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ABDIEV TOLLIBOY BAYDULLAYVICH

UMUMLASHGAN ANALITIK FUNKSIYALARNI DAVOM ETTIRISH MASALASI

5A 460102 – Differensial tenglamalar

Magistr akademik darajasini olish uchun yozilgan

DISSERTATSIYA

Ish ko’rib chiqildi va himoyaga
ruxsat berildi.

“Differensial tenglamalar”
kafedrasi mudiri:
________prof. A.Begmatov

Ilmiy rahbar:
________prof. T.Ishankulov

M.O’.

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MUNDARIJA

KIRISH

Masalaning qo'yilishi. Tekislikda maxsus sohalarda umumlashgan Koshi-Riman sistemasi uchun nokorrekt Koshi masalasining yechilish shartini olishdan iborat.

Mavzuning dolzarbligi. Umumlashgan Koshi-Riman sistemasi gidrodinamika, mexanika,geometriya masalalarida keng qo'laniladi. Shuning uchun bu sistema yechimini davom ettirish masalasi matematik fizikaning dolzarb masalalaridan hisoblanadi.

Ishning maqsad va vazifalari. Umumlashgan Koshi-Riman sistemasi uchun Koshi masalasining yechilish shartlarini tekshirish. Bu sistema uchun Fok-Kuni teoremasini o'rnatish dissertatsiyaning asosiy maqsadi hisoblanadi.

Ilmiy tadqiqot metodlari. Dissertatsiya kompleks o'zgaruvchili funksiyalar nazariyasining metodlaridan keng foydalanilgan.

Ishning ilmiy ahamiyati. Dissertatsiyaning asosiy natijasi-umumlashgan analitik funksiya uchun Fok-Kuni teoremasining o'xshatmasi yangi natija bo'lib, elliptik sisitemalar yechimini davom ettirishda qo'llanilishi mumkin.

Ishning amaliy ahamiyati. Dissertatsiya nazariy harakterga ega bo'lib uning natijalari elliptik sistemalar nazariyasida tatbiq etilishi mumkin.

Ishning tarkibi: Dissertatsiya uch bob va o'n ikki paragrfdan iborat bo'lib.
I bob. Koshi teoremasi va integral formulasi.
II bob. Tekislikda Koshi-Riman sistemasi uchun Koshi masalasi, soha chegarasining bo'laklarida berilgan funksiyalarni soha ichiga golomorf davom ettirish masalasi.
III bob. Asosiy qism hisoblanib, bu bobda chegaraning qismida berilgan funksiyani sohaga umumlashgan analitik funksiya sifatida davom ettirish masalasi hamda umumlashgan analitik funksiyalarni davom ettirish masalalari qaraladi.

Kompleks tekislikni E orqali, $C_\alpha(E) - E$ tekislikda Gyo'lder shartini qanoat-lantiruvchi funksiyalar to'plamini belgilaymiz. Quyidagi shartlarni

$$f(z)\in L_P(E_1),\ |z|^{-2}f\left(\frac{1}{z}\right)\in L_P(E_1),\ E_1=\{z:|z|<1\}$$

qanoatlantiruvchi $f(z)$ funksiyalar to'plamini $L_{P,2}$ – orqali belgilaymiz[4].

D_ρ - kompleks tekislikdagi $G_\rho:\frac{\pi}{2}-\frac{\pi}{2\rho}<\arg z<\frac{\pi}{2}+\frac{\pi}{2\rho}\ (\rho>1)$ burchakdan va G_ρ ichida yotuvchi silliq S egri chiziqdan iborat bo'lakli silliq chegaralangan bir bog'lamli soha bo'lsin. $U_{p,2}(A,B,D_\rho)$ -orqali D_ρ -sohada

$$\frac{\partial W}{\partial \bar{z}}+A(z)W+B(z)\overline{W}=0,\qquad \frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right),\qquad (0.4.1)$$

tenglamaning yechimlar to`plamini belgilaymiz, bu yerda $A,B\in L_{P,2}(E)\cap C_\alpha(E),\ P>2.\ A(z)\equiv 0,\ B(z)\equiv 0$ da $U_{P,2}(A,B,D_\rho)$ to'plami $U(D_\rho)$ bilan ustma-ust tushadi.

