $\delta > 0$ bo‘lsa o‘suvchi tebranishni, $w = 0$ bo‘lganda a - davri harakat bo‘ladi. uning spektral

$$\int_0^\infty e^{-st} e^{\alpha t} dt = \int_0^\infty e^{-(s-\alpha)t} dt = \frac{1}{s-a}, \ \operatorname{Re} s > R \qquad (1.2.7)$$

Misol 1.2.4. $f(t) = \cos wt$ bo‘lgan holni qaraymiz.Bu hol uchun $f(t) = u(t)\cos wt$ deyish mumkin. $\cos wt = \dfrac{e^{iwt} + e^{iwt}}{2}$ bo‘lgani uchun misol.3. ga ko‘ra

$$\int_0^\infty e^{-st} \cos wt\ dt = \frac{1}{2}\int_0^\infty e^{-st}\left(e^{iwt} + e^{-iwt}\right)dt = \frac{1}{2}\left(\frac{1}{s - iw} + \frac{1}{s^2 + w^2}\right) (1.2.8)$$

formulaga ega bo‘lamiz. Bu integral yaqinlashishi uchun. $\operatorname{Re} s > \operatorname{Re}(w) \operatorname{Re} s > re(-iw)$ bo‘lishi kerak.Ya'ni $\operatorname{Re} s > 0$ bo‘lishi kerak.

Misol 1.2.5. $f(t) = \sin wt$ ya’ni $f(t) = u(t)\sin wt$ bo‘lgan holni qaraymiz. Bunda ham xuddi misol.1.2.4 dagidek

$$\int_0^\infty e^{-st} \sin wt\ dt = \frac{1}{2i}\int_0^\infty e^{-st}\left(e^{iwt} + e^{-iwt}\right)dt =$$
$$= \frac{1}{2i}\left(\frac{1}{s - iw} + \frac{1}{s + w}\right) = \frac{w}{s^2 + w^2}, \quad \operatorname{Re} s > 0 \qquad (1.2.9)$$

formulaga kelamiz.

§1.3.0riginalga ko‘ra tasvirni topish

Teorema 1.3.1 Agar $f(t) \leftarrow F(p)$ bo‘lsa ,u holda $a>0$ bo‘lganda

$$f(at) \leftarrow \frac{1}{a}F\left(\frac{p}{a}\right) \qquad (1.3.1)$$

bo‘ladi.

Isbot. Laplas almashtirishiga ko‘ra

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt = at = \tau \quad bo'lsa \quad t = \frac{\tau}{a} \;\; dt = \frac{d\tau}{a}$$

$$= \int_0^\infty e^{-s\frac{\tau}{a}} f(\tau)\frac{d\tau}{a} = \frac{1}{a}\int_0^\infty e^{-s\frac{\tau}{a}} f(\tau)d\tau = \frac{1}{a}F\left(\frac{s}{a}\right)$$

shuning uchun $f(at) \leftarrow \frac{1}{a}F\left(\frac{s}{a}\right)$ formula o‘rinli. O‘xshashlik teoremasini $F(as) \rightarrow \frac{1}{a}f\left(\frac{t}{a}\right)$ moslik uchun ham isbotlash mumkin. Haqiqatdan ham

$$L\left\{\frac{1}{a}\left(\frac{t}{a}\right)\right\} = \int_0^\infty e^{-st}\frac{1}{a}f\left(\frac{t}{a}\right)dt$$

bu tenglikdan $\tau = \frac{t}{a}$ desak, $t = a\tau, dt = ad\tau$ bo‘ladi. Shuning uchun oxirgi integral qo‘yidagicha bo‘ladi.

$$\int_0^\infty e^{-st}\frac{1}{a} f\left(\frac{t}{a}\right)dt = \int_0^\infty e^{sa\tau} f(\tau)\,d\tau = F(as)$$

Demak,

$$F(as) \to \frac{1}{a} f\left(\frac{t}{a}\right) \qquad (1.3.2)$$

Teorema 1.3.2. (Siljish teoremasi) Agar $f(t) \leftarrow F(s)$ bo‘lsa, u holda $a > 0$ bo‘lganda

$$f(t-a) \leftarrow e^{as}F(s) \qquad (1.3.3)$$

bo‘ladi.

Isbot. $L\{f(t-a)\} = \int_0^\infty e^{-st} f(t-a)\,dt$ bu tengliklarda $z = 1-a$ desak $t = z + a, dt = dz$. Shuning uchun oxirgi integralni qo‘yidagicha yozish mumkin

$$\int_0^\infty e^{-st} f(t-a)\,dt = \int_0^\infty e^{-s(z-a)} f(z)\,dz = \int_0^\infty e^{-sz}e^{-sa} f(z)\,dz =$$

$$= e^{-sa}\int_0^\infty e^{-sz} f(z)\,dz = e^{-sa}F(s)$$

Demak,

$$f(t-a) \leftarrow e^{-as}F(s)$$

$$f(t) = t^2 \text{bo‘lsa}$$

$$f(t-1) = (t-1)^2 \leftarrow e^{-s}F(s)$$

(1.3.3) formulaga asosan $t^2 \leftarrow \frac{2!}{s^{2+1}} = \frac{2}{s^3}$ shuning uchun

$$(t-1)^2 \leftarrow e^{-s}\frac{2}{s^3}$$

$$(t-4)^2 \leftarrow e^{-4s}\frac{2}{s^3}$$

teorema-1.3.2dan foydalanib tasvirning jadvalini kengaytirish mumkin. Biz qo‘yida siljish teoremasining boshqacha ko‘rinishini ham keltiramiz.

Teorema1.3.3. (Ikkinchi siljish teoremasi) Agar $f(t) \leftarrow F(s)$ bo‘lsa u holda

$$f(t+a) \leftarrow e^{as}\left(F(s) - \int_0^a e^{-st} f(t)dt\right) \quad a>0 \quad (1.3.4)$$

munosabato‘rinli.

Isbot. $L\{f(t+a)\} = \int_0^\infty e^{-st} f(t+a)dt$

$t+a=z$ desak $t=z-a \qquad dt=dz$ Shuning uchun

$$\int_0^\infty e^{-st} s(t+a)dt = \int_0^\infty e^{-s(z-a)} f(t+a)dz = \int_0^\infty e^{-s(z-a)} f(z)dz -$$

$$-\int_0^a e^{-s(z-a)} f(z)dz = e^{sa}\int_0^\infty e^{-sz} f(z)dz - e^{sa}\int_0^a e^{-sz} f(z)dz =$$

$$= e^{sa}\left[\int_0^\infty e^{-sz} f(z)dz \int_0^a e^{-sz} f(z)dz\right] = e^{sa}\left[F(s) - \int_0^a e^{-sz} f(z)dz\right]$$

teorema isbot bo‘ ldi.

Teorema1.3.4. (So‘nish teoremasi)Agar $f(t) \leftarrow F(s)$ bo‘lsa, u holda istalgan$a$ da

$$e^{-at} f(t) \leftarrow F(s+a) \qquad (1.3.5)$$

bo‘ladi.

Isbot.

$$L\{e^{-at}f(t)\}=\int_0^\infty e^{-st}e^{-at}f(t)dt=\int_0^\infty e^{-(s+a)t}f(t)dt=F(s+a)$$

demak, original $f(t)$ ni e^{-at} ga ko‘paytirish tasvirda s - erkli o‘zgaruvchini a - ga siljitishga olib kelar ekan.

Misol 1.3.1 $f(t)=e^{-at}\sin wt$ originalning tasvirini toping.

$$\sin wt \to \frac{w}{s^2+w^2}$$

bo‘lgani uchun teorema-1.3.3- ga asosan

$$e^{-at}\sin wt \to \frac{w}{(s+a)^2+w^2}$$

Teorema1.3.5.(Parametr bo‘yicha differensiallash)Agar $f(t,x)\to F(s,x)$ bo‘lsin, u holda

$$\frac{\partial f(t,x)}{\partial x}\to\frac{\partial F(s,x)}{\partial x} \qquad (1.3.6)$$

bo‘ladi.

Isbot. $L\{f(t,x)\}=\int_0^\infty e^{-st}f(t,x)dt=F(s,x)$ bo‘ lgani uchun

$$\frac{\partial F(s,x)}{\partial x}=\int_0^\infty e^{-st}\frac{\partial f(t,x)}{\partial x}$$

bo‘ladi. Bundan

$$\frac{\partial f(t,x)}{\partial x}\leftarrow\frac{\partial F(p,x)}{\partial x}$$

hosil bo‘ladi. Teorema isbot bo‘ldi.

Misol 1.3.2. $f(t)=t^n a^{at}$ original tasvirini toping.

Yechish. $e^{at} \leftarrow \dfrac{1}{p-a}$ bo‘lgani uchun teorema-5 dan foydalanidigan bo‘lsak oxirgi moslikning har ikkala tomonini a bo‘yicha differensiallab, qo‘yidagilarni hosil qilamiz.

$$te^{at} \leftarrow \frac{1}{(p-a)^2}; \quad t^2e^{at} \leftarrow \frac{2!}{(p-a)^3};$$
$$t^3e^{at} \leftarrow \frac{3!}{(p-a)^4};\ldots,\ t^ne^{at} \leftarrow \frac{n!}{(p-a)^{n+1}};$$

Teorema 1.3.6.(Original uchun differnsiallash teoremasi) Agar $f(t) \leftarrow F(p)$ bo‘lib $f'(t)$ ham original bo‘lsa, u holda

$$f'(t) = pF(p) - f'(0) \tag{1.3.7}$$

Isbot. $L\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt$ integralni bo‘laklab integrallaymiz.

$u = e^{-st}$ desak, $du = -se^{-st}dt$, $dv = f'(t)dt$ uchun $v = f(t)$ bo‘ladi. Shu sababli

$$f'(t) \leftarrow f(t)e^{-st}\Big|_0^\infty + S\int_0^\infty f(t)e^{-st}dt$$

oxirgi tenglikdan

$$f'(t) \leftarrow f(s) - f(0)$$

bu teoremani takrorlab $f''(t), f'''(t),\ldots, f^{(n)}(t)$ originallarning tasvirini aniqlaymiz

$$\begin{aligned}
&f''(t) \leftarrow s[sF(s) - f(0)] - f'(0) = s^2F(s) - sf(0) - f'(0) \\
&f'''(t) \leftarrow s[s^2F(s) - sf(0) - f'(0)] - f''(0) = s^3F(s) - s^2f(0) - sf'(0) - f''(0) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&f^{(n)}(t) \leftarrow s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \ldots - f^{(n-1)}(0)
\end{aligned} \tag{1.3.9}$$

agar $f(0) = f'(0) = \dots = f^{(n-1)}(0) = 0$ bo‘lsa

$$f'(t) \leftarrow sF(s)$$
$$f^{(n)}(t) \leftarrow s^n F(s) \qquad (1.3.10)$$

munosabatlarga kelamiz.

Biz teorema-1.3.6 da $f'(t)$ funksiya originalga ega bo‘lgan holni o‘rgandik.Bu holda $f(t)$ ham originalga ega bo‘ladi. Lekin teskari har doim to‘g‘ri bo‘lavermaydi. $f(t)$ funksiyaning Laplas almashtirishi mavjud bo‘lsa, $f'(t)$ funksiya tasvirga ega bo‘lmasligi mumkin. Masalan. $\ln t$ funksiya uchun hosilasi $1/t$ uchun $L\{1/t\}$ mavjud emas. Agar biz har bir $t > 0$ da $t^n(t)$ hosilaning mavjudligiga e'tibor berilmasdan tasvirni izlaydigan bo‘lsak noto‘g‘ri natijalarga kelishimiz mumkin.

Birlik sakrashga ega bo‘lgan $u(t-a)$ funksiya uchun tasvirni topishga to‘xtalamiz.

$$u(t-a) = \begin{cases} 1, & agar \quad t > 0 \\ 0, & agar \quad t < 0 \end{cases}$$

ko‘rinib turibdiki $u(t-a)$ funksiya $t > 0$ da differensiallanuvchi emas. Bunga e’tibor bermasdan teorema.1.2.6 ni qo‘llaymiz.

$$F(s) = L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\,dt = \int_0^\infty e^{-st}\,dt = \frac{e^{-as}}{s}, \qquad \operatorname{Re} s > 0$$

bo‘lganligi uchun

$$L\{u'(t-a)\} = sF(s) - f(+0) = e^{-as} \qquad (1.3.11)$$

Endi bu tasvirni topish uchun $u'(t-a)$ ni topib, unga Laplas almashtirishini qo‘llaymiz.

$$u'(t-a)=\begin{cases} 0, & agar\ t\neq a \\ aniqlanmagan, & agar\ t=a \end{cases} \qquad (1.3.12)$$

shuning uchun

$$L\{u'(t-a)\}=0 \qquad (1.3.13)$$

chunki $u'(t-a)$ funksiya faqat bitta nuqtada aynan nolga teng bo‘lgan funksiyadan farqlanayapti.Bunday funksiyalarning integrallari teng. Demak, biz $u(t-a)$ funksiya hosilasining tasviri uchun ehtiyotsizlik bilan teorema.1.3.6 ni qo‘llab, hosil qilgan (1.3.11) natijamiz noto‘g‘ridir.

Biz (1.3.10) ni hosil qilganimizda $f(t)$ funksiyaning $f^n(t)$ hosilasi $t>0$ da mavjud ekanligini talab qildik. $t=0$ nuqtada $f^n(t)$ hosila mavjud bo‘lmagan hol ham uchraydi. Bu holda ham bu formula o‘rinli bo‘ladi.

$$f(t)=t^{\frac{1}{2}} \qquad bo'lsa, \qquad f'(t)=\frac{1}{2}t^{-\frac{1}{2}}, \quad f(+0)=0$$

mavjud emas. Bu funksiya uchun

$$L(f)=\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}, \quad L(f')=\frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}, \quad \Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

bo‘lgani uchun

$$\frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}=s\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}$$

ko‘rinishli to‘g‘ri natijaga kelamiz.

Teorema.1.3.7.(Tasvirni differensiallash teoremasi) Agar $f(t)\leftarrow F(s)$ bo‘lsa, u holda

$$-t\,f(t)\leftarrow F'(s) \qquad (1.3.14)$$

formulao‘rinli.

Isbot. (1.3.14) formulani isbotlash uchun

$$F(s)=\int_0^\infty f(t)e^{-st}dt$$

funksiyani s bo yicha differensiallaymiz.

$$F'(s)=\int_0^\infty f(t)(-t)e^{-st}dt$$

bundan

$$-t\,f(t)\leftarrow F'(s)$$

munosabat kelib chiqadi.

Shunday qilib, tasvirni differensiallash uchun, originalni $-t$ ga ko‘paytirish kerak ekan.(1.3.14) formula tadbiqiga doir misol keltiramiz.

$f(t)=1$bo‘lsa

$$1\leftarrow\frac{1}{s}$$

bu munosabatni o‘ng tomonini s bo‘yicha differensiallaymiz.

$$-t\leftarrow-\frac{1}{s^2}\quad yoki\quad t\leftarrow\frac{1}{s^2}$$

$$-t^2\leftarrow-\frac{2}{s^3}\qquad yoki\quad t^2\leftarrow\frac{2}{s^3},\ldots,$$

$$t^n\leftarrow\frac{n!}{s^{n+1}}$$

oxirgi formulani biz (1.3.10) formulada ham keltirgan edik.

Teorema.1.3.8.(Originalni integrallash haqidagi teorema) Agar $f(t)$ funksiyaning tasviri $F(s)$ bo‘lsa, u holda

$$\int_0^t f(t)dt \leftarrow \frac{F(s)}{s} \qquad (1.3.15)$$

munosabato‘rinli bo‘ladi.

Isbot. $f(t)$ funksiya tasvirga ega bo‘lganligi uchun uning integrali ham tasvirga ega. Shuning uchun biz

$$\int_0^t f(\tau)d\tau \leftarrow P(s)$$

desak, bu munosabatga $\int_0^t f(\tau)d\tau$ tasvir uchun differensiallash formulasi (1.3.7) qo‘llaymiz

$$\left(\int_0^t f(\tau)d\tau\right)' \leftarrow sp(s) - \int_0^0 f(\tau)d\tau$$

oxirgi munosabatdan

$$\left(\int_0^t f(\tau)d\tau\right)'_t = 0 \quad \int_0^0 f(\tau)d\tau = 0$$

bo‘lgani uchun $f(t) \leftarrow Sp(s)$ munosabat hosil bo‘ladi. Endi $F(t) \leftarrow F(s)$ ekanligini hisobga oladigan bo‘lsak

$$Sp(s) = F(s) \text{ yoki } p(s) = \frac{F(s)}{s}$$

bu (1.3.8)-teoremaning isboti.

***Teorema 1.3.9.* (**Tasvirni integrallash teoremasi)

Agar $\int_s^{\infty} F(z)dz$ integral yaqinlashuvchi bo‘lsa va $f(t) \leftarrow F(s)$ bo‘lsa u holda

$$\frac{f(t)}{t} \leftarrow \int_s^{\infty} F(z)dz$$

munosabato‘rinlidir. Bu qoidaga asosan tasvirni (s,∞) oraliqda integrallash uchun orginalni t-ga bo‘lishga mos kelarekan.

Isbot. Tasvirni differensiallash haqidagi teorema (1.3.8)-da tasvirni differensiallashga originalni –t ga ko‘paytirishga teng edi.Tasvirning integrallash amali teorema-(1.3.8) dagi tasvirni differensiallashga teskari amal bo‘lgani uchun tasvirni integrallash originalni t-ga bo‘lishga mos bo‘ladi.Bu qatordan foydalanib tasvir jadvalini kengaytirish mumkin.