Masalaning qo'yilishi: $\varphi(z)\in C(S)$ funksiyaga qanday shartlar qo'yilganda shunday $W(z)\in U_{P,2}(A,B,D_\rho)\cap C(D_\rho\cup S)$ funksiya mavjud bo'ladiki, uning S dagi chegaraviy qiymatlari $\varphi(z)$ bilan ustma-ust tushadi, yani $W(z)=\varphi(z),\ z\in S$ bo'ladi.

Umumlashgan analitik funksiyalar nazariyasida I.N. Vekuaning muvofiqlik teoremasi bor. Bu teoremaga ko'ra $z=x+iy$ kompleks o'zgaruvchining har bir analitik funksiyasiga umumlashgan Koshi-Riman tenglamasining yagona yechimi mos keladi. D_ρ sohaga nisbattan S egri chiziqning Karliman funksiyasini qaraymiz.

$$\Phi_\sigma(z,\zeta)=(\zeta-z)^{-1}exp[\sigma((-i\zeta)^\rho-(-iz)^\rho)]$$

Bu yerda σ-haqiqiy musbat sonly parameter, $\zeta=\xi+i\eta$, $(-iz)^\rho$ sifatida D_ρ sohada bu funksiyaning $z=i$ da birga teng bir qiymatli tarmog'ini tushunamiz. I.N. Vekua teoremasiga ko'ra $\frac{1}{2}\Phi_\sigma$ va $\frac{1}{2i}\Phi_\sigma$ funksiyalarga (3.4.1) tenglamaning z o'zgaruvchi bo'yicha $X_j^\sigma(z,\zeta)$ $(j=1,2)$ yechimlari mos keladi. Quydagi funksiyalarni qaraymiz.

$$\Omega_1^\sigma(z,\zeta)=X_1^\sigma(z,\zeta)+iX_2^\sigma(z,\zeta),\ \Omega_2^\sigma(z.\zeta)=X_1^\sigma(z,\zeta)-iX_2^\sigma(Z,\zeta).$$

Teorema-1. $W\in U_{P,2}(A,B,D_P)\cap C(\bar{D}_P)$ va $W(z)=\varphi(z),\ z\in S$ bo'lsin. U holda davom ettirishning quyidagi teng kuchli formulalari o'rinli

$$W(z)=\lim_{\sigma\to\infty}\frac{1}{2\pi i}\int_S\left[\Omega_1^{\sigma}(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2^{\sigma}(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta}\right],\quad z\in D_\rho, \tag{0.4.2}$$

$$W(z)=\frac{1}{2\pi i}\int_S\left[\Omega_1(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta}\right]+\int_0^{\infty}J(z,\sigma)d\sigma,\quad z\in D_\rho, \tag{0.4.3}$$

bu yerda

$$J(z,\sigma)=\frac{1}{2\pi i}\int_S\left[\gamma_1^{\sigma}(z,\zeta)\varphi(\zeta)d\zeta-\gamma_2^{\sigma}(z,\zeta)\overline{\varphi(\zeta)}d\zeta\right],\quad \gamma_j^{\sigma}(z,\zeta)=\frac{\partial}{\partial\sigma}\Omega_j^{\sigma}(z,\zeta).$$

Teorema-2. $\varphi(\zeta)\in L(S)\cap C\left(\overset{0}{S}\right)$ bo'lsin. U holda $W(z)=\varphi(z),\ z\in S$ shartni qanoatlantiruvchi $W\in U_{P,2}(A,B,D_\rho)\cap C\left(D_\rho\cup\overset{0}{S}\right)$ funksiya mavjud bo'lishi uchun

$$\left|\int_0^{\infty}J(z,\sigma)d\sigma\right|<\infty$$

integralning har bir $K\subset G_\rho$ kompaktda tekis yaqinlashuvchi bo'lishi zarur va yetarli. Agar bu shart bajarilsa, u holda davom ettirish (3.4.2) va (3.4.3) teng kuchli formulalar bilan amalga oshiriladi.