Bu paragrafda qo‘llanilayotgan amallar bitta funksiyaga original bo‘lgan hol uchun qaraladi. Endi bir nechta funksiyalar kombinatsiyasi uchun qo‘llanaladigan amallarga to‘xtalamiz.Bir nechta originallar yigindisining tasviri harbir originallar tasvirlarning yig‘indisiga tengligiga ishonch hosil qilish mumkin. Tasvirlar ko‘paytmasini olish uchun orginallarning qanday

kombinasiyasini olish kerakligini aniqlaymiz.Buning uchun originallarning qo‘yidagi integral kombinasiyasini aniqlaymiz

$$\int_0^t f_1(\tau) f_2(t-\tau) d\tau \quad (1.3.16)$$

bu integral kombinatsiyasiga $f_1(t)$ va $f_2(t)$ larning o‘ramasi deb aytiladi, hamda uni $f_1 * f_2$ kabi yoziladi.O‘rama ham xuddi ko‘paytmagidek komutativlik hamda assosiativlik xossasiga bo‘ysinadi $f_1 * f_2 = f_2 * f_1$ yoki

$$\int_0^t f_1(\tau) f_2(t-\tau)\,d\tau = \int_0^t f_2(\tau) f_1(t-\tau)\,d\tau \qquad (1.3.17)$$

tenglikga ega bo‘lamiz. Bu keltirilganlarga asosan ushbu teoremani keltiramiz.

Teorema1.3.10 (Borel teoremasi).Agar $f(t) \leftarrow F(s)$, $g(t) \leftarrow G(s)$ bo‘lsa, u holda tasvirlarning ko‘paytmasi uchun originallar o‘ramasi mos keladi, ya'ni

$$f * g \leftarrow F(s)G(s) \qquad (1.3.18)$$

Isbot.Funksiyalar o‘ramasi ichida Laplas integralini yozamiz.

$$L(f \times g) = \int_0^\infty \left[\int_0^t f(\tau) g(t-\tau)\,d\tau \right] e^{-st}\,dt \qquad (1.3.20)$$

endi (1.3.20) takroriy integralda integrallash tartibini o‘zlashtiramiz.

$$\int_0^\infty dt \int_0^t f(\tau) g(t-\tau) e^{-st}\,dt = \int_0^\infty d\tau \int_0^\infty f(\tau) g(t-\tau) e^{-st}\,dt$$

o‘ng tomonidagi ichki integralda $t-\tau = t_1,\ dt = dt_1$ belgilash kiritamiz. Natijada

$$f \times g = \int_0^\infty f(\tau) f(\tau) e^{-st}\,d\tau \int_0^\infty g(t_1) e^{-st_1}\,dt_1$$

tenglikni hosil qilamiz. Bundan

$$f * g \leftarrow F(s)G(s)$$

munosabatga olib keladi.

Misol1.3.3 $F(s) = \dfrac{sw}{(s^2+w^2)^2}$ funksiyaga mos originalni o‘rama yordamida aniqlang. Buning uchun $F(s)$ ko‘paytma ko‘rinishda yozamiz.

$$F(s)=\frac{s}{s^2+w^2}\cdot\frac{w}{s^2+w^2}=F_1(s)\cdot F_2(s)$$

tasvirlar jadvaliga asosan

$$F_1(s)=\frac{s}{s^2+w^2}\to f(t)=\cos wt,$$

$$F_2(s)=\frac{s}{s^2+w^2}\to g(t)=\sin wt.$$

teorema.1.3.9 ga asosan

$$F(s)=F_1(s)\cdot F_2(s)\to f*g$$

shuning uchun

$$F(s)\to\int_0^t \sin w\tau\cos w(t-\tau)d\tau=\frac{1}{2}\int_0^t[\sin wt+\sin(2w\tau-wt)]d\tau=$$
$$=\frac{1}{2}t\sin wt-\frac{1}{2w}[\cos wt-\cos wt]d\tau=\frac{t\sin wt}{2}$$

Demak,

$$F(s)=\frac{sw}{(s^2+w^2)^2}\to\frac{t\sin wt}{2}$$

§1.4. Tasvirga ko‘ra originalni topish

Tasvir bo‘yicha originalni topishda tasvirlar jadvalidan va qo‘yidagi yoyish teoremasidan foydalanamiz.

Teorema.1.4.1.(Yoyishning birinchi teoremasi)Agar $f(t)$ funksiyaning $F(s)$ tasvirni $1/s$ ning darajalari bo‘yicha $\frac{1}{|s|} < R$ yaqinlashuvchi

$$F(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \ldots + \frac{a_n}{s^{n+1}} + \ldots \qquad (1.4.1)$$

qatorga yoyish mumkin bo‘lsa, u holda $f(t)$ originalning quyidagi formula bo‘yich topish mumkin.

$$f(t) = a_0 + a_1\frac{t}{1!} + a_2\frac{t^2}{2!} + \ldots + \frac{t^n}{n!} + \ldots. \qquad (1.4.2)$$

bu qator $t > 0$ da yaqinlashuvchi va $t < 0$ bo‘lganda $f(t) = 0$ deb olinadi.

Yuqorida keltirilgan teoremalardan va asosiy elementlar funksiyalar tasviri jadvaldan foydalanib ba’zi funksiyalarning tasvirini topamiz.

Isbot.Teoremani isbotlash uchun quyidagi uchta shartlarning bajarilishini ko‘rsatishimiz kerak.

a) (1.4.2) qator ham barcha t larda yaqinlashuvchi;

b) Uning $f(t)$ yig‘indisi $|f(t)| \leq Me^{s_0 t}$ shartni qanoatlantiradi.

c) $f(t)$ va $F(s)$ funksiyalar orasida

$f(t) \leftarrow F(s)$

operatsion moslik o‘rinli.

(1.4.2) qator $F(s)$ funksiyaning Loran qatoridan iborat. Bu qatorning $|s| > \frac{1}{R}$ da yaqinlashuvchi bo‘lgani uchun, yaqinlashish sohasidagi ixtiyoriy s da qator yaqinlashishining zaruriylik shartiga asosan

$$\lim_{n\to\infty} \frac{|a_n|}{|s|^{n+1}} = 0$$

bo‘lishi kerak. Shuning uchun $|s| \geq \frac{1}{R_n} \geq \frac{1}{R}$ shartni qanoatlantiradigan istalgan s uchun

$$\frac{a_n}{|s|^{n+1}} \leq |a_n| R^{n+1} \leq M \qquad (1.4.3)$$

tengsizlik bajarilishi lozim.(1.4.3) tengsizliklardan (1.4.2) qator koeffisentlarining modullari uchun

$$|a_n| \leq \frac{M}{R^{n+1}}$$

tengsizliklaro‘rinlidir. R_1 ni R ga istalgancha yaqin qilib tanlab $|a_n|$ uchun

$$|a_n| \leq \frac{M}{R^{n+1}}$$

baholashlarni hosil qilamiz. Shunday qilib biz (1.3.2) qator hadlari uchun $t > 0$ bo‘lganda

$$\left| a_n \frac{t^n}{n!} \right| \leq \frac{M}{R^{n+1}} \cdot \frac{t^n}{n!}$$

baholashlarni hosil qilamiz. Endi

$$\sum_{n=0}^{\infty} \frac{M}{R^{n+1}} \cdot \frac{t^n}{n!} = \frac{M}{R} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t}{R} \right)^n$$

qator barcha t lar uchun yaqinlashganidan uning yig‘indisi uchun $\frac{M}{R} e^{\frac{t}{R}}$ funksiyani topamiz. Bu hol (1.4.2) qatorni barcha t lar uchun yaqinlashishini va uning $f(t)$ yig‘indisini absalyut qiymati bo‘yicha majorant qator yigindisidan ortiq emasligini isbotlashdek, yani

$$|f(t)| < \frac{M}{R} e^{\frac{t}{R}}$$

bular asosida $f(t)$ funksiyaning a) va b) shartlarni qanoatlantirishini ko‘ramiz.

s) shartni bajarilishi uchun Laplas almashtirishining chiziqliligidan foydalanishimiz etarli. Yani

$$\sum_{n=0}^{k} \frac{a_n}{s^{n+1}} \to \sum_{n=0}^{k} \frac{a_n t^n}{n!}$$

bu munosabatda $|s| > \frac{1}{R}$ uchun $k \to \infty$ da limitga o‘tsak

$$F(s) \to f(t)$$

munosabat isbotlanadi.

Teorema1.4.2. (Yoyishning ikkinchi teoremasi)Agar $F(s)$ bir qiymatli funksiya bo‘lib, bu funksiya tenglamaning chekli qismida chekli sondagi $s_1, s_2,...,s_r$ maxsus nuqtalarga ega bo‘lsa, u holda

$$f(t)=\sum_{k=1}^{n}\operatorname{Res}_{s}\left[e^{st}F(s); S_k\right] \qquad (1.4.4)$$

formulao‘rinli. Agar $F(s)=\frac{P_m(s)}{Q_n(s)}$ ko‘rinishda bo‘lsa, bunda $P_m(s), Q_n(s)$ lar ko‘phadlardan iborat bo‘lib $(m<n)$ $s_1, s_2,...,s_r$ nuqtalar $Q_n(s)$ ning $l_1, l_2,...,l_r$ karrali nollari bo‘lsa $(l_1+l_2+...+l_r=n)$ bu holda

$$f(t)=\sum_{k=1}^{r}\frac{1}{(l_k-1)!}\lim_{s\to s_k}\left[(s-s_k)^{l_k}F(s)e^{st}\right]^{(l_k-1)} \qquad (1.4.5)$$

xususiy holda $s_1, s_2,...,s_r$ lar Q_n ning oddiy nollari bo‘lsa, u holda

$$f(t)=\sum_{k=1}^{n}\frac{P_m(s_k)}{Q_n'(s_k)}e^{s_k t} \qquad (1.4.6)$$

formulao‘rinli bo‘ladi.

Misol 1.4.1. $F(s)=\frac{1}{s}\cos\frac{1}{s}$ funksiya originalni topamiz.Buning uchun $\frac{1}{s}e^{st}\cos\frac{1}{s}$ funksiyani $s=0$ nuqtadagi chegirmasini topamiz.Bu chegirma funksiyaning Loran qatoriga yoyilmasining $\frac{1}{s}$ qatnashgan hadning koeffisiyentidan iborat. Shuning uchun $\frac{1}{s}e^{st}\cos\frac{1}{s}$ funksiyaning Loran qatoriga yoyamiz.

$$\frac{1}{s}e^{st}\cos\frac{1}{s}=\frac{1}{s}\left(1+st+\frac{s^2t^2}{2!}+\ldots+\frac{s^nt^n}{n!}+\ldots\right)\cdot$$
$$\cdot\left(\frac{1}{s}-\frac{1}{2!s^2}+\frac{1}{4!s^4}+\ldots+(-1)^n\frac{1}{(2n)!s^{2n}}\right)=$$
$$\left(1+st+\frac{s^2t^2}{2!}+\frac{s^3t^3}{3!}+\frac{s^4t^4}{4!}+\ldots+\frac{s^nt^n}{n!}+\ldots\right)\cdot$$
$$\cdot\left(\frac{1}{s}-\frac{1}{2!s^3}+\frac{1}{4!s^5}+\ldots+(-1)^n\frac{1}{(2n)!s^{2n+1}}\right)$$

bu ko‘paytmadan $\frac{1}{s}$ koeffisenti $f(t)=1-\frac{t^2}{(2!)^2}+\frac{t^4}{(4!)^2}+\ldots+(-1)^n\frac{t^{2n}}{((2n)!)^2}$ funksiyadan iborat.Shunday qilib $f(t)=\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{(n!)^2}$ ko‘rinishga ega.

Misol 1.4.2. $F(s)=\sin\frac{1}{s}$ funksiyaning originalini toping. Buning uchun $e^{st}\sin\frac{1}{s}$ funksiyaning $s=0$ nuqtadagi chegirmasini topish kerak.Bu chegirma funksiya Loran qatorining $\frac{1}{s}$ hadining koeffisentidan iborat.Shuning uchun $e^{st}\sin\frac{1}{s}$ funksiya Loran qatorini topamiz.

$$e^{st}\sin\frac{1}{s}=\left(1+st+\frac{s^2t^2}{2!}+\frac{s^3t^3}{3!}\ldots+\frac{s^nt^n}{n!}+\ldots+\right)$$
$$\cdot\left(\frac{1}{s}-\frac{1}{3!s^3}+\frac{1}{5!s^5}+\ldots+(-1)^n\frac{1}{(2n+1)!s^{2n+1}}\right)$$

bu ko‘paytmadan

$$f(t)=1-\frac{t^2}{2!3!}+\frac{t^3}{4!5!}+\ldots+\frac{t^{2n}}{(2n)!(2n+1)!}$$

yoki

$$f(t)=\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!(2n+1)!}$$

Misol 1.4.3. $F(s)=\frac{1}{s}e^{\frac{1}{s^2}}$ tasvirning originalini toping. Huddi oldingi misoldagidek $\frac{1}{s}e^{st}e^{\frac{1}{s^2}}$ funksiyaning Loran qatoriga yoyishimiz kerak.

$$\frac{1}{s}e^{st}e^{\frac{1}{s^2}}=\left(1+st+\frac{s^2t^2}{2!}+\ldots+\frac{s^nt^n}{n!}+\ldots\right)\cdot$$
$$\cdot\left(\frac{1}{s}+\frac{1}{s^3}+\frac{1}{2!s^5}+\ldots+\frac{1}{n!s^{2n+1}}+\ldots\right)=$$
$$=1+\frac{t^2}{2!}+\frac{t^4}{4!2!}+\ldots\frac{t^{2n}}{n!(2n)!}+\ldots=\sum_{n=0}^{\infty}\frac{t^{2n}}{n!(2n)!}$$

Misol 1.4.4. $F(s)=\frac{s^2+s+1}{s(s^4-1)}$ tasvirning originalini toping. Bunda $F(s)=\frac{P_2(s)}{Q_5(s)}$ ko‘rinishda bo‘lib $P_2(s)=s^2+s+1$ va $Q_5(s)=s(s^4-1)$ dan iborat. Tasvir maxrajini nollari $s_1=0,\ s_2=1,\ s_3=-1,\ s_4=i,\ s_5=-i$. Bu holda $F(s)$ tasvir yoyilmasi

$$F(s)=\frac{A_1}{s}+\frac{A_2}{s-1}+\frac{A_3}{s+1}+\frac{A_4}{s-i}+\frac{A_5}{s+i}$$

bunda $A_1,\ A_2,\ A_3,\ A_4$ va A_5 koeffisentlar (1.4.6) formulaga asosan $A_i=\frac{P(s_i)}{Q'(s_i)}$ sonlardan iborat.

$$A_1=\frac{P(s_i)}{Q'(s_i)}=\frac{P(0)}{Q'(0)}=-1,\quad A_2=\frac{P(1)}{Q'(1)}=\frac{3}{4},$$
$$A_3=\frac{P(-1)}{Q'(-1)}=\frac{1}{4},\quad A_4=\frac{P(i)}{Q'(i)}=\frac{i}{4},\quad A_5=\frac{P(-i)}{Q'(-i)}=-\frac{i}{4}$$

shuning uchun (1.3.6) formuladan

$$f(t) = -1e^{0t} + \frac{3}{4}e^{1\cdot t} + \frac{1}{4}e^{-1\cdot t} + \frac{i}{4}e^{1\cdot t} - \frac{i}{4}e^{-1\cdot t} =$$

$$= -1 + \frac{1}{4}\left(3e^{t} + e^{-t}\right) - \frac{1}{2}\cdot\frac{e^{it} - e^{-it}}{2i} = -1 + \frac{1}{4}\left(3e^{t} + e^{-t}\right) - \frac{1}{2}\sin t$$

Misol1.4.5. $F(s) = \dfrac{1}{s^4 - 1}$ funksiyaning originalini toping. Bu funksiyaning maxsus nuqtalarini topamiz. $s^4 - 1 = 0$ tenglamaning nollari $s_{1,2} = \pm 1,\ s_{3,4} = \pm i$. Maxsus nuqtalar oddiy bo‘lganligi uchun (1.3.6) formulalardan foydalanish mumkin.

Bunda $P_M(s) = 1,\ Q_M(s) = S^4 - 1,\ Q_M^{'}(s) = 4s^3$ shuning uchun

$$F(s) = \sum_{k=1}^{n}\frac{1}{s_k^3} = \frac{1}{4}\left(e^{t} - e^{-t} + \frac{e^{it}}{j^3} + \frac{e^{-it}}{(-i)^3}\right) = \frac{1}{2}\frac{e^{t} - e^{-t}}{2} -$$
$$-\frac{1}{2}\frac{e^{it} - e^{-it}}{2i} = \frac{1}{2}\left(sh\,t - \sin t\right)$$

Misol 1.4.6. $F(s) = \dfrac{s}{s^2 + 4s + 5}$ funksiyaning originalini toping. $F(s)$ ning maxsus nuqtalarini topamiz.

$$s^2 + 4s + 5 = 0\quad D_1 = 4 - 5 = -1,\quad s = -2 \pm i,\quad s_1 = -2 - i,$$
$$s_2 = -2 + i,\quad P_1(s) = s,\quad Q_2(s) = s^2 + 4s + 5,\quad Q'(s) = 2s + 4$$

Maxsus nuqtalarning oddiy bo‘lganligi uchun (1.3.6) formuladan foydalanamiz.

$$f(t) = \sum_{k=1}^{2}\frac{P_1(s_K)}{Q_2^{'}(s_K)}e^{s_k t} = -\frac{2 - i}{2(-2 - i) + 4}e^{(-2-i)t} + \frac{-2 + i}{2(-2 + i) + 4}e^{(-2+i)t} = e^{-2t}\left[\frac{2 + i}{2i}e^{-it} + \frac{-2 + i}{2i}e^{it}\right] =$$
$$= e^{-2t}\left[e^{it} + e^{-it} - 2\frac{e^{it} - e^{-it}}{2i}\right] = e^{-2t}\left(\cos t - 2\sin t\right)$$

natijada originalning ko‘rinishini topish oson

Misol 1.4.7. $F(s)=\dfrac{1}{s^2+2s+5}$ tasvirning originalini toping.

s^2+2s+5 ko‘phadlarning ildizlari -1+2i, $-1-2i$ bo‘lgani uchun uning sodda kasrlarga yoyilmasi

$$\frac{1}{s^2+2s+5}=-\frac{1}{4i}\left[\frac{1}{s+1+2i}-\frac{1}{s+1-2i}\right]\text{bo‘ladi.}$$

Shuning uchun

$$f(t)=-\frac{1}{4i}\left[e^{(-1-2i)t}-e^{(-1+2i)t}\right]=-\frac{1}{2}e^{-t}\left[\frac{e^{-2it}-e^{2it}}{2i}\right]=\frac{1}{2}e^{-t}\times$$
$$\times\left[\frac{e^{2it}-e^{-2it}}{2i}\right]=\frac{1}{2}e^{-t}\sin t$$

§1.5. Operatsion hisobning tadbiqlari

Operatsion hisobning tadbiqlarini o‘zgarmas koeffitsentli tenglamalar va ular sistemasini yechishdagi tadbiqlarini qaraymiz. $x(t)$ funksiya

$$x^{(n)}+a_1x^{(n-1)}+....+a_nx=f(t) \qquad (1.5.1)$$

tenglamasini

$$x(0)=x_0,\ x'(0)=x_0',...,x^{(n-1)}(0)=x_0^{(n-1)} \qquad (1.5.2)$$

shartni qanoatlantiruvchi yechimi bo‘lsin. (1.5.1) tenglamaning har ikkala tomoniniga Laplas almashtirishini qo‘llaymiz. Natijada $f(t)$ originalni tasviri $F(s)$, $x(t)$ originalni tasviri $X(s)$ va $Q(s)$ orqali $x_0,x_0',...,x_0^{(n-1)}$ larga bog‘liq ko‘phadni belgilasak,

$$(s^n+a_1s^{n-1}+...+a_n)X(s)+Q(s)=F(s)$$

ko‘rinishli $X(s)$ noma'lumli bog‘liq bo‘lgan operator tengalamaga ega bo‘lamiz. $Ls = s^n = a_1 s^{n-1} + ... + a_n$ ko‘phadga harakteristik ko‘phad dyemiz. Bundan

$$X(s) = \frac{F(s) - Q(s)}{Ls}.$$

Biz izlayotgan (1.5.1), (1.5.2) masalaning yechimi $X(s)$ tasvirning originalidan iborat. Huddi shu usul bilan sistema uchun quyidagi Koshi masalasini ham yechishimiz mumkin.

Misol 1.5.1. $x'' + 2x + x = e^{-t}$, $x(0) = 0$, $x'(0) = 0$

Yechish; $x(t) \leftarrow X(s)$ bo‘lsin. Originalni differensiallash teoremasidan foydalanamiz.

$$x'(t) \leftarrow sX(s) - x_0,\ x''(t) \leftarrow s^2 X(s) - s x_0 - x_0',$$
$$x(0) = 0,\ x'(0) = 0$$

bo‘lgani uchun

$$x'(t) \leftarrow sX(s),\ x''(t) \leftarrow s^2 X(s).$$

Endi $e^{-t} \leftarrow \frac{1}{s+1}$ ekanligidan

$$s^2 X(s) + 2sX(s) + X(s) = \frac{1}{s+1}$$

funksiyanal tenglamaga kelamiz. Bunda

$$X(s) = \frac{\frac{1}{s+1}}{s^2 + 2s + 1} = \frac{1}{(s+1)^3}$$

Bu funksiyaning originali berilgan masalani yechimi bo‘ladi. $x(t) \leftarrow X(s)$ demak

$$x(t) = \frac{1}{2} t^2 e^{-t}$$

Misol 1.5.2. $x'' + 3x' = e^{-3t}$, $x(0) = 0, x'(0) = -1$ Koshi masalasini yeching.

Bu tenglamani har ikkala tarafiga Laplas almashtirishini qo‘llaymiz va originalni differcnsiallash formulasidan foydalanamiz.

$$x'(t) \leftarrow sX(s) - sx(0) = sX(s)$$

$$x''(t) \leftarrow s^2 X(s) - sx_0 - x_0 s^2 X(s) + 1$$

bo‘lgani uchun tenglamaga Laplas operatorini qo‘llash natijasida

$$s^2 X(s) + 1 + 3sX(s) = \frac{1}{s+3}$$

operator tenglamaga kelamiz. Bunda

$(s^2 + 3s)X(s) = \dfrac{1}{s+3} - 1$ bo‘lgani uchun $X(s)$ tasvir quyidagicha topiladi.

$$X(s) = \frac{1}{(s^2+3s)(s+3)} - \frac{1}{s^2+3s}$$

yoki

$$X(s) = -\frac{s+2}{s(s+3)^2} = \left(\frac{A}{s} + \frac{B}{s+3} + \frac{D}{(s+3)^2}\right)$$

yoyilmaga kelamiz. Bunda $A = \dfrac{2}{3},\ B = -\dfrac{2}{3},\ D = \dfrac{1}{3}$ ekanligiga ishoch hosil qilish mumkin. Shuning uchun

$$X(s) = -\frac{2}{9}\cdot\frac{1}{s} + \frac{2}{9}\cdot\frac{1}{s+3} - \frac{1}{3}\cdot\frac{1}{(s+3)^2}$$

bu funksiya originalini topib misol.3 ning yechimini topamiz.

$$x(t) = -\frac{2}{9} + \frac{2}{9}\cdot e^{-3t} - \frac{1}{3}t\,e^{-3t}$$

Misol 1.5.3. $x''' + 3x' = te^{-t}$, $\quad x(0) = x'(0) = x''(0) = 0$, Koshi masalasini yeching.

Yechish. Tenglamaning har ikkala tomoniga Laplas almashtirishini qo‘llaymiz va Koshi shartlaridan foydalanib quyidagi operator tenglamaga kelamiz.

$$s^3X(s)-3s^2X(s)+3sX(s)+X(s)=\frac{1}{(s+1)^2}$$

bundan

$$(s^3-3s^2+3s)X(s)=\frac{1}{(s+1)^2}$$

$$(s+1)^2X(s)=\frac{1}{(s+1)^2}\ \text{yoki}\ X(s)=\frac{1}{(s+1)^5}$$

Endi $X(s)=\frac{1}{(s+1)^5}$ funksiyaning originalini topish kerak. Buning uchun

$$t^n e^{at}\leftarrow\frac{n!}{(s-a)^{n+1}}$$

formuladan foydalanamiz. Natijada

$$x(t)=\frac{t^4}{4!}e^{-t}$$

originalni hosil qilamiz.

Misol 1.5.4. $\int_0^t \cos(t-\tau)x(\tau)d\tau=t\cos t$ ko‘rinishli Volterraning birinchi tur integral tenglamasini yechamiz.

Yechish. $x(t)\leftarrow X(s)$ bo‘lsin.

$$\cos t\leftarrow\frac{s}{s^2+1},\quad t\cos t\leftarrow\frac{s^2+1}{(s^2-1)^2},$$

bo‘lgani uchun o‘rama xossasiga asosan

$$\int_0^t\cos(t-\tau)x(t)\leftarrow\frac{sX(s)}{s^2+1}$$

natijada biz quyidagi operator tenglamaga kelamiz:

$$\frac{sX(s)}{s^2+1}=\frac{s^2+1}{(s^2-1)^2}$$

yoki

$$X(s)=\frac{s^2+1}{s(s^2+1)}=\frac{2s}{s^2+1}-\frac{1}{s}$$

$$x(t)=2\cos st-1$$

Misol 1.5.5. $x''+x=\sin t+\int_0^t \sin(t-\tau)x(\tau)d\tau,\quad x(0)=0,\ x'(0)=1$

Koshi masalasini yeching.

Yechish. $x(t)\leftarrow X(s),\ x''\leftarrow s^2X(s)-1$

$$\sin t\leftarrow\frac{1}{s^2+1},\ \int_0^t \sin(t-\tau)x(\tau)d\tau\leftarrow\frac{1}{s^2+1}X(s)$$

shuning uchun $X(s)$ tasvirga nisbatan quyidagi operator tenglamasiga kelamiz

$$(s^2+1)X(s)-1=\frac{1}{s^2+1}+\frac{1}{s^2+1}X(s)\ \text{yoki}\ \left[(s^2+1)^2-1\right]X(s)=s^2+2$$

bundan $X(s)=\dfrac{1}{s^2}\Rightarrow x(t)=t$

Misol 1.5.6. Quyidagi Abel integral tenglamasini yeching. $\int_0^t\dfrac{x(\tau)d\tau}{\sqrt{t-\tau}}=\pi$

$$\pi\leftarrow\frac{\pi}{s},\ \frac{1}{\sqrt{\pi\tau}}\leftarrow\frac{1}{\sqrt{s}}\Rightarrow\frac{1}{\sqrt{t}}\leftarrow\frac{\sqrt{\pi}}{\sqrt{s}}\ \text{shuning uchun}$$

$$\frac{\sqrt{\pi}}{\sqrt{s}}X(s)=\frac{\pi}{s},\ X(s)=\frac{\sqrt{\pi}\sqrt{s}}{s}=\frac{\sqrt{\pi}}{\sqrt{s}}$$

bu tasvirning originali

$$x(t)=\frac{1}{\sqrt{t}}$$

II BOB.**Umumlashgan funksiyalar**

§2.1.Umumlashgan funksiyalarning ta'rif va xossalari

Fizik jarayonlarni oʻrganishda biz taqsimlangan massa, taqsimlagan zaryad tushunchalari bilan birga nuqtaga joylashtrilganmassa, nuqtaga qoʻyilgan kuch, nuqtadagi manba funksiyasi tushunchalari bilan tanishmiz.Nuqtaga joylashtrilgan massa tushunchasini qanday qilishimiz mumkinligini qaraymiz.

Birlik massa$[x_0, x_0+\varepsilon)$ oraliqqa taqsimlangan boʻlsin. Bu taqsimlanishning massa zichligini $\delta_\varepsilon(x, x_0)$ deb belgilasak, bu massa zichlik uchun

$$\int_{x_0}^{x_0+\varepsilon} \delta_\varepsilon(x, x_0)dx = 1$$

formula hosil boʻladi. $\delta_\varepsilon(x, x_0)$funksiyani$[x_0, x_0+\varepsilon)$dan tashqarida nol qilib aniqlaydigan boʻlsak, bu funksiya uchun

$$\delta_\varepsilon(x, x_0) = 0, \quad \forall x \equiv [x_0, x_0+\varepsilon)$$

$$\int_{-\infty}^{\infty} \delta_\varepsilon(x, x_0)dx = \int_{x_0}^{x_0+\varepsilon} \delta_\varepsilon(x, x_0)dx = 1$$

tenglikka kelamiz.

Agar biz birlik massani x_0 nuqtaga tortadigan boʻlsak, bu birlik massaning zichligi $\delta_\varepsilon(x, x_0)$ uchun limitga oʻtish kattasida nuqtaga tortilgan massa zichligi uchun

$$\lim_{\varepsilon\to 0} \delta_\varepsilon(x, x_0) = \begin{cases} 0, & \forall x \neq x_0 \\ \infty, & x = x_0 \end{cases} = \delta(x - x_0).$$

munosabatga kelamiz. Massani nuqtaga tortganda oʻzgarmas boʻlib qolganligi uchun

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\delta_\varepsilon(x,x_0)dx = \lim_{\varepsilon\to 0}\int_{x_0}^{x_0+\varepsilon}\delta_\varepsilon(x,x_0)dx = 1 = \int_{-\infty}^{\infty}\delta\ \ (x-x_0)dx$$

tengliklarga kelamiz. Klassik nuqtayi nazardan kelib chiqadigan bo‘lsak bu limit mavjud emas. Bu holda $\delta\ \ (x-x_0)$ elamentni qanday tushunish kerak degan savol paydo bo‘ladi. Limitik element $\delta\ \ (x-x_0)$ni tushunish uchun quydagi ta’rifni keltrib o‘tamiz.

Ta’rif.$u(x)$funksiya$[a,b]$ oraliqda integrallanuvchi $u_\varepsilon(x)$ funksiyalarning bo‘sh limiti deb aytiladi, agar ixtiyoriy $f(x)\epsilon C[a,b]$ funksiyalar uchun

$$\lim_{\varepsilon\to 0}\int_a^b u_\varepsilon(x)f(x)dx = \int_a^b u\ \ (x)f(x)dx \quad (2.1.1)$$

limit mavjud bo‘lsa. Agar $u(x)$ funksiya $\{u_\varepsilon(x)\}$ sinfg tegishli bo‘lmasa, bu sinfni $u(x)$bilan kengaytiriladi va$u(x)$ funksiyaga umumlashgan funksiyadeyiladi.

Bunda (2.1.1)tenglikni$u(x)\cdot f(x)$funksiya integralining tarifidan iborat bo‘ladi. Bu tarifdan foydalanib $\delta(x-x_0)$ funksiyani $\delta_\varepsilon(x,x_0)$ funksiyalarning bo‘sh limiti deb qarash mumkin.Haqiqattan ham $f(x)$ funksiya o‘qdauzluksiz bo‘lsa, $\delta_\varepsilon(x,x_0)\geq 0$ funksiya integrali uchun o‘rta quymat formulasidan foydalanib quyidagiga kelamiz.

$$\lim_{\varepsilon\to 0}\int_{x_0}^{x_0+\varepsilon}\delta_\varepsilon(x,x_0)f(x)dx = \lim_{\varepsilon\to 0}f(x_{o'rta})\int_{x_0}^{x_0+\varepsilon}\delta_\varepsilon(x,x_0)dx = f(x_0)$$

shunday qilib, $\delta(x-x_0)$ bo‘sh limit mavjud bo‘lib, biz

$$\int_{-\infty}^{\infty} \delta(x, x_0) f(x) dx = \int_{x_0}^{x_0+\varepsilon} \delta_\varepsilon(x - x_0) f(x) dx = f(x_0), \qquad \forall \varepsilon > 0$$

tenglikni yozamizki bu tenglikni $\delta(x - x_0) f(x)$ funksiyaning integrali tarifidan iborat bo‘ladi. Agar bunda $f(x) = 1$bo‘lsa, u holda

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = \int_{x_0}^{x_0+\varepsilon} \delta(x - x_0) f(x) dx = 1, \ \ \forall \varepsilon > 0$$

(2.1.1) tenglik orqalianiqlangan $u(x)$ funksiyaga umumlashgan funksiya deymiz.$\delta(x - x_0)$umumlashgan funksiya bo‘lib bu funksiya quydagi shartlarni qanoarlantiradi:

1) $\delta(x - x_0) = \begin{cases} 0, & agar x \neq x_0 \\ x, & agar x = x_0 \end{cases}$,

2) $\int_a^b \delta(x - x_0) f(x) dx = \begin{cases} f(x_0), & agar \ \ x_0 \epsilon [a, b] \\ 0\ , & agar \ \ x_0 \ \bar{\in} \ [a, b] \end{cases}$.

Bunda $f(x)$funksiya x_0 nuqtaning atrofida uzluksiz funksiyalardan iborat bo‘lib,$\delta(x - x_0)$ funksiya Dirakning δ funksiyasi deb aytiladi. Bo‘sh limiti Dirak δ funksiyasidan iborat bo‘ladi$\delta(x - x_0)$ funksiyalarni har xil usullar bilan tuzish mumkin, bunda 1) va 2) shartlar bajarilishi kifoya. Bu funksiyalarni

$$\delta_\varepsilon(x, x_0) = \begin{cases} \dfrac{1}{\varepsilon}, & x \epsilon [x_0, x_0 + \varepsilon) \\ 0, & x \ \bar{\in} \ [x_0, x_0 + \varepsilon) \end{cases}$$

ko‘rinishda olish ham mumkin.

Tar'if. $u(x)$ va $v(x)$ umumlashgan funksiyalar $[a,b]$ oraliqda teng deb aytiladi, agar ixtiyoriy $f(x)\epsilon C[a,b]$ funksiyalar uchun

$$\int_a^b u(x)f(x)dx = \int_a^b v(x)f(x)dx$$

tenlik bajarilsa.

Bo'sh limiti Dirak δ -funksiyasidan iborat bo'lgan funksiyalarni δ -shaklli funksiyalar deymiz. Yuqoridagi ta'rifga asosan hamma $\delta -$shaklli funksiyalar o'zaro tengdir.

$$\delta_n(x,x_0) = \frac{2}{l}\sum_{k=1}^{n} sin\frac{k\pi}{l}xsin\frac{k\pi}{l}x$$

funksiyalarni $[0,l]$ oraliqda $\delta -$shaklli ketma-ketlik ekanligini ko'rsatamiz. $f(x)$ funksiya x_0 nuqta atrofida uzluksiz bo'lgan $\{sin\frac{k\pi}{l}x\}$ sistema bo'yicha yaqinlashuvchi Fure qatoriga yoyiladigan funksiya bo'lgan. Bu tenglikni $f(x)dx$ ga ko'paytrib $(0,l)$ oraliqda integrallaymiz va $n \to \infty$ da limitga o'tamiz

$$\lim_{n\to\infty}\int_0^l \delta_n(x,x_0)f(x)dx = \lim_{n\to\infty}(\frac{2}{l}\int_0^l f(x)sin\frac{k\pi x}{l}dx)sin\frac{k\pi x_0}{l}$$

bu tenglikning o'ng tomoni $f(x)$ funksiyaning x_0 nuqtada $f(x)$ uzluksiz bo'lganligidan

$$\lim_{n\to\infty}\int_0^l \delta_n(x,x_0)f(x)dx = f(x_0)$$

bu tenglikdan $\delta_n(x,x_0)$ ko'rinishdagi funksiyalardan iborat. Shuning uchun

$$\delta(x-x_0)=\frac{2}{l}\sum_{k=1}^{\infty} sin\frac{k\pi}{l}x sin\frac{k\pi}{l}x_0 \quad (2.1.2)$$

bo‘lib, (2.1.2) formula $\delta(x-x_0)$ Dirak δ funksiyaning Fure qatori (2.1.2) qator orqalibekilar ekan. Huddi shunday $[-l,l]$ oraliqda Dirak δ funksiyaning Fure qatori,

$$\delta(x-x_0)=\frac{1}{2l}+\frac{1}{l}\sum_{k=1}^{\infty} cos\frac{k\pi x}{l}cos\frac{k\pi x_0}{l}+sin\frac{k\pi x}{l}sin\frac{k\pi x_0}{l}$$

ko‘rinishda bo‘ladi.