0-teoremadan $A(z)\equiv 0,\ B(z)\equiv 0$ va $\rho=1$ bo'lganda quyidagi teorema kelib chiqadi.

Teorema. (Fok-Kuni). $\varphi(z)\in L(S)\cap C(\overset{0}{S})$ bo'lsin. S da $W(\zeta)=\varphi(\zeta),\ \zeta\in\overset{0}{S}$ shartni qanoatlantiruvchi $W(z)\in U(D_1)\cap C(D_1\cup\overset{0}{S})$ funksiya mavjud bo'lishi uchun

$$\left|\int_0^{\infty}\left[\int_S\varphi(\zeta)\exp\left[i\sigma(\zeta-z)\right]d\zeta\right]d\sigma\right|<\infty$$

integralning har bir $K\subset C\{\operatorname{Im} z>0\}$ kompaktda tekis yaqinlashuvchi bo'lishi zarur va yetarli.

I BOB. Koshi teoremasi va integral formulasi.

1.1-§. Koshi teoremasi.

1.1.1-Teorema. Agar bir bog'lamli G sohada $f(z)$ funksiya analitik bo'lsa, u holda G da yotuvchi har qanday Γ yopiq kontur bo'ylab $f(z)$ funksiyadan olingan integral nolga teng bo'ladi. [10]:

$$\oint_{\Gamma} f(z)dz = 0.$$

Agar qo'shimcha shart - $f'(z)$ ning $\overline{G}$ da uzluksizligi talab qilinsa, bu teoremaning o'rinli ekani, Dalamber-Eyler shartlari va Grin formulasiga asosan bevosita kelib chiqadi. Haqiqatdan ham, matematik analiz kursidan ma'lumki, agar $P(x,y)$, $Q(x,y)$, $\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial y}$ lar yopiq $\overline{G}$ sohada uzluksiz bo'lsa, u holda ushbu

$$\oint_{\Gamma} Pdx + Qdy = \iint_{G} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Grin formulasi o'rinlidir, bundagi G yopiq konturning ichki qismidan iborat. Ravshanki

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

ning uzluksizligidan

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}$$

hosilalarning, shuningdek *u(x,y)* va *v(x,y)* funksiyalarning uzluksizligi kelib chiqadi. [10] da bu teorema quyidagicha isbotlangan:

Grin formulasidan foydalanib, quyidagilarni hosil qilamiz:

$$\oint_{\Gamma} f(z)dz = \oint_{\Gamma} (udx - vdy) + i\oint_{\Gamma} (vdx + udy) = \iint_{G} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy + i\iint_{G} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy.$$

Dalamber-Eyler shartlariga asosan:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

U holda oxirgi tenglikning o'ng tomoni nolga teng bo'ladi, ya'ni

$$\oint_{\Gamma} f(z)dz=0.$$

Bu teoremani $f'(z)$ ning uzluksizligini talab qilmasdan ham isbot qilish mumkin, u birinchi marta E.Gursa tomonidan isbotlangan.[10].

1.2-§. Koshining integral formulasi va Koshi tipidagi integral.

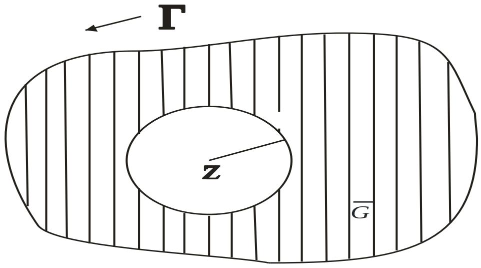
Chegarasi Γ chiziqdan iborat bo'lgan yopiq $\overline{G}$ sohada bir qiymatli va analitik $f(z)$ funksiya berilgan bo'lsin. Bu degan so'z $\overline{G}$ ni o'z ichiga olgan biror G' sohaning har bir nuqtasida $f(z)$ funksiya aniq chekli hosilaga ega degan so'zdir. Sohaning ichidan ixtiyoriy bir z nuqtani olaylik va bu nuqtani markaz qilib G ichida ρ radiusli γ aylana chizaylik. U holda Γ va γ lar bilan chegaralangan ikki bo'g'lamli sohada (halqada) ushbu