Bir o‘zgaruvchili umumlashgan funksiyalar $C_0^{\infty}(R)$ orqali finit va silliq funksiyalar sinfini belgilaymizki, bundagi har bir $\varphi(x)$ funksiyaquyidagi shartlarni qanoatlantiradi:

1) $\varphi(x)\epsilon C_0^{\infty}(R)$

2) $\varphi(x)=0 \quad agar \ |x|\geq A$

bunda$A$ son φ funksiyaga bog‘liq. Ihtiyoriy uzluksiz $u(x)$funksiya bilan $\varphi\epsilon C_0^{\infty}(R)$ funksiyalar skalyar ko‘paytmasini

$$< u(x),\varphi(x) > = \int_{-\infty}^{\infty} u(x)\varphi(x)dx \quad (2.1.3)$$

formula orqali aniqlaymiz. Bu integral yaqinlashuvchidir, chunki $\varphi(x)\equiv 0, \ agar \ |x|\geq A$bo‘lganda.

$$< u(x),\varphi(x) > = \int_{-A}^{A} u(x)\varphi(x)dx \quad (2.1.4)$$

(2.14) ning o‘ng tomoni xos integraldan iborat. Berilgan uzluksiz $u(x)$ funksiya uchun

$$\{< u, \varphi >\}, \varphi \epsilon C_0^\infty(R) \quad (2.1.5)$$

sonlarni qaraymiz. Bu sonlardan foydalanib $u(x)$ funksiyani tuzish mumkinligini qaraymiz. Bu funksiyani

$$u(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \varphi(\frac{x-y}{\varepsilon}) u(y) dy = \lim_{\varepsilon \to 0} < \varphi_\varepsilon^y(y), u(y) > \quad (2.1.6)$$

formula orqali berilishini isbotlaymiz. Bunda

$$\varphi_\varepsilon^y(y) = \frac{1}{\varepsilon} \varphi(\frac{x-y}{\varepsilon}) \epsilon C_0^\infty(R)$$

bo‘lib,

1) $\varphi(y) \equiv 0, \; agar \;\; |y| \geq 1$

2) $\int_{-1}^{1} \varphi(y) dy = 1$

(2.1.6) ni hisbotlash uchun $\frac{x-y}{\varepsilon} = z$ belgilash kiritamiz. Natijada

$$u(x) = \lim_{\varepsilon \to 0} \int_{-1}^{1} \varphi(z) u(x - \varepsilon z) dz \quad (2.1.7)$$

$u(x)$funksiyaning uzluksizligidan foydalanib:

$$\lim_{\varepsilon \to 0} \int_{-1}^{1} \varphi(z) u(x - \varepsilon z) dz = \int_{-1}^{1} \varphi(z) \lim_{\varepsilon \to 0} u(x - \varepsilon z) dz$$
$$= \int_{-1}^{1} \varphi(z) u(x) dz = u(x) \int_{-1}^{1} \varphi(z) dz = u(x)$$

(2.1.3) formulaga asosan $\forall \varphi_1, \varphi_2 \epsilon C_0^\infty(R)$ uchun

$$< u, \varphi_1 + \varphi_2 > = < u, \varphi_1 > + < u, \varphi_2 >.$$

Abatrant raviashda olingan

$$l_{\varphi} = < u, \varphi > \quad (2.1.8)$$

lar uchun biror $u(x) \epsilon C(R)$ funksiya mos kelishi uchun bu sonlar

$$l_{\varphi_1+\varphi_2} = l_{\varphi_1} + l_{\varphi_2}, \quad \forall \varphi_1, \varphi_2 \epsilon C_0^{\infty}(R) \quad (2.1.9)$$

shartning bajarilishi zarur.

Ta'rif. $\varphi_n \xrightarrow[n\to\infty]{C_0^{\infty}} \varphi$ yaqinlashish quydagini anglatadi:

1) $\varphi_n(x)$ funksiyalar $x \epsilon R$ da $\varphi \ (x)$ ga yaqinlashadi. Huddi shunday

$$\varphi_n^{(k)}(x) \Rightarrow \varphi^{(k)(x)}, \ x \epsilon R \ agar \ \ n \to \infty \quad (2.1.10)$$

Hamda φ_n lar uchun $[-A, A], \ \forall n = 1,2, \ldots$ bo'lganda

$$\varphi_n(x) \equiv 0, \qquad agar \ \ |x| \geq A \quad (2.1.11)$$

bu holda $[-A, A]$ segment $\varphi_n(x)$ larning yurituvchisi deyiladi.

$\{l_{\varphi}\}$ sonlar to'plamiga mos keluvchi $u(x) \epsilon C(R)$ funksiya mavjud bo'lishi yetarli bo'lmasligi mumkin. Bundan tashqari yana quydagi uzluksizlik turdagi

$$< u, \varphi_n > \xrightarrow[n\to\infty]{} < u, \varphi >, \ agar \varphi_n \xrightarrow{C_0^{\infty}} \varphi \quad (2.1.12)$$

shart bajarilishi kerak. (2.1.10) yaqinlashishni biz quyidagicha tushunamiz:

$$< u, \varphi_n > \equiv \int_{-A}^{A} u(x)\varphi_n(x)dx \to \int_{-A}^{A} u(x)\varphi(x)dx = < u, \varphi >, \quad (2.1.13)$$

shunday qilib, (2.1.8) ni qanoatlantiruvchi $u(x) \epsilon C(R)$ funksiya mavjud bo'lishi uchun

$$l_{\varphi_n} \to l_\varphi, \ agar \quad \varphi_n \xrightarrow{C_0^\infty} \varphi, \qquad (2.1.14)$$

shartning bajarilishi kerak.

$\{l_\varphi\}$larga asosan (2.1.8) shartni qanoatlantiruvchi $u(x)\epsilon C(R)$ funksiyani,agarda mavjud bo‘lsa yagona ravishda aniqlash mumkin, lekin $\{l_\varphi\}$ ga mos keluvchi $u(x)$bo‘lmasligi mumkin. Shunday qilib, (2.1.9) va (2.1.14) shartlarning bajarilishi uzluksiz $u(x)$funksiyaning mavjudligining zaruriy sharti bo‘lib, yetarli emas.

Ta'rif. Ixtiyoriy $l = \{l_\varphi\}$,$\varphi\epsilon C_0^\infty(R)$ to‘plam (2.1.9) va (2.1.14) shartni qanoatlantirsa, biz uni umumlashgan funksiya deymiz.Soddalik kiritish maqsadida $D = D(R) = C_0^\infty(R)$ belgilash kiritamiz. Shunday qilib (2.1.9) va (2.1.14) shartni qanoatlantiruvchi $\{l_\varphi\}$tanlanma yani umumlashgan funksiya funksianal analiz nuqtai nazaridan $D(R)$ da aniqlangan chiziqli uzluksiz funksionaldan iborat ekan. Demak, umumlashganfunksiya $D(R)$ning qo‘shma fazosi $D'(R)$ ning elementidan iborat ekan:

$$l = \{l_\varphi\}\epsilon D'(R); \ l(\varphi) \equiv l_\varphi, \ \forall\varphi\epsilon D(R) (2.1.15)$$

bu aytilganlarga asosan $D'(R)$ hamma umumlashgan funksiyalarning fazosidan iborat ekan.

$\{l_\varphi\}$umumlashgan funksiya uchun l funksionalning φ dagi qiymati $l(\varphi)$ deb olsak, u $< l(x), \varphi(x) >$ skalyar ko‘paytmadan iborat.

$$l_\varphi = l(\varphi) =< l(x), \varphi(x) > (2.1.16)$$

(2.1.16) da $l(x)$ funksiyaning x nuqtadagi qiymati bo‘lmasdan u simvoldan iborat.

$$l_\varphi = \varphi(0), \ \ \forall\varphi\epsilon C_0^\infty(R) \quad (2.1.17)$$

umumlashgan funksiya Dirakning δ $-$funksiyasidan iborat:

$$\delta_\varphi =< \delta(x), \varphi(x) >= \varphi(0), \quad \forall\varphi\epsilon D(R) \quad (2.1.18)$$

Har qanday umumlashgan funksiya (2.1.3) formula orqali uzluksiz funksiyadan hosil qilinmaydi.Shunday umumlashgan funksiyadan biri Dirakning δ funksiyasidan iborat.Umumlashgan funksiyaga misollar keltiramiz.

$$1) \ < \delta_k(x), \varphi(x \ \ ) >= \varphi^{(k)}(0), \quad \forall\varphi\epsilon D(R)$$

buning uchun biz $\delta_k(x)\epsilon D'(R)$ ekanligini ko‘rsatishimiz kerak.

$$< \delta_k(x), \varphi_1(x) + \varphi_2(x) >=< \delta_k(x), \varphi_1(x) > + < \delta_k(x), \varphi_2(x) > \\ = \varphi_1^{(k)}(0) + \varphi_2^{(k)}(0)$$

huddi shunday (2.1.14) shartning bajarilishini ko‘rsatish qiyin emas.

2) Xevisayd funksiyasi

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$< \theta(x), \varphi(x) >\equiv \int_{-\infty}^{\infty} \theta(x)\varphi(x)dx = \int_0^\infty \varphi(x)dx$$

$\theta(x)\epsilon D'(R)$ ekanligini ko‘rsatamiz.

.

$< \theta(x), \varphi_1(x) + \varphi_2(x) >= \int_{-\infty}^{\infty} \theta(x)(\varphi_1(x) + \varphi_2(x))dx = \int_0^{\infty}(\varphi_1(x) + \varphi_2(x))dx = \int_0^{\infty} \varphi_1(x)dx + \int_0^{\infty} \varphi_2(x)dx$

demak, $\theta(x)$ (2.1.9) shartni qanoatlantirar ekan.

(2.1.14) shartni bajarilishini

$$< \theta(x), \varphi_n(x) >\equiv \int_0^A u(x)\, \varphi_n(x)dx \underset{n\to 0}{\longrightarrow} \int_0^A \varphi(x)u(x)dx =< u, \varphi >$$

tengliklardan kelib chiqadi. Umumlashgan funksiyalar ustida bajariladigan bazi amallarni keltiramiz.

1) Agar $u_1(x),\ u_2(x)$ uzluksiz funksiyalar bo‘lsa, u holda

$$< u_1(x) + u_2(x), \varphi(x) >=< u_1, \varphi > +< u_2, \varphi >,\ \forall \varphi \epsilon D(R) \quad (2.1.19)$$

(2.1.19) ga asosan quyidagi tarifni keltiramiz.

Ta’rif. Agar $l^1,\ l^2 \epsilon D'(R)$bo‘lsa,

$$(l^1 + \ l^2)_\varphi = l_\varphi^1 + l_\varphi^2,\quad \forall \varphi \epsilon D(R) \quad (2.1.20)$$

2) Umumlashgan funksiyalarni songa ko‘paytirish.

Agar $u(x) \epsilon C(R)$va$\alpha \epsilon R$ bo‘lsa, (2.1.3) formuladan

$$< \alpha u(x), \varphi \geq \alpha < u(x), \varphi >,\quad \forall \varphi \epsilon D(R) \quad (2.1.21)$$

(2.1.21) formuladan quyidagi tarifni keltirib chiqaramiz.

Ta’rif.$l(x) \epsilon D'(R)$va$\alpha \epsilon R$ uchun

$$< \alpha l(x), \varphi \geq \alpha < l, \varphi >,\quad \forall \varphi \epsilon D(R) \quad (2.1.22)$$

3) Umumlashgan funksiyalarni silliq funksiyalarga ko‘paytirish.

Agar $g(x)\epsilon C^{\infty}(R)vau(x)\epsilon C(R)$ bo‘lsa, (2.1.3) formuladan

$$< g(x)u(x),\varphi >=< u, g(x)\varphi(x) >, \qquad \forall\varphi\epsilon D(R) \quad (2.1.23)$$

0xirgi tenglikdan foydalanib quyidagi tarifni beramiz.

Ta'rif. Agar $l(x)\epsilon D'(R)$bo‘lsa,

$$< g(x)l(x),\varphi >=< l, g(x)\varphi(x) >, \qquad \forall\varphi\epsilon D(R) \quad (2.1.24)$$

(2.1.24) ning o‘ng tomoni $g(x)\varphi(x)\epsilon D(R)$bo‘lgani uchun manoga ega.

$x\delta(x)$ni hisoblaymiz.

$< x\delta(x),\varphi(x) >=< \delta(x), x\varphi(x) >= x|_{x=0} = 0$

$\int_{-\infty}^{\infty} g(x)\delta(x)\varphi(x)dx = g(0)$bo‘lgani uchun

$$g(x)\delta(x) = g(0)\delta(x) \quad (2.1.25)$$

tengliko‘rinli

4) Umumlashgan funksiyada siljish.

Agar $u(x)\epsilon C(R)$va$a\epsilon R$ bo‘lsa

$$\int_{-\infty}^{\infty} u(x-a)\varphi(x)dx = \int_{-\infty}^{\infty} u(y)\varphi(y+a)dy, \qquad \forall\varphi\epsilon D(R)$$

bu tenglikka asosan quyidagi tarifni kiritamiz.

$$< l(x-a),\varphi(x) >=< l(y),\varphi(y+a) > \quad (2.1.26)$$

Misol .

$$< \delta(x-a), \varphi(x) > = < \delta(y), \varphi(y+a) > = \varphi(a) \quad (2.1.27)$$

5) Umumlashgan funksiyalar argumentida mashtabni o‘zgarish.

Agar $u(x) \epsilon C(R)$va$n \neq 0$ bo‘lsa

$$\int_{-\infty}^{\infty} u(kx)\varphi(x)dx = \frac{1}{|k|}\int_{-\infty}^{\infty} u(y)\varphi(\frac{y}{k})\,dy \quad (2.1.28)$$

(2.1.28) tenglikdan quyidagi tarifni keltiramiz:

Ta’rif. Agar $f(x)\epsilon D'(R)$bo‘lsa, $k \neq 0$ bo‘lganda

$$< f(kx), \varphi(x) > = \frac{1}{|k|} < f(y), \varphi\left(\frac{y}{k}\right) >$$

oxirgida$\delta(kx) = f(kx)$ deb olsak

$$\delta(kx) = \frac{1}{|k|}\delta(x), \quad k \neq 0 \qquad (2.1.29)$$

(2.1.29) tenglikda$k = -1$ deb olamiz.

$$\delta(-x) = \delta(x) \quad (2.1.30)$$

oxirgi tenglik $\delta(x)$ funksiyani juft ekanligini ko‘rsatadi.

$$< \delta(y-x), \varphi(y) > = \varphi(x) \quad (2.1.31)$$

bo‘lgani uchun .$\delta(y-x) I\varphi = \varphi$birlik operatorning integral yadrosidan iborat. Chiziqli algebrada birlik matritsiyaning elamentlari kronekerning δ $-$simvoli δ_{ij} dan iborat bo‘lgani uchun Dirak (2.1.18) funksionalni δ funksiya deb atagan.

6) Umumlashgan funksiyalarni differensiallash.

Agar $u(x)\epsilon C^1(R)$bo‘lsa$(\varphi(x) \equiv 0, \ |x| > A)$

$$\int_{-\infty}^{\infty} u'(x)\varphi(x)dx = u\varphi|_{-A}^{A} - \int_{-A}^{A} u(x)\varphi'(x)dx$$

$$= -\int_{-\infty}^{\infty} u(x)\varphi'(x)dx, \qquad (2.1.32)$$

tenglik bajariladi, chunki $\varphi(A) = \varphi(-A) = 0.$

(2.1.32) tenglikdan quyidagi tarifni keltiramiz.$u(x)\epsilon D'(R)$uchun

$$< u'(x), \varphi(x) \geq < u(x), \varphi'(x) >, \quad \forall\varphi\epsilon D(R) \qquad (2.1.33)$$

(2.1.33) yenglikdan ixtiyoriy umumlashgan funksiya hosilaga ega ekanligi kelib chiqadi.

Ta'rif.$u(x)\epsilon D'(R)$uchun

$$< u'(x), \varphi(x) > = < u(x), \varphi'(x) >, \quad \forall\varphi\epsilon D(R)$$

shunday qilib, ixtiyoriy umumlashgan funksiya ixtiyoriy tartibli hosilaga ega bo‘lib, n −chi tartibli hosilasi uchun

$$< u^{(n)}(x), \varphi(x) > = (-1)^n < u(x), \varphi^{(n)}(x) >, \quad \forall\varphi\epsilon D(R)$$

Umumlashgan funksiyaning hosilasiga doir misollar keltiramiz.

1)

$< sin'(x), \varphi(x) > = -\int_{-\infty}^{\infty} \sin(x)\varphi'(x)dx = -[\sin(x)\varphi(x)|_{-A}^{A} - \int_{-A}^{A} \cos(x)\varphi(x)dx] = \int_{-A}^{A} \cos(x)\varphi(x)dx = < cosx, \varphi(x) >$

demak, $sin'(x) = cosx$

2) $\theta'(x)$funksiya hosilasini topamiz.

$$< \theta'(x), \varphi(x) > = -< \theta(x), \varphi'(x) > = -\int_0^\infty \varphi'(x)dx = -\varphi(x)|_0^\infty = \varphi(0)$$

$$= < \delta(x), \varphi(x) >$$

shunday qilib,

$$\theta'(x) = \delta(x) \quad (2.1.34)$$

Endi $u(x)$bo‘lakli silliq funksiya bo‘lgan hol uchun differensiallash masalasini qaraymiz.$u(x)$funksiya$x < a$ va $x > a$ da C^1 ga tegishli bo‘lib,$x = a$ nuqtada birinchi turdagi uzulishga ega bo‘lsin, yani bir tomonli $u(a \mp 0)$bir tomonli limitlar mavjud. Bu holda quyidagi formula o‘rinli

$$u'(x) = \{u'(x)\} + h\delta(x-a); \quad h = u(a+0) - u(a-0) \qquad (2.1.35)$$

Umumlashgan funksiyalar uchun operatsion hisob.