$$\frac{f(\zeta)}{\zeta-z}$$

funksiya analitik bo'ladi, chunki $\zeta\neq z$. [10].

Shu sababli, Koshi teoremasiga asosan, tashqi Γ kontur bo'ylab olingan integral ichki γ bo'ylab olingan integralga teng bo'ladi (1.2.1-chizma):

$$\oint_{\Gamma}\frac{f(\zeta)d\zeta}{\zeta-z}=\oint_{\gamma}\frac{f(\zeta)d\zeta}{\zeta-z} \qquad (1.2.1)$$



1.2.1-chizma

Berilgan $f(z)$ funksiya $\overline{G}$ sohada analitik bo'lgani sababli tabiiyki, o'sha sohada uzluksiz ham bo'ladi. U holda har qanday istalgancha kichik $\varepsilon > 0$ son olingan bo'lmasin, shunday $\delta > 0$ son mavjudki, γ aylananing ixtiyoriy ζ nuqtasi uchun $|\zeta - z| = \rho < \delta$ bo'lganda

$$|f(\zeta) - f(z)| < \varepsilon$$

tengsizlik o'rinli bo'ladi.

Endi, integralning xossalaridan foydalanib, quyidagi ayirmani tekshiramiz (baholaymiz):

$$\left| \oint_{\gamma} \frac{f(\zeta)d\zeta}{\zeta - z} - \oint_{\gamma} \frac{f(z)d\zeta}{\zeta - z} \right| = \left| \oint_{\gamma} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \right| < 2\pi\varepsilon.$$

O'ng tomondagi ε istagancha kichik musbat sondan iborat bo'lgani uchun chap tomondagi ayirmaning limiti nolga tengdir. Ikkinchi tomondan

$$\oint_{\gamma} \frac{f(z)d\zeta}{\zeta - z} = f(z) \oint_{\gamma} \frac{d\zeta}{\zeta - z} = f(z) 2\pi i.$$

Demak,

$$\lim_{\rho \to 0} \left[\oint_{\gamma} \frac{f(\zeta)d\zeta}{\zeta - z} - \oint_{\gamma} \frac{f(z)d\zeta}{\zeta - z} \right] = \lim_{\rho \to 0} \oint_{\gamma} \frac{f(\zeta)d\zeta}{\zeta - z} - f(z) 2\pi i = 0.$$

Bundan

$$\lim_{\rho \to 0} \oint_{\gamma} \frac{f(\zeta)d\zeta}{\zeta - z} = f(z) 2\pi i. \qquad (1.2.2)$$

Agar (1.2.1) tenglikning ikki tomonidan ρ ni nolga intiltirib limitga o'tilsa quyidagi tenglik hosil bo'ladi:

$$\lim_{\rho \to 0} \oint_{\Gamma} \frac{f(\zeta)d\zeta}{\zeta - z} = \lim_{\rho \to 0} \oint_{\gamma} \frac{f(\zeta)d\zeta}{\zeta - z} = f(z) 2\pi i.$$

Γ chiziq bo'ylab olingan integral ρ ga bog'liq bo'lmagani sababli limit belgisini tashlab yozish mumkin, natijada Koshi formulasi deb ataluvchi ushbu tenglikka ega bo'lamiz:

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\zeta)d\zeta}{\zeta - z}. \qquad (1.2.3)$$

Tenglikning o'ng tomonidagi ifoda Koshi integrali deyiladi.

Koshi formulasining mohiyati shundaki, u G sohaning ichki z nuqtasi *f(z)* funksiya qiymatini