(2.1.35) da$u'(x), u(x)$ umumlashgan funksiyaning hosilasi, $\{u'(x)\}$ esa $x \neq a$bo‘lganda uzluksiz funksiya bo‘lib, $u(x)$ funksiyaning oddiy hosilasidan iborat. $\{u'(x)\}$funksiya umumlashgan $u'(x)$ hosilaning regulyar qismi deyiladi.

Misollar keltiramiz.

a) $u(x) = \theta(x)$bo‘lsa$a = 0, \{\theta'(x)\} \equiv 0$ chunki $x \neq 0$ bo‘lganda $\theta'(x) = 0,$

$h = \theta(0+) - \theta(0-) = 1$shuning uchun (2.1.35) formuladan

$$\theta'(x) = \delta(x) \quad (2.1.36)$$

bu formula (2.1.34) bilan ustma-ust bo‘ladi.

b) $|x|''$umumlashgan hosilani topamiz

$$|x|' = \{|x|'\} + 0\delta(x) = sgn(x) \equiv \begin{cases} 1, & agar x > 0 \\ -1, & agar x < 0 \end{cases} \quad (2.1.37)$$

$$|x|'' = (sgn(x))' = \{sgn' x\} + 2\delta(x) 2\delta(x) \qquad (2.1.38)$$

Endi (2.1.35) formulani isbotlaymiz.$\varphi(x) \in C_0^\infty(R)$bo‘lganda

$$\langle u',\varphi\rangle = -\langle u,\varphi'\rangle = -\int_{-\infty}^{+\infty} u\varphi' dx - \int_a^{+\infty} u\varphi' dx = -u\varphi|_{-\infty}^{a-0} - u\varphi|_{-a+0}^{\infty}$$

$$+\int_{x\neq a} u'\varphi dx = -u(a-0)\varphi(a-0) + u(a+0)\varphi(a+0) + +\langle\{u'\},\varphi\rangle$$

bo‘lib, bu tenglik (2.1.35) dan iborat.Endi ko‘paytmani differensiallash formulasini keltiramiz.$g(x) \in C^\infty(R)$va$u(x) \in D'(R)$ bo‘lsin. (2.1.17)va (2.1.33) formulalarga asosan

$$\langle (g(x)u(x))',\varphi\rangle = -\langle g(x)u,\varphi'\rangle = -\langle u(x), g(x)\varphi'(x)\rangle =$$

$$= -\int_{-A}^{A} \left(u(x)g(x)\right)'\varphi(x)dx \ =$$

$$= \int_{-A}^{A} \left(u'(x)g(x) + u(x)g'(x)\right)\varphi(x)dx \qquad (2.1.39)$$

ushbu tenglikdan

$$(g(x)u(x))' = u'(x)g(x) + u(x)g'(x) \text{(2.1.40)}$$

ko‘rinishli ko‘paytmaning differensiallash fomulasini hosil qilamiz.

$$\left(\frac{d}{dx} + \lambda\right)\left(\theta(x)e^{-\lambda x}\right) \text{(2.1.41)}$$

hosilani hisoblaymiz.

$$\frac{d}{dx}\left(e^{-\lambda x}\theta(x)\right) = -\lambda e^{-\lambda x}\theta(x) + e^{-\lambda x}\theta'(x) = \lambda e^{-\lambda x}\theta(x) + \delta(x) \quad (2.1.42)$$

(2.1.42) ning har ikkala tomaniga$\lambda\theta(x)e^{-\lambda x}$ni qo‘shamiz, natijada

$$\left(\frac{d}{dx} + \lambda\right)\left(\theta(x)e^{-\lambda x}\right) = \delta(x) \text{(2.1.43)}$$

tenglik hosil bo‘ladi.

Endi $\mathcal{W} \neq 0$bo‘lganda

$$\left(\frac{d}{dx^2}^2 + \mathcal{W}^2\right)\left(\theta(x)\frac{sin\mathcal{W}x}{\mathcal{W}}\right) \text{(2.1.44)}$$

ifodani hisoblaymiz. (2.1.40) va (2.1.25) formulalardan foydalanamiz

$$\frac{d}{dx}\left(\frac{sin\mathcal{W}x}{\mathcal{W}}\theta(x)\right) = \cos(\mathcal{W}x)\,\theta(x) + \frac{sin\mathcal{W}x}{\mathcal{W}}\theta'(x) =$$

$$= \cos(\mathcal{W}x)\,\theta(x)\frac{sin\mathcal{W}x}{\mathcal{W}}\delta(x) = \cos(\mathcal{W}x)\,\theta(x) + \frac{sin\mathcal{W}0}{\mathcal{W}}\delta(x) =$$
$$= \cos(\mathcal{W}x)\,\theta(x) \tag{2.1.45}$$

(2.1.45) ni yana bir marta differensiallaymiz.

$$\frac{d^2}{dx^2}\left(\frac{sin\mathcal{W}x}{\mathcal{W}}\theta(x)\right) = \frac{d}{dx}(\cos(\mathcal{W}x)\,\theta(x)) =$$
$$= -\mathcal{W}\sin(\mathcal{W}x)\,\theta(x) + \cos(\mathcal{W}x)\,\theta'(x)$$
$$= -\mathcal{W}\sin(\mathcal{W}x)\,\theta(x) + \delta(x) \tag{2.1.46}$$

(2.1.46) tenglikning har ikkala tomoniga $\mathcal{W}^2\theta(x)\frac{sin\mathcal{W}x}{\mathcal{W}}$ni qo‘shib, quyidagi tenglikni hosil qilamiz

$$\left(\frac{d}{dx^2}^2 + \mathcal{W}^2\right)\left(\theta(x)\frac{sin\mathcal{W}x}{\mathcal{W}}\right) = \delta(x) \tag{2.1.47}$$

(2.1.43) va (2.1.47) formulalarni (2.1.35) formuladan foydalanib hisoblash mumkinligini qaraymiz. (2.1.35) formulada$a = 0, h = 1$bo‘lganda

$$\frac{d}{dx}\left(e^{-\lambda x}\theta(x)\right) = -\lambda e^{-\lambda x}\theta(x) + \delta(x).$$

Oxirgi tenglikning har ikkala tomoniga $\lambda e^{-\lambda x}\theta(x)$ni qo‘shamiz. Natijada (2.1.43) tenglikni hosil bo‘ladi.Endi (2.1.35) formulada $a = 0, h = 0$ deymiz.

$$\frac{d}{dx}\left(\theta(x)\frac{sin\mathcal{W}x}{\mathcal{W}}\right) = \theta(x)\cos(\mathcal{W}x) + \ 0 \cdot \delta(x) = \theta(x)\cos(\mathcal{W}x)$$

$\theta(x)\cos(\mathcal{W}x)$ni hosilasini hisoblash uchun (2.1.35) formulada $a = 0, h = 1$ bo‘ladi

$$\frac{d^2}{dx^2}\left(\theta(x)\frac{sin\mathcal{W}x}{\mathcal{W}}\right) = \frac{d}{dx}(\theta(x)\cos(\mathcal{W}x)) = -\theta(x)\sin(\mathcal{W}x) \ + \delta(x).$$

Bu tenglikning har ikkala tomoniga $\mathcal{W}^2\theta(x)\frac{sin\mathcal{W}x}{\mathcal{W}}$ funksiyani qo‘shish natijasida (2.1.47) tenglikni hosil qilamiz.

(2.1.43) formuladan$x \neq 0$bo‘lganda$\delta(x) = 0$ bo‘lgani uchun quyidagi bir jinsli tenglamani hosil qilamiz.

$$\left(\frac{d}{dx} + \lambda\right)\left(\theta(x)e^{-\lambda x}\right) = 0 \;\; agar x \neq 0 \tag{2.1.48}$$

demak$\theta(x)e^{-\lambda x}$ funksiya (2.1.48) bir jinsli tenglamani yechimi bo‘ladi va uning sakrashi $a = 0$ nuqtada $h = 1$$y = \theta(x)\frac{sinWx}{w}$ funksiya esa

$$\left(\frac{d^2}{dx^2 + W^2}\right)\theta(x)\frac{sinWx}{W} = 0, \qquad agar x \neq 0 \tag{2.1.49}$$

bir jinsli tenglamaning yechimi bo‘ladi hamda $y(x) \neq \theta(x)$ funksiya $a = 0, h = 1$ sakrashga ega. $y = \theta(x)\frac{sinWx}{w}$funksiyaga$x = 0$ da uzluksiz bo‘ladi. Shuning uchun$y = \theta(x)\frac{sinWx}{w}$ funksiya uchun $x = 0$ nuqtada quyidagi shartlarga kelamiz.

$$\begin{cases} y(-0) = y(0+) \\ y'(0+) - y'(0-) = 1 \end{cases} \tag{2.1.50}$$

shunday qilib, (2.1.43) va (2.1.47) formulalarda ularning regulyar qismi nolga aylanar ekan.

§2.2.Differensial tenglamalarning fundamental yechimlari

Tartibi n bo‘lgan chiziqli A operatorni quydagicha kiritamiz:

$$A = A\left(\frac{d}{dx}\right) = \sum_{k=0}^{n} a_k \frac{d^k}{dx^k}, \qquad a_n \neq 0 \quad (2.2.1)$$

murakkab funksiyaning hosilasini hisoblashga asosan

$$\frac{d^k}{dx^k}(u(x-y)) = u^{(k)}(x-y) \quad (2.2.2)$$

formula o‘rinli ekanligini bilamiz. Shuning uchun

$$A\left(\frac{d}{dx}\right)(u(x-y)) = (Au)(x-y) \quad (2.2.3)$$

bu operatorning fundamental yechimi deb, shunday $\varepsilon(x)$ umumlashgan funksiyagaaytamizki, bu funksiya

$$A\left(\frac{d}{dx}\right)\varepsilon(x-y) = \delta(x) \quad (2.2.4)$$

tenglamani qanoatlantiradi.

Bu yerda $\delta(x)$ funksiya Dirakning δ – funksiyasidan iborat.

Yuqorida keltirilgan umumlashgan funksilardan foydalangan holda eng sodda differensial A operator uchun ularning fundamental yechimlarini keltiramiz.

1) $A = \frac{d}{dx}$ bo‘lganda $\varepsilon(x) = \theta(x)$

2) $A = \frac{d^2}{dx^2}$ bo‘lganda $\varepsilon(x) = \frac{|x|}{2}$

3) $A=\frac{d}{dx}+\lambda$ bo'lganda $\varepsilon(x)=\theta(x)e^{-\lambda x}$

4) $A=\frac{d^2}{dx^2}+\mathcal{W}^2$ bo'lganda $\varepsilon(x)=\theta(x)\frac{\sin\mathcal{W}x}{\mathcal{W}}$

har bir qanoatlangan A differensial operatorning fundamental yechimi cheksiz ko'p bo'lishi mumkin.

Agar (2.2.1) operatorning fundamental yechimi topilgan bo'lsa, bu holda bir jinsli bo'lmagan

$$A\left(\frac{d}{dx}\right)u(x)=f(x)\quad(2.2.5)$$

tenglamani yechimi fundamental yechim yordamida topish mumkin. Haqiqatdan ham A operatorning fundamental yechimi $\varepsilon(x)$va$f(x)$ ning o'ramasi (2.2.5) tenglamaning xususiy yechimi bo'ladi.

$$u(x)=\int_{-\infty}^{\infty}\varepsilon(x-y)f(y)dy=(\varepsilon^{*}f)(x)$$

$$=\int_{-\infty}^{\infty}\varepsilon(y)f(x-y)dy\qquad(2.2.6)$$

Agar $f(x)=0,\ |x|\geq consf$va$f(x)\in C(R)$ agar $f(x)\in C^n(R)$ bo'lsa (2.2.6) ni tekshirish oson.

$$A\left(\frac{d}{dx}\right)(u(x)=\int_{-\infty}^{\infty}\varepsilon(y)A\left(\frac{d}{dx}\right)f(x-y)dy=\langle\varepsilon(y),A\left(\frac{d}{dx}\right)f(x-y)\rangle=$$
$$\langle A\left(\frac{d}{dy}\varepsilon(y),f(x-y)\right)\rangle=\langle\delta(y),f(x-y)\rangle=f(x)$$

(2.2.7)

Misollar keltiramiz.

$$1)\ \frac{d}{dx}u(x)=f(x)\quad(2.2.8)$$

tenglama uchun $\varepsilon(x-y)=\theta(x-y)$ bo'lib, (2.2.6) formuladan foydalanib

$$u(x)=\int_{-\infty}^{\infty}\theta(x-y)f(y)dy=\int_{-\infty}^{0}f(y)dy\quad(2.2.9)$$

$$2)\frac{d^2}{dx^2}u(x)=f(x)\quad(2.2.10)$$

ushbu tenglamaning xususiy yechimi

$$u(x) = \int_{-\infty}^{\infty} \frac{|x-y|}{2} f(y)dy \quad (2.2.11)$$

Endi (2.2.1) ko‘rinishdagi differensial operator uchun fundamental yechimni aniqlaymiz.

$u_0(x)$ funksiya $x \geq 0$ bo‘lganda quyidagi Koshi masalasining yechimi bo‘lsin.

$$\begin{cases} A\left(\frac{d}{dx}\right)u_0(x) = 0, \quad x > 0 \\ u_0(0) = 0 \\ ------------ \\ u_0^{(n-2)}(0) = 0 \\ u_0^{(n-1)}(0) = \frac{1}{a_n} \end{cases} \quad (2.2.12)$$

shartlar asosida aniqlaymiz. Bu holda

$$\varepsilon(x) = \begin{cases} u_0(x), & x > 0 \\ 0, & x < 0 \end{cases} \quad (2.2.3)$$

(2.2.13) funksiya A operatorning fundamental yechimi bo‘ladi. Buni isbotlash uchun(2.2.13) dan

$$u'(x) = \{u'(x)\} + h\delta(x-a); \quad h = u(a+0) - u(a-0)$$

ekanligini ko‘rsatish yetarli.

Misollar qaraymiz.

$$1) \qquad 3u'' - u' = \delta(x), \qquad x \in R \quad (2.2.14)$$

tenglamani yeching.

Buning uchun $3u'' - u' = 0$ tenglamaning xarakteristik tenglamasini topamiz

$$3\lambda^2 - \lambda = 0 \qquad \Leftrightarrow \lambda(3\lambda - 1) = 0, \quad \lambda_1 = 0 \quad (2.2.15)$$

$$\lambda_2 = \frac{1}{3} u_0(x) = C_1 + C_2 e^{\frac{x}{3}}$$

(2.2.15) dagi C_1 va C_2 koeffitsentlari $u_0(0) = 0,\ u_0'(0) = \frac{1}{3}$ shartlardan topamiz:

$$\begin{cases} C_1 + C_2 = 0\,, & C_1 = 1 \\ \frac{1}{3}C_2 = -\frac{1}{3}, & C_2 = -1 \end{cases} \quad (2.2.16)$$

(2.2.13) va (2.2.16) ga asosan A operatorning fundamental yechimini topamiz

$$u(x) = \theta(x)\left(e^{\frac{x}{3}} - 1\right)$$

2) Quyida $n=3$ bo‘lganda

$$\frac{d^3}{dx^3} - \frac{d}{dx} = \delta(x) \quad (2.2.17)$$

tenglamadan $\frac{d^3}{dx^3} - \frac{d}{dx}$ operatorning fundamental yechimini topamiz.

Buning uchun

$$u^{III} - u^{I} = 0 \quad (2.2.18)$$

tenglamaning xarakteristik tenglamasini tuzamiz. Bu xarakteristik tenglama $\lambda^3 - \lambda = 0$ dan iborat bo`lib, uning ildizlari

$$\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 1 \quad (2.2.19)$$

xarakteristik sonlardan iborat bo‘lib, (2.2.18) ga asosan (2.2.17) tenglamaning umumiy yechimi quydagicha bo‘ladi.

$$u_0(x) = C_1 + C_2 e^{-x} + C_3 e^{x} \quad (2.2.20)$$

bundagi C_1, C_2 va C_3 koeffitsentlarni (2.2.12) tenglamalar sistemasidan topamiz:

$$\begin{cases} u_0(0) = 0, \\ u_0'(0) = 0, \\ u_0''(0) = 1. \end{cases} \Rightarrow \begin{cases} C_1 + C_2 + C_3 = 0, \\ -C_2 + C_3 = 0, \\ C_2 + C_3 = 1. \end{cases} \Rightarrow C_1 = -1, C_2 = C_3 = \frac{1}{2}$$

(2.2.17) differensial operator uchun uning fundamental yechimi

$$u(x) = \theta(x) u_0(x) = \theta(x)(-1 + \frac{1}{2}e^{-x} + \frac{1}{2}e^{x})$$

ko‘rinishda bo‘ladi. Bunda $\theta(x)$ Xevisayd funksiyasidaniborat.

Quyidagi to`rtinchi tartibli differensial tenglamani qaraylik.

$$u^{(IV)} + 5u'' + 4 = \delta(x) \quad (2.2.21)$$

oxirgi $A = \frac{d^4}{dx^4} + 5\frac{d^2}{dx^2} + 4$ operatorning fundamental yechimini topamiz.

(2.2.21) tenglamaga mos bo‘lgan bir jinsli tenglama

$$u^{(IV)}+5u''+4=0 \quad (2.2.22)$$

ko‘rinishda bo‘ladi.Bu tenglamaning xarakteristik tenglamasi

$$\lambda^4+5\lambda^2+4=0$$

bo‘ladi.Uning xarakteristiksonlari $\lambda_{1,2}=\pm i, \lambda_{3,4}=\pm 2i$ lardan iborat.

Endi (2.2.21) tenglamaning umumiy yechimini topamiz

$$u_0(x)=C_1\cos x+C_2\sin x+C_3\cos 2x+C_4\sin 2x \quad (2.2.23)$$

(2.2.23) tenglamadagi C_1, C_2, C_3 va C_4 koeffitsentlarni (2.2.12) tenglamalar sistemasidan foydalanib topamiz.

$$\begin{cases} u_0(0)=0, \\ u_0'(0)=0, \\ u_0''(0)=0, \\ u'''(0)=1. \end{cases} \begin{cases} C_1+C_3=0, \\ C_2+2C_4=0, \\ -C_1-4C_3=0, \\ -C_2-8C_4=1. \end{cases} \Rightarrow C_1=C_3=0, C_2=\frac{1}{3}, C_4=-\frac{1}{6}$$

Demak , (2.2.21) tenglamadan berilgan operatorning fundamental yechimi

$$u(x)=\theta(x)u_0(x)=\theta(x)(\frac{1}{3}\sin x-\frac{1}{6}\sin 2x)$$

ko‘rinishda bo‘ladi.

$$u''(x)-3u'(x)+2u(x)=f(x), \quad x\in R \qquad (2.2.24)$$

$$f(x)\in C(R), \ f(x)=0, \qquad agar|x|>const$$

tenglamaning xususiy yechimini topamiz.

Buning uchun A operatorning fundamental yechimini topamiz.

$$\varepsilon''(x)-3\varepsilon'(x)+2\varepsilon(x)=\delta(x) \qquad (2.2.25)$$

(2.2.25) tenglamaning chap qismining xarakteristik tenglamasini yechamiz

$$\lambda^2-3\lambda+2=0 \quad \Rightarrow D=9-8=1, \qquad \lambda_{1,2}=\frac{3\mp 1}{2}, \qquad \lambda_1=1, \ \lambda_2=2$$

$$u_0(x)=C_1e^x+C_2e^{2x} \qquad (2.2.26)$$

boshlanfg‘ich shartlardan foydalanib C_1 va C_2 o‘zgarmaslarni aniqlaymiz

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 + 2C_2 = 1 \end{cases} \Rightarrow \begin{matrix} C_1 = -1 \\ C_2 = 1 \end{matrix} \Rightarrow u_0(x) = e^{2x} - e^x$$

$$\varepsilon(x) = \theta(x)(e^{2x} - e^x) \quad (2.2.27)$$

Endi (2.2.7) formuladan foydalanib xususiy yechimni topamiz

$$u(x) = \varepsilon \times f(x) =$$
$$= \int_{-\infty}^{\infty} \theta(x-y)\left(e^{2(x-y)} - e^{x-y}\right) f(x)dy$$
$$= \int_{-\infty}^{x} \left(e^{2(x-y)} - e^{x-y}\right) f(y)dy$$

5)
$u''' + u' = f(x), \ x \in R, \quad f(x) \in C(R), \ f(x) = 0, \ \ agar|x| > const$

Tenglamaning xususiy yechimi uchun formula keltiring .buning uchun quyidagi Koshi masalasini yechamiz.

$$\begin{cases} u_0''' + u_0' = 0, \quad x > 0 \\ u_0(0) = 0 \\ u_0'(0) = 0 \\ u_0''(0) = 1 \end{cases}$$

$u_0''' + u_0' = 0$differensial tenglamaning xarakteristik tenglamasi $k(k^2 + 1) = 0$ bo‘ladi.Bundan $k_1 = 0, \ k_{2,3} = \pm i$ekanligidan

$$u_0(x) = C_1 + C_2 cosx + C_3 sinx$$

umumiy yechim kelib chiqadi.C_1, C_2, C_3o‘zgarmaslarni

$$u_0(0) = 0, u_0'(0) = 0, u_0''(0) = 1$$

shartlardan topamiz:

$$\begin{cases} C_1 + C_2 = 0 \\ C_3 = 0 \\ -C_2 = 1 \end{cases} \Rightarrow C_1 = 1, C_2 = -1, C_3 = 0.$$

Demak,

$$u_0(x) = 1 - cosx.$$

Endi fundamental yechimni aniqlaymiz:

$$\varepsilon(x) = \theta(x)(\ 1 - cox)$$

izlanayotgan$u(x)$ yechim $\varepsilon(x)$va$f(x)$ funksiyalarning o‘ramasidan iborat

$$u(x) = \varepsilon \times f(x) =$$

$$= \int_{-\infty}^{\infty} \theta(x-y)(1 - \cos(x-y)\ )f(x)dy$$

$$= \int_{-\infty}^{x} (1 - \cos(x-y)\,)f(y)dy.$$

III BOB. Umumlashgan funksiyalar uchun Laplas almashtirishlari va uning tadbiqlari

§3.1.Umumlashgan funksiyalar uchun Laplas almashtirishlari

Fransuz matematigi Loran Shvarst tomonidan 1945-yilda kiritilgan taqsimot tushunchasi funksiya tushunchasining umumlashmasidan iborat.

Bu yangi tushuncha muhandisning sohasida sermahsul bo‘lib, fizika va tehnikada uchraydigan ba’zi qarama-qarshi obrozlarga , masalan Dirakning δ funksiyasiga aniq matematik ma’no berishiga imkon berdi, bundan tashqari, klassik matematik analizda to‘planib qolgan ko‘pchilik qiyinchiliklardan qutilishga imkon berdi.

Agar funksiya klassik tushunchasini yangi nuqtai nazardan qarasak,taqsimot tushunchasining oson tushuntirish mumkin . Odatdagicha tushunishda $y = f(x)$ funksiya deganda uning alohida qiymatlar x ning alohida qiymatlari orasidagi moslik tushunilib, bunday tushunishga yo‘zgaruvchi miqdorning aniq qiymatini, masalan, har bir alohida vaqt momenti uchun qo‘llanishni o‘rnatish imkoniyati haqidagi fizik tasavvurga mos keladi.

Shu bilan birga haqiqatda o‘lchash bilan aniqlanuvchi narsa kuchlanishningo‘zi emas, uning o‘lchash asbobiga tayanish hisoblanadi. Har bir o‘lchash faqat berilgan nuqtani o‘z ichiga oluvchi biror oraliq bo‘yicha o‘rtaga yoki o‘rtaga vaznlashgan qiymatni beradi, bunda $\varphi(x)$ vazn funksiya va o‘lchov oralig‘i o‘lchov asbobiga bog‘liq.$\varphi(x)$funksiyani biz kelgusida asosiy deb ataymiz. Shunday qilib, o‘lchashlar tegishli mos (katta yoki kichik) oraliqlarga tarqatilgan

$$\int f(x)\varphi(x)dx,$$

integralning qiymatlarini beradi. Bunday integrallashda $f(x)$ funksiyaning alohida qiymatlari emas, $f(x)$qiymatlari o‘zgarish butun jarayon boshqacha aytganda barcha x lar ustidan $f(x)$ taqsimot muhim rol o‘ynaydi.

$f(x)$funksiya butun $-\infty < x < +\infty$ son o‘qida aniqlangan bo‘lsin, qisqalik uchun R^1 deb belgilaymiz. Barcha keying tasdiqlar umumiy holda o‘rinli bo‘lishi uchun faraz qilamiz $f(x)$ funksiya har bir chekli oraliqda integrallanuvchi yoki qabul qilinganidek Lebek manosida lokal integrallanuvchi bo‘lsin deb faraz qilamiz.

Bu faraz umumiy mazmun qurish mahsadida keltiriladi, tastiq tadbiqlarda uchraydigan funksiyalar hamma vaqt Riman manosida integrallanuvchi va shuning uchun injiner umumiyroq tushuncha kiritishdan xafsiramasa ham bo‘ladi.

Asosiy $\varphi(x)$ funksiyalar shunday tanlanadiki , ular ustida amallar hech qanday qiyin- chilik tug‘dirmasin.Buning uchun har bir $\varphi(x)$ funksiya butun R^1 o‘qda aniqlangan bo‘lsada kichik biror,chekli oraliqdan tashqarida (u turlichaφfunksiyalar uchun turlicha bo‘lishi mumkin.) nolg teng degan faraz kiritilib keyin $\varphi(x)$ funksiya cheksiz differensiallanuvchi deb qabul qilinadi.

Bundan kelgusi uchun zarur natija kelib chiqadi, $\varphi(x)$ funksiya va uning barcha hosilalari $\varphi(x)$ aniqlangan chekli oraliqlar uchlarida nolga teng chunki bu nuqtalarda “tashqi” hosilalari nolga teng.

Barcha bu φlar tanlanmasi $\mathcal{D}$ orqali belgilanuvchi φfunksiyalar fazosini tashkil etadi.

φfunksiya haqida aytish faraz natijasiga ko‘ra

$$\int_{-\infty}^{+\infty} f(x)\varphi(x)dx$$

integralni haqiqatdan faqat chekli oraliqqa qo'llash mumkinligi kelib chiqadi, shuning uchun cheksizlikda yaqinlashish bo'yicha har qanday qiyinchilik paydo bo'lmaydi, demak, integral ihtiyoriy f funksiya va $\mathcal{D}$ fazodagi har qanday φ funksiya uchun mavjud.Bu integralni $f va \varphi$ kabi " ichki ko'paytma" kabi tushunish mumkin, bu esa oddiy belgilashdan foydalanishga

$$\int_{-\infty}^{+\infty} f(x)\varphi(x)dx = \langle f, \varphi \rangle$$

belgilash kiritishga imkon beradi.

Shunday qilib , agar f funksiya tayinlangan bo'lsa, u holda ,$\mathcal{D}$ fazodagi har bir φ funksiyaga biror sonni mos qo'yish mumkin. Bu xolatni quyidagicha bayon etish mumkin.

f funksiya$\mathcal{D}$ fazoda $\langle f, \varphi \rangle$ funksionalni aniqlaydi bu funksional chiziqlilik xossasiga ega, ya'ni

$$\langle f, \varphi_1 + \varphi_2 \rangle = \langle f, \varphi_1 \rangle + \langle f, \varphi_2 \rangle \qquad (3.1.1)$$

Shvarst nazariyasining asosiy g'oyasi$f$ funksiyani$\langle f, \varphi \rangle$funksional qiymatlari bilan ifodalashdan iborat. Bu g'oya naqadar chuqur natijalargaolib kelishini bu funksiya differensiallanuvchi bo'lmagan holda ham har lokal integrallanuvchi $f(x)$ funksiya uchun hosilani kiritishga imkon berishidan ko'rinadi. Haqiqatdan, agar$f(x)$ klassik manoda lokal integrallanuvchi $f(x)$ hosilaga ega bo'lsa, u holda yuqorida aytilganlarga ko'ra bu hosila

$$\langle f', \varphi \rangle = \int_{-\infty}^{+\infty} f'(x)\varphi(x)dx$$

funksional bilan ifodalanadi. Agar bu integralni bo'laklab integrallash bilan soddalashtirsak, u holda oraliqlar chegaralari beradigan hissalar yo'qoladi, chunki unda φ yo'qoladi va faqat

$$\langle f', \varphi \rangle = -\int_{-\infty}^{+\infty} f(x)\varphi'(x)dx = -\langle f, \varphi' \rangle$$

qoladi. Bu tenglikning o‘ng qismi har bir lokal integrallanuvchi f funksiya uchu aniq bir manoga ega va $\mathcal{D}$ fazodagi funksionaldan iborat .

Shuning uchun bunday har bir f funksiyaga umumlashganhosilani keltrib qo‘yishdan o‘z–o‘zidan paydo bo‘ladi. Umumlashgan hosilani oddiy hosiladan farqlash uchun uni D orqali belgilaymiz shunday qilib biz

$$\langle Df, \varphi \rangle = -\langle f, \varphi' \rangle \tag{3.1.2}$$

deb yozishimiz mumkin.

Shunga etibor qaratamizki bunday shartning ko‘rsatilishi faqat Df ning qanday qiymati har bir φ funksiya bilan mos qo‘yilishini o‘rnatadi . Demak ,Df umumlashgan hosila klassik $f'(x)$ hosiladan farqli ravishda son manosiga ega emas.

Misol 3.1.1. $u(x)$ birlik sakrashga ega funksiya bo‘lsin, ya’ni

$$u(x) = \begin{cases} 0, & agar x \leq 0 \\ 1, & agar x > 0 \end{cases}$$

klassik ma’noda bu funksiya differensiallanuvchi emas, chunki $x = 0$ nuqtada uzilishga ega taqsimotlar nazariasi nuqtai nazaridan bu holat boshqacha izohlanadi.Haqiqatdan,

$$\langle Du, \varphi \rangle = -\langle u, \varphi' \rangle = -\int_{-\infty}^{+\infty} u(x)\varphi'(x)dx = -\int_0^{\infty} \varphi'(x)dx = \varphi(0) \tag{3.1.3}$$

Demak, Du umumlashgan hosilani ifoda bo‘yicha funksional $\mathcal{D}$ fazodagi har bir φ funksiya uchun $\varphi(x)$ qiymatga ega.Bu nimaniangalatishini keyinchalik qaraymiz.

Ko‘rsatilganjarayonni tasvirlab biz f funksiyaning $D^k k-$ umumlashgan hosilasiga asosan

$$\langle D^k f, \varphi \rangle = (-1)^k \langle f, \varphi^{(k)} \rangle (k = 1,2, \dots) \tag{3.1.4}$$

har bir lokal integrallanuvchi funksiya yetarlicha ko‘p marta umumlashgan manoda differensiallanuvchi bo‘lishi mumkin.

Misol 3.1.2.

$$\langle D^k f, \varphi\rangle = (-1)^k \langle f, \varphi^{(k)}\rangle = (-1)^k \int_0^{\infty} \varphi^{(k)}(x)dx == (-1)^{k-1}\varphi^{(k-1)}(0)$$

differensiallanuvchanlikning yangi tushunchasi eski tushunchani qamrab oladi ya'ni agar $f(x)$ funksiya $k-$ tartibli$f^{(k)}(x)$hosilaga ega bo'lsa, u holda bu hosila $D^k f$ umumlashgan hosila aniqlagan funksionalni aniqlaydi.

Taqsimot. Yuqoridagi tushunchalarklassik analizdan zamonaviy taqsimotlar nazariyasiga o'tishga tayyorgarlik maqsadiga ega edi. Xususiy holdan f funksiya bilan aniqlanuvchi va $\mathcal{D}$ fazodagi har bir φ funksiyaga $\langle f, \varphi\rangle$ qiymatni mos qo'yuvchi funksionalga asoslanib endi $\mathcal{D}$ dagi har bi φ funksiyaga $\langle T, \varphi\rangle$ deb belgilanuvchi sonni mos qo'yadigan Tabstrak funksionalga qaraymiz.Bunda bu funksiyaning chiziqli [yuqoridagi (1) munosabat kabi] va uzluksiz bo'lsin deb talab qilamiz. Oxirgi xossa maxsus tasvirlashni talab etadi.

Bunda biz odatdagidek funksiyaning uzluksizlik ta'ifiga tayanamiz.$g(x)$funksiya$x$ nuqtada uzluksiz deb aytiladi gar x ga yaqinlashuvchi har bir $x_n(n = 0,1, \dots)$ ketma-ketlik uchun

$$\lim_{n\to\infty} g(x_n) = g(x)$$

tenglik o'rinli bo'lsa. "$n \to \infty$da$x_n x$ ga yaqinlashadi yoki belgilashlardagi

$$\lim_{n\to\infty} (x_n) = x$$

yozuvning ma'nosi klassik analizdan ma'lum.

$\mathcal{D}$fazoda$\lim_{n\to\infty} \varphi_n = \varphi$ ifodaning ma'nosiga kelsak, u holda bu manoni dastlab aniqlashtrish lozim, quyidagi ta'rifni kiritamiz.

Ta'rif: $\mathcal{D}$fazodagi φ_n ketma-ketlik $n \to \infty$ da $\mathcal{D}$ fazodagi φ ga yaqinlashadi, agar tayinlangan$T$kesmadan tashqarida yotuvchi barcha φ_n lar nolga teng bo'lsa va har bir alohida $k = 0,1, \dots$ lar uchun $n \to \infty$ da $\varphi_n^{(k)}$ lar biror$I$ kesmada $\varphi^{(k)}(x)$ ga intilsa.

Bu yaqinlashish tushunchasini $x_n \to x$ klassik analiz tushunchasidan aniqroqlash uchun

$$\lim_{n\to\infty}(\mathcal{D})\varphi_n = \varphi$$

deb yozamiz. Endi odatdagi $g(x)$ funksiya uzluksizligi tushunchasiga o‘xshash quyidagi ta’rifni kiritamiz.

Ta’rif: T funksional $\mathcal{D}$ fazoda φ elamentda uzluksiz deb ataladi, agar φ ga yaqinlashuvchi har bir φ_n ketma-ketlik uchun

$$\lim_{n\to\infty}\langle T,\varphi_n\rangle = \langle T,\varphi\rangle$$

tenglik o‘rinli bo‘lsa.

Endi biz quyidagi tastiq uchun barcha zarur tushunchalarga egamiz. $\mathcal{D}$ fazoning barcha elementlarida uzluksiz bo‘lgan, $\mathcal{D}$ fazoda aniqlangan chiziqli T funksional ***taqsimot*** deb aytiladi. Bu abstrak tushunchaga misollar keltiramiz.

Lokal integrallanuvchi $f(x)$ funksiya bilan aniqlangan $\langle f,\varphi\rangle$ funksional taqsimot ekanligini isbotlash mumkin. Bunday taqsimotni regulyar taqsimot deb aytamiz va uni klassik ma’nodagi f funksiyadan farqlash zarur bo‘lgan holda $[f]$ deb belgilaymiz.

Barcha taqsimotlar to‘plamini $\mathcal{D}'$ orqli belgilanadigan fazo deb qarash mumkin ($\mathcal{D}$ fazoga nisbatan ikki yoqlama fazo). Bu fazo barcha regulyar taqsimotlarni asosiy shubhasiz boshqa elementlarini o‘z ichiga oladi buni biz faqt misolda ko‘rsatamiz.

Yuqorida biz o‘rnatgan $u(x)$ birlik sakrashning umumlashgan hosilasi

$$\langle Du,\varphi\rangle = \varphi(0)$$

xossaga ega. Du –funksional, chunki har bir φ ga sonni mos qo‘yadi, ravshanki, Du funksional faqat chiziqli, balki uzluksiz, chunki φ_0 yuqorida aniqlangan manod φ ga yaqinlashsa, u holda $\varphi_n(0)$ trival ravishda $\varphi(0)$ ga intiladi. Demak, Du taqsimot, uni δ harfi bilan belgilaymiz.

Buning uchun

$$\delta = Du \quad (3.1.5)$$

$$\langle\delta,\varphi\rangle = \varphi(0) \quad (3.1.6)$$

munosabatlar o‘rinli, shunga qaramay δ shubhasiz regulyar taqsimot bo‘lmaydi.

Haqiqatdan , agarδ ,$\delta(x)$funksiyadan iborat bo‘lganda edi, u holda

$$\int_{-\infty}^{+\infty} \delta(x)\varphi(x)dx = \varphi(0) \qquad (3.1.7)$$

tengliko‘rinli bo‘lishi kerak edi, lekin bunday xossaga esa lokal integrallanuvchi $\delta(x)$funksiya mavjud emas.

Endi tushunarliki, nima? Uchun (3.1.6) tenglama bilan aniqlangan taqsimotni Dirak tomonidan kiritilgan obrazni belgilash uchun qabul qilingan δ- harfi bilan belgiladik bu obroz (3.1.7) munosabat bilan ifodalangan xossaga ega bo‘lishi lozim , ravshan ediki , bunday obraz klasik manodagi funksiya ko‘rinishida mavjud bo‘lishi mumkin emas.

Bu obrazni limit tushunchasi ko‘rinishida tasavvur etiladi,tabiatan bu haqiyqatda mavjud bo‘lmagan obraz sifatida funksiya klassik analiz qoidalari asosida amallar bajarilganda aniq matematikaning mustahkam asosini targ‘ib qilgan bo‘ladilar. Aksincha δ ni taqsimot deb tushinib unga eski “implus” nomini bersa ham bo‘lar edi, biz oldin $\delta(x)$ “funksiyadan” biror kutgan natijaga erishishga imkon beruvchi ziddiyatsiz matematik obektni qo‘lga kiritamiz . $\varphi(x)$funksiyadan$\varphi(0)$ qiymatni “chiqarish” ni endi (3.1.6) tenglik manosida tushunarliki, bunda$\langle\delta,\varphi\rangle$ obrazni integral sifatida tushinish kerak emas.

(3.1.5) tenglik oldin δ Dirak “funksiya” siga xos deb olingan xossani ham aniq ifodalaydi, yani $\delta - u(x)$ funksiyaning “hosilasi” hisoblanadi, bunda bu hosila $x = 0$ nuqtadan tashqari barcha joyda nolga teng va $x = 0$nuqtada cheksizlikka teng.

Endi hosilani umumlashgan hosila manosida tushunish lozim.δtaqsimotning umumlashmasi

$$\langle\delta_h,\varphi\rangle = \varphi(h) \qquad (3.1.8)$$

tenglik bilananiqlanuvchi $\delta_h$taqsimot hisoblanadi.$\delta_h$taqsimot oldingi joyida

postulatlangan$x = 0$nuqtada emas $x = h$ nuqtada amal qiluvchi implus tushunchasiga (δ funksiyaning shiljishi) mos keladi.

Umumiy nazariyaga qaytamiz

$$\langle T_1 + T_2, \varphi\rangle = \langle T_1, \varphi\rangle + \langle T_2, \varphi\rangle$$

tenglik bilan aniqlanuvchi taqsimotlar yig'indisidan tashqari ikkita taqsimot ko'paytmasi qizziqtiradi. Regulyar taqsimotlar uchun maxsus

$$\langle f_1 f_2, \varphi\rangle = \int_{-\infty}^{+\infty} f_1(x) f_2(x)\varphi(x)dx$$

tenglik bilan aniqlanishi lozim , lekin bu integral hamma vaqt manoga ega emas, chunki ikkita lokal integrallanuvchi funksiyalar ko'paytmasi hamma vaqt lokal integrallanuvchi bo'lavermaydi (misol: $x^{-1/2} \cdot x^{-1/2} = x^{-1}$). Shuning uchun ikkita taqsimt ko'paytmasi umumiy ko'rinishida tariflanishiga yo'l qo'ymaydi, taqsimotlardan biri funksiya bo'lgan hol bilan chegaralanishga to'g'ri keladi.

T taqsimotning$\alpha(x)$ funksiyasiga ko'paytmasi R^1 da cheksiz diffferensiallanuvchi bo'lib

$$\langle \alpha(x)T, \varphi\rangle = \langle T, \alpha\varphi\rangle \qquad (3.1.9)$$

tenglik bilan aniqlanadi. Quyidagiga etiborni qaratamiz:$\alpha(x)\mathcal{D}$fazoga tegishli deb faraz qilinganligi uchun bu fazoga $\alpha(x)$ ham tegishli bo'ladi, xususiy holda $T = f$ bo'lsa, ko'paytma ta'rifidan

$$\int_{-\infty}^{+\infty} [\alpha(x)f(x)]\varphi(x)dx = \int_{-\infty}^{+\infty} f\ (x)[\alpha(x)\varphi(x)]dx$$

$trivaltenglikkaolibkelinadixususiyholdaT = \delta$bo'lsa,

$$\langle \alpha(x)\delta, \varphi\rangle = \langle \delta, \alpha\varphi\rangle = \alpha(0)\varphi(0) = \langle \alpha(0)\delta, \varphi\rangle$$

ni olamiz demak

$$\alpha(x)\delta = \alpha(0)\delta \qquad (3.1.10)$$

bu munosabat avval δ Dirakt funksiyasi uchun o'z-o'zidan tushinarli deb hisoblanar edi va bunda har bir $\alpha(x)$funksiya uchuntaqsimot haqidagi tushuncha asosida mos analizni qurish uchun tabiiyki taqsimotlar uchun differensiallash tushunchasini kiritish zarur. Buni (3.1.4) munosabat asosida bajarish mumkin . Bu munosabat funnksiyalar uchun o'rnatilgan bo'lsada uni birdaniga taqsimotlar xoli uchun ham qo‘llash mumkin T taqsimotning $k-$ tartibli hosilasi

$$\langle D^kT, \varphi(x)\rangle = (-1)^k\langle T, \varphi^{(k)}(x)\rangle \qquad (3.1.11)$$

tenglik bilan aniqlanadi.Hosilaning bunday ta’riflashda har bir taqsimot yetarlicha ko‘p marta differensiallanishi mumkin.

Misol 3.1.3. $T = \delta$ uchun (3.1.6) formulaga ko‘ra

$$\langle D^k\delta, \varphi(x)\rangle = (-1)^k\langle \delta, \varphi^{(k)}(x)\rangle = (-1)^k\varphi^{(k)}(0).$$

ga ega bo‘lamiz . Shunday qilib avval noaniq mavjudmas deb hisoblagan δ taqsimot hosilasining aniq ta’rifini oldik (dipol deb nomlanuvchi v. k).umumiy holda biz hosilalarni D simvol bilan belgilaymiz lekin δ taqsimot hosilalari uchun fizikada 10 yillar davomida qabul qilingan odatga yuz tutib oddiy hosilalar uchun qabul qilingan ya’ni yuqori indeks yordamida belgilashdan foydalanamiz . Bunda bazida bir vaqtda qavslarda R^1 dagi mos o‘zgaruvchini ham ko‘rsatamiz, masalan $\delta(x)$, $\delta^{(k)}(x)$ yoki $\delta(t)$, $\delta^{(k)}(t)$ unga mos ravishda δ_h o‘rniga $\delta(x-h)$ deb yozamiz.

Agar $f(x)$ funksiya butun R^1 o‘qda k marta differensiallanuvchi bo‘lsa, u holda $k-chi$ $f^{(k)}(x)$ hosila bilan aniqlanuvchi regulyar taqsimot (uning lokal integrallanuvchanlik fazosida) $D^k\mathrm{f}$ umumlashgan hosila bilan ustma – usttushadi.Bunga $\langle\delta^{(k)}, \varphi\rangle$ ni integral ko‘rinishda yozib ,bu integralni k marta bo‘laklab integrallash bilan oson ishonch xosil qilish mumkin.

$f^{(k)}(x)$ va D^kf orasida bog‘lanish mavjud, agar quyidagi tasdiqlarda ko‘p uchraydigan xol o‘rinli bo‘lsa. $f(x)$ funksiya $f^{(k)}(x)$ mavjud bo‘lmagan bitta yagona a maxsus nuqtaga ega shu bilan birga $x < a$ va $x > a$ da $f(x)$ $k-$ tartibgacha hosilalarga ega, ya’na $f(x), f'(x), \ldots, f^{(k-1)}(x)$, $x = a$ nuqtada faqat birinchi tur uzilishga ega, demak,ular $x = a$ dan chapda va o‘ngda

$$f(a-0), f'(a-0), \ldots, f^{(k-1)}(a-0);$$
$$f(a+0), f'(a+0), \ldots, f^{(k-1)}(a+0)$$

limit qiymatlarga ega .Oldin aytilganidek, $f^{(k)}(x)$, $f(x)$ ning $k-$hosilasini bildiradi , $x = a$ nuqtadan tashqarida barcha joyda mavjud.

U holda

$$D^k f = \left[f^{(k)}(x)\right] + \left[f^{(k-1)}(a+0) - f^{(k-1)}(a-0)\right]\delta(x-a)$$
$$+ \left[f^{(k-2)}(a+0) - f^{(k-2)}(a-0)\right]\delta'(x-a) + \cdots$$

$$+\left[f\ \ (a+0) - f\ \ (a-0)\right]\delta^{(k-1)}(x-a) \qquad (2.1.12)$$

munosabato'rinli .

Birinchi qismda biz $[f^{(k)}(x)$] uchun to'liq belgiashni munosabatning ikkala qismiga faqat taqsimotlar kirishini aniq ko'rsatish maqsadida qo'lladik haqiyqatda taqsimot taqsimotga teng bo'lishi mumkin.Bazida qisqalik uchun bu taqsimot$I$ ochiq oraliqda funksiyaga teng deb ham aytiladi . Bu bilan quyidagicha manotushuniladi.$\langle I, \varphi\rangle$taqsimot$I$ oraliqdan tashqarida nolga teng bo'lgan har bir φ uchun $\langle f, \varphi\rangle$ ga teng shu manoda masalan δtaqsimot nol funksiyaga $-\infty < x < 0$ va $0 < x < \infty$ oraliqlarda teng deb atash mumkin . Haqiqatdan, agar φ masalan $-\infty < x < 0$ oraliqdan tashqarida ya'ni$0 < x < \infty$ nolga teng bo'lsa, u holda $\langle \delta, \varphi\rangle = \varphi(0) = 0$ ikkinchi tomondan $\langle 0, \varphi\rangle = 0$ demak, $\langle \delta, \varphi\rangle = \langle 0, \varphi\rangle$ ya'ni $\delta = 0$ $(-\infty < x < 0$ oraliqda).

Klassik Laplas almashtrishi faqat $t \geq 0$ da aniqlangan funksiyalarga qo'llab bo'ladi, agar butun to'qda aniqlangan funksiyalarni qarasak (bu maqsadga mofiq, masala teskari Laplas almashtrishini bajaruvchi kompleks integraldan foydalanishda) u holda $t < 0$ da ularni nolga teng deb qabul qilish lozim. Taqsimotlarning Laplas bo'yicha almashtrishga kelsak, bu almashtrishning manosiga taqsimotlar uchun qo'llab bo'ladi. Bu xossani aniq bayon etish uchun quyidagi tushunchani kiritamiz**.** R^1da uzluksiz $\varphi(\mathrm{x})$ funksiyaning tashuvchisi deb $\varphi(\mathrm{x}) \neq 0$ shartni qanoatlantruvchi eng kichik yopiq to'plamda aytiladi yani u limit nuqtalar bilan to'ldirilgan $\varphi(\mathrm{x}) \neq 0$ shartni qanoatlantruvchi x ning qiymatlar to'plamidan iborat. Boshqacha aytganda , $\varphi(x)$ funksiyaning tashuvchisi tashqarisida $\varphi(\mathrm{x}) = 0$ bo'lgan eng kichik yopiq to'plam hisoblanadi.

Taqsimotlar uchun shunga o'xshash tushunchani bayon etish uchun avvalo taqsimot uchun funksiyaning nolga tengligi haqidagi tushunchaga o'xshash tushunchani kiritish zarur.

Taqsimot oraliqda nolga teng deb qabul qlinadi, agar $\langle T,\varphi\rangle = 0$ tashuvchisi I da yotuvchi har qanday φ funksiya uchun bajarilsa. Bunday tushunchaning kritilishi quyidagi tarifni berishga imkon beradi.

Taqsimot tashuvchisi tashqarisida$T = 0$ bo‘lgan eng kichik yopiq to‘plamdan iborat.

Misol 3.1.4.$\langle\delta,\varphi\rangle = \varphi(0)$taqsimot tashuvchilari $|x| > 0$ ochiq oraliqda yotgan barcha φ lar uchun nolga teng, chunki bu holda $\varphi(0) = 0$. Demak, $|x| > 0$ ochiq oraliqda $\delta = 0$, $|x| > 0$ ga to‘ldiruvchi to‘plam tashqarisida $\delta = 0$ bo‘lgan eng kichik yopiq to‘plam hisoblanadi. Umumiy qilib δ tashuvchi sifatida yagona $x = 0$ nuqtaga ega. Shu holda $\delta^{(k)}$ uchun ham o‘rinli bo‘ladi.

Taqsimot butun R^1 son o‘qida aniqlangan, lekin Laplas almashtirishi faqat tashuvchilari $t \geq 0$ yarim o‘qda yotgan taqsimotlar uchun aniqlangan. Bu taqsimotlar fazosi $\mathcal{D}'_+$ deb belgilanadi.Bu faqat $t < 0$ da nolga teng funksiya taqsimot sifatida tushiniladigan taqsimotga tegishli.

Taqsimotning tasvirini $\mathfrak{L}$ da barcha $\mathcal{D}'_+$ fazoning biror qism fazosi uchun to‘g‘ri keladigan turli usullar bilan aniqlash mumkin. Biz bunday chekli tartibli taqsimotlarga qo‘llab bo‘ladigan qilib tayinlaymiz. R^1da$h(t)$ funksiya chekli $k-$ tartibli umumlashgan hosilasi bo‘lgan T taqsimotlarga ya’ni $T = D^{(k)}h(t)$ taqsimotga aytiladi.

Misol 3.1.5. Malumki, $\delta = Du$. Lekin bu tenglik δ ni chekli tartibga ega taqsimot deb hisoblash uchun yetarli emas, chunki $u(t)$funksiyadan integralni qurish zarur bunda

$$\mathrm{h(t)} = 0 \text{ agar} t \leq 0$$

$$h(t) = t \text{agar } t > 0.$$

bu funksiya R^1 da uzluksiz $t = 0$ da differensiallanuvchi emas, lekin mavjud. Haqiqatdan,

$$\langle Dh,\varphi\rangle = -\langle h,\varphi'\rangle = -\int_0^{+\infty} t\varphi'(t)dt = -t\varphi(t)|_0^{+\infty} + \int_0^{+\infty} \varphi(t)dt =$$

$$= 0 + \int_{0}^{+\infty} \varphi(t)dt = \int_{-\infty}^{+\infty} u(t)\varphi(t)dt = \langle u, \varphi\rangle$$

shuning uchun $Dh = u(t)$ ga demak (3.1.5) formulaga ko‘ra

$$D^2h = Du = \delta \qquad (3.1.13)$$

shunday qilib δ taqsimot chekli tartibga ega ekan.

Chekli tartibli taqsimotda $\mathcal{D}'_+$ fazo tartibiga kiruvchi va elementlari sifatidagi shartlarni qanoatlantiruvchi $\mathcal{D}'_0$ qism fazoni belgilaymiz. Qaralayotgan taqsimotlarni aniqlovchi $h(t)$ funksiyalar $t < 0$ da nolga teng, ya’ni

$$h(t) = 0,\ t < 0 \qquad (3.1.14)$$

va bundan tashqari, bu funksiyalar uchun

$Res > \sigma$da$\mathfrak{L}\{h(t)\ \ \}$ (3.1.15)

mavjud. Agar T taqsimot $\mathcal{D}'_0$ ga tegishli va $T = \mathcal{D}^k\, h(t)$ bo‘lsa, u holda bu taqsimotning $\mathfrak{L}$ −tasviri

$$\mathfrak{L}\{T\} = s^k\mathfrak{L}\{h(t)\} \qquad (3.1.16)$$

tenglik bilan aniqlanadi. O‘ng qismi $Res > \sigma$ da mavjud va unda analitik funksiyadan iborat.

Misol 3.1.6. Bizga ma’lumki δ taqsimot chekli tartibga ega, bunda $\delta = D^2h$,. bu yerda h −oldingi misollardagi (3.1.14) va (3.1.15) shartlarni qanoatlantiruvchi funksiya. Demak, δ taqsimot $\mathfrak{L}$ −tasvir

$$\mathfrak{L}\{\delta\} = s^2\mathfrak{L}\{h\} = s^2\mathfrak{L}\{t\} = s^2 \cdot \frac{1}{s^2} = 1 \qquad (3.1.17)$$

ga teng taqsimot ekan. Malumki, $\delta(t - t_0) = D^2h(t - t_0)$ demak

$$\mathfrak{L}\{\delta(t - t_0)\} = s^2\mathfrak{L}\{h(t - t_0)\} = s^2e^{-t_0s}\mathfrak{L}\{h(t)\} = e^{-t_0s}$$

(3.1.18)

yana biz

$$\mathfrak{L}\{\delta^{(k)}\} = \mathfrak{L}\{D^{k+2}h(t)\} = s^{k+2}\mathfrak{L}\{h\} = s^k \quad (3.1.19)$$

§3.2. Laplas almashtirishining xossalari

Klassik Laplas almashtirishda amallar akslantirishning muhim qoidasi bo‘lib, differensiallash teoremasi (V –qoida) xisoblanadi. Shunga o‘xshash qoida taqsimotlar uchun quyidagicha ifodalanadi

V' –**qoida**. Agar $\mathfrak{L}\{T\} = F(s)$mavjud bo‘lsa, u holda

$D^nT \leftarrow s^nF(s)$

bo‘ladi.Bu yerda D^nT –umumlashgan hosila

Isbot. Agar $T = D^k h(t)$bo‘lsa, u holda $D^nT = D^{n+k}h(t)$ shuning uchun

$\mathfrak{L}\{T\} = F(s) = s^k\mathfrak{L}\{h\}va\mathfrak{L}\{D^nT\} = s^{n+k}\mathfrak{L}\{h\}$

Demak,

$\mathfrak{L}\{D^nT\} = s^nF(s)$

V'-qoida klasik V –qoidadan taqsimotlar uchun hech qanday manoga ega emas.

Boshlang‘ich qiymatlar yo‘qligi bilan farq qiladi, chunki qandaydir joyda taqsimot aniq qiymatga ega emas. Shunga qaramay V'-qoida $T, f(t)$ funksiyadan iborat bo‘lmaganda ham V-qoida bilan ustma –ust tushadi.

Haqiqatdan, $f(t)$, $\mathcal{D}'_+$ dagi taqsimot sifatida nol deb aniqlanadi , shuning uchun chap tomonli $f, f', \ldots$, qiymatlar $t > 0$ da odatdagi ma’noda mavjud va $t \to +0$ da $f(+0), f'(+0), \ldots$, limitik qiymatlarga ega bo‘ladi. U holda (12) formulaga ko‘ra

$D^nf = f^{(n)} + f^{(n-1)}(+0)\delta + \ldots + f(+0)\delta^{(n-1)}$ (3.2.1)Agar $\mathfrak{L}\{f^{(n)}\}$ ham, $\mathfrak{L}\{f\}$ ham klassik ma’noda mavjud bo‘lsa, u holda ular taqsimotlar nazariyasi manosida ham mavjud. Shuning uchun (3.2.1) tenglikka Laplas Laplas almashtirishini qo‘llab (3.1.17) va (3.1.19) tengliklarni etiborga olib, V' – qoida asosida

$$s^n\mathfrak{L}\{f\} = \mathfrak{L}\{f^{(n)}\} + f^{(n-1)}(+0) + \ldots + f(+0)s^{n-1}$$

ni olamiz, ya'ni V-qoida qo'llanishidagi natijaga kelamiz.

Misol 3.2.1. $T = u(t)$ bo'lsa, u holda $DT = \delta V'$ qoidaga asosan

$$\mathfrak{L}(\delta) = s\mathfrak{L}\{u\} = s \cdot \frac{1}{s} = 1$$

ni beradi .

Misol 3.2.2. $T = u(t - t_0)$ bo'lsin , demak $DT = \delta(t - t_0)V -$ qoidani $u(t - t_0)$ funksiyag qo'llab bo'lmaydi deyilgan, chunki bu funksiya t_0 nuqtada differensiallanuvchi emas. Lekin agar $u(t - t_0)$ ni taqsimot deb tushunsak va V' qoidani qo'llasak

$$\mathfrak{L}\{Du(t - t_0)\} = \mathfrak{L}\{\delta(t - t_0)\} = s\mathfrak{L}\{u(t - t_0)\} = s \cdot \frac{e^{-t_0 s}}{s} = e^{-t_0 s}$$

natija olinadi.

O'rama. $T_1 = D^{k_1}h_1(t)$ va $T_2 = D^{k_2}h_2(t)$, $\mathcal{D}'_0$ fazodagi ikita taqsimot bo'lsin, bunday taqsimotlar uchun ularning o'ramasi

$$T_1 * T_2 = D^{k_1+k_2}[h_1(t) * h_2(t)] \quad (3.2.2.)$$

tenglik bilan aniqlanadi. Bunda $h_1 * h_2$ ifodani klassik ma'noda tushinish lozim ya'ni

$$h_1 * h_2 = \int_0^t h_1(\tau)\, h_2(t - \tau)d\tau \qquad (3.2.3)$$

taqsimotlarning bunda o'ramasi hamma vaqt mavjud va $t < 0$ da nolga teng lokal integrallanuvchi f_1 va f_2 funksiyalar bilan aniqlanganda klassik o'rama bilan ustma-ust tushadi.

Isbot uchun

$$h_1(t) = \int_0^t f_1(\tau)d\tau = f_1 * 1, h_2(t) = \int_0^t f_2(\tau)d\tau = f_2 * 1$$

funksiyalarni tuzamiz, ular barcha t larda uzluksiz va $t<0$ da nolga teng.

Kelgusi uchun quyidagi umumiy teorama kerak bo‘ladi. $g(t)$funksiya barcha t lar uchun aniqlangan local integrallanuvchi funksiyasi bo‘lsa u holda

$$G(t)=\int_0^t g(\tau)d\tau=g*1$$

uchun$DG=g$ munosabat o‘rinli. Bu yerda $g,[g]$ taqsimot sifatida tushiniladi.

Haqiqatdan,

$$\langle DG,\varphi\rangle=-\langle G,\varphi'\rangle=\int_{-\infty}^{+\infty} G(\tau)\varphi'(\tau)d\tau$$

integralga umumlashgan bo‘laklab integrallashni qo‘llaymiz. $\varphi(t)$biror chekli oraliqdan tashqarida nolga tengligini xisobga olib

$$\langle DG,\varphi\rangle=-\int_{-\infty}^{+\infty} g(\tau)\varphi(\tau)d\tau=\langle g,\varphi\rangle,\qquad DG=g$$

ni olamiz, bu teoremaga ko‘ra

$$Dh_1=f_1,\ \ Dh_2=f_2$$

demak, (3.2.1) ta’rifga ko‘ra isbotlangan teoremaga asosan

$$T_1*T_2=D^2(h_1*h_2)=D^2(f_1*1*f_2*1)=D^2(f_1*f_2*1*1)=f_1*f_2$$

(3.2.1) o‘ramaga Laplas almashtirishini qo‘llab, $V'-IX-$qoidalarga asosan

$$\mathfrak{L}\{T_1*T_2\}=s^{k_1+k_2}\mathfrak{L}\{h_1\}\cdot\mathfrak{L}\{h_2\}=s^{k_1}\mathfrak{L}\{h_1\}\cdot s^{k_2}\mathfrak{L}\{h_2\}=\mathfrak{L}\{T_1\}\cdot\mathfrak{L}\{T_2\}$$

ni olamiz. Shunday qilib quyidagi teoramani isbotladik.

Teorema. (O‘rama teoremasi) Agar $T_1,T_2-\mathcal{D}_0'$ fazodagi taqsimotlar bo‘lsa va $\mathfrak{L}\{T_1\}=F_1(s)$,

$\mathfrak{L}\{T_2\}=F_2(s)$bo‘lsa$T_1*T_2\leftarrow F_1(s)\cdot F_2(s)$ moslik o‘rinli bo‘ladi.

Misol 3.2.3.Ttaqsimot$\mathcal{D}_0'$ fazoga tegishli bo‘lsa, u holda

$$\mathfrak{L}\{T\ \ * \delta\} = \mathfrak{L}\{T\ \ \} \cdot \mathfrak{L}\{\delta\} = \mathfrak{L}\{T\ \ \} \cdot 1 = \mathfrak{L}\{T\ \ \}$$

demak,

$$T * \delta = T$$

tengliko‘rinli ya’ni δ –o‘ramalashgan (birlik elamenti) rolini o‘ynaydi.

§3.3.Laplas operatorining fundamental yechimlari

Biz yuqorida biro‘zgaruvchili umumlashgan funksiyalar va ularning xossalari bilan tanishgan edik. Ko‘po‘lchovli funksiyalar uchun umumlashgan funksiyalar va ular ustida qo‘llaniladigan amallar shularga o‘xshash bajariladi.

$$\langle \delta(x), \varphi(x) \rangle = \varphi(0), \forall \varphi \epsilon C_0^\infty(R^n)$$

$$\langle \frac{\partial}{\partial x_j} u(x), \varphi(x) \rangle = -\langle u(x), \frac{\partial \varphi}{\partial x_2} \rangle$$

$$\Delta_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \quad (3.3.1)$$

(3.3.1) Laplas operatori

$u = -\frac{1}{4\pi} \frac{1}{|x|}$, bunda

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad (3.3.2)$$

Endi $\Delta_3 \left(-\frac{1}{4\pi} \frac{1}{|x|}\right) = \delta(x)$ ekanligini ko‘rsatamiz. Bu holda bu funksiya Laplas operatorining umumlashgan funksiyalar fazosida fundamental yechimi bo‘ladi.

Isbot. $\langle \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) \frac{1}{|x|}, \varphi \rangle = -4\pi \varphi(0),\ \forall \varphi \epsilon C_0^\infty(R^n)$ (3.3.3)

ekanligini ko‘rsatish kerak.

Hosila ta’rifiga asosan

$$\langle \frac{\partial^2}{\partial x_1^2} \cdot \frac{1}{|x|}, \varphi \rangle = \langle \frac{1}{|x|}, \frac{\partial^2 \varphi}{\partial x_1^2} \rangle = \int \frac{1}{|x|} \frac{\partial^2 \varphi}{\partial x_1^2} dx \quad (3.3.4)$$

(3.3.4) formula bilan topiladi.

Oxirgi integral

$$\int \frac{1}{|x|}\frac{\partial^2\varphi}{\partial x_1^2}dx = \lim_{\varepsilon\to 0+}\int_{|x|\geq\varepsilon}\frac{1}{|x|}\frac{\partial^2\varphi}{\partial x_1^2}dx \qquad (3.3.5)$$

Quyidagi belgilashlarni kiritamiz.

$$a_\pm = a_\pm(x_2,x_3) = \begin{cases}\pm\sqrt{\varepsilon^2 - x_2^2 - x_3^2},\ x_2^2+x_3^2\leq\varepsilon^2\\ 0,\ \ x_2^2+x_3^2\geq\varepsilon^2\end{cases} \qquad (3.3.6)$$

(3.5) integralni takroriy integral ko‘rinishda yozamiz vax_1 bo‘yicha bo‘laklab integrallaymiz

$$\int_{|x|\geq\varepsilon}\frac{1}{|x|}\frac{\partial^2\varphi}{\partial x_1^2}dx = \int_{R^2}\left(\int_{-\infty}^{a_-(x_2,x_3)}\frac{1}{|x|}\frac{\partial^2\varphi}{\partial x_1^2}dx_1 + \int_{a_+(x_2,x_3)}^{+\infty}\frac{1}{|x|}\frac{\partial^2\varphi}{\partial x_1^2}dx_1\right)dx_2dx_3 =$$
$$\int_{R^2}\left(\frac{1}{|x|}\frac{\partial\varphi}{\partial x_1}\Big|_{a_+}^{a_-} - \int_{-\infty}^{a_-}\left(\frac{\partial}{\partial x_1}\frac{1}{|x|}\right)\frac{\partial\varphi}{\partial x_1}dx_1 - \int_{a_+}^{+\infty}\left(\frac{\partial}{\partial x_1}\frac{1}{|x|}\right)\frac{\partial\varphi}{\partial x_1}dx_1\right)dx_2dx_3$$

(3.3.7)

$x_2^2+x_3^2\leq\varepsilon^2$bo‘lganligi uchun $x_1 = a_-$ yoki a_+ lar uchun $|x| = \varepsilon$ bo‘ladi va quyidagi ko‘rinishni oladi,

$$\frac{\partial}{\partial x_1}\frac{1}{|x|}\Big|_{x_1=a_\pm} = -\frac{1}{|x|^2}\frac{\partial|x|}{\partial x_1}\Big|_{a\pm} = -\frac{1}{\varepsilon^2}\frac{x_1}{|x|} =$$
$$= -\frac{1}{\varepsilon^2}cos\alpha_1 \qquad (3.3.8)$$

bunda$\alpha_1 - x$ vektor bilan ox_1o‘q orasidagi burchak, shuning uchun

$$dx_2dx_3 = dScos\alpha_1 \qquad (3.3.9)$$

bunda$dS - |x| = \varepsilon$ sferaning sirti.

Oxirgi tenglikga asosan (3.3.9) ni quyidagicha yozamiz

$$\int\limits_{|x|\geq\varepsilon}\frac{1}{|x|}\frac{\partial^2}{\partial x_1^2}dx =$$
$$= -\frac{1}{\varepsilon}\int\limits_{|x|=\varepsilon}\frac{\partial\varphi}{\partial x_1}dScos\alpha_1 - \int\left(\frac{\partial}{\partial x_1}\frac{1}{|x|}\right)\frac{\partial\varphi}{\partial x_1}dx \qquad (3.3.10)$$

bunday integrallashni yana bir marta x_1 bo‘yicha takrorlab, quyidagi tenglikka kelamiz

$$\int \left(\frac{\partial}{\partial x_1}\frac{1}{|x|}\right)\frac{\partial\varphi}{\partial x_1}dx =$$

$$= \int_{R^2}\left(\left(\frac{\partial}{\partial x_1}\frac{1}{|x|}\right)\varphi|_{a_+}^{a_-} - \int_{-\infty}^{a_-}\left(\frac{\partial^2}{\partial x_1^2}\frac{1}{|x|}\right)\varphi dx_1 - \int_{a_+}^{+\infty}\left(\frac{\partial^2}{\partial x_1^2}\frac{1}{|x|}\right)\varphi dx_1\right)dx_2 dx_3$$

$$= \frac{1}{\varepsilon^2}\int_{|x|=\varepsilon}\frac{x_1}{|x|}\varphi dS cos\alpha_1 - \int_{|x|\geq\varepsilon}\left(\frac{\partial^2}{\partial x_1^2}\frac{1}{|x|}\right)\varphi dx \qquad (3.3.11)$$

Bunda$dx_2 dx_3 = \frac{1}{\varepsilon^2}\int_{|x|=\varepsilon}\frac{x_1}{|x|}dS cos\alpha_1 - \int_{|x|\geq\varepsilon}\left(\frac{\partial^2}{\partial x_1^2}\frac{1}{|x|}\right)\varphi\, dx$

Oxirgini (3.3.10) ga qo‘yamiz

$$\int_{|x|\geq\varepsilon}\frac{1}{|x|}\frac{\partial^2\varphi}{\partial x_1^2}dx = -\frac{1}{\varepsilon}\int_{|x|=\varepsilon}\frac{\partial\varphi}{\partial x_1}\, dS cos\alpha_1 - \frac{1}{\varepsilon^2}\int\frac{x_1}{|x|}\varphi dS cos\alpha_1 +$$

$$+ \int_{|x|\geq\varepsilon}\left(\frac{\partial^2}{\partial x_1^2}\frac{1}{|x|}\right)\varphi\, dx \qquad (3.3.12)$$

bu formulalardan $\frac{\partial^2\varphi}{\partial x_2^2}$ va $\frac{\partial^2\varphi}{\partial x_3^2}$ lar uchun ham o‘rinli bu uchala integrallarni qo‘shib, quyidagini hosil qilamiz

$$\int_{|x|\geq\varepsilon}\frac{1}{|x|}\Delta_3\varphi\, dx = -\frac{1}{\varepsilon}\int_{|x|=\varepsilon}\left(\frac{\partial\varphi}{\partial x_1}cos\alpha_1 + \frac{\partial\varphi}{\partial x_2}cos\alpha_2 + \frac{\partial\varphi}{\partial x_3}cos\alpha_3\right) dS -$$

$$-\frac{1}{\varepsilon^2}\int_{|x|=\varepsilon}\frac{x_1 cos\alpha_1 + x_2 cos\alpha_2 + x_3 cos\alpha_3}{|x|}\varphi dS$$

$$+ \int_{|x|\geq\varepsilon}\left(\Delta_3\frac{1}{|x|}\right)\varphi\, dx \qquad (3.3.13)$$

Endi $\cos\alpha_k = \frac{x_k}{|x|}$ bo‘lganligi uchun

$$x_1 cos\alpha_1 + x_2 cos\alpha_2 + x_3 cos\alpha_3 = \frac{x_1^2 + x_2^2 + x_3^2}{|x|} = |x| \qquad (3.3.14)$$

$x \neq 0$ bo‘lganda $y = \frac{1}{|x|}$ Laplas tenglamasining fundamental yechimi bo‘lganligi uchun

$$\Delta_3 \frac{1}{|x|} = 0,\ x \neq 0 \quad (3.3.15)$$

bo‘lganligi uchun, (3.3.13) dan (3.3.3) kelib chiqadi. (3.3.14) va (3.3.15) ga asosan quyidagi ko‘rinishga ega bo‘ladi.

$$\int\limits_{|x|\geq\varepsilon} \frac{1}{|x|}\Delta_3\varphi dx = -\frac{1}{\varepsilon}\int\limits_{|x|=\varepsilon}\left(\frac{\partial\varphi}{\partial x_1}cos\alpha_1 + \frac{\partial\varphi}{\partial x_2}cos\alpha_2 + \frac{\partial\varphi}{\partial x_3}cos\alpha_3\right)dS -$$

$$-\frac{1}{\varepsilon^2}\int\limits_{|x|=\varepsilon}\varphi dx \qquad (3.3.16)$$

Bunda $\varepsilon \to 0+$ da

$$\int \frac{1}{|x|}\Delta_3\varphi dx = -4\pi\varphi(0) \quad (3.3.17)$$

Demak, $-\frac{1}{4\pi|x|}$ funksiya Laplas tenglamasining umumlashgan funksiyalar fazosidagi fundamental yechimi ekan.

Xulosa

Magistratura ishini bajarishda operatsion hisobning asosiy usullari to‘liq ravishda o‘rganilib chiqildi. Differensial tenglamalarni fundamental yechimini topishda, umumlashgan funksiyalar ahamiyati o‘rganildi va uning tadbiqiga oid misollar o‘rganildi. Differensial tenglamalarning umumlashgan yechimlari bo‘lgan, taqsimotlarni umumlashgan hosilalari o‘rganilib ularga doir misollar keltirilgan.Umumlashgan funksiyalar uchun Laplas almashtrishlari o‘rganilgan.

Tasvir va originallarni qo‘llanishi umumlashgan funksiyalar fazosida olib borilgan. Ushbu dissertatsiyada differensial tenglamalarning regulyar bo‘lmagan yechimlarini topishda tavsiya etilgan usul samarali ekanligi ishning asosiy moxiyatini tashkil etadi.

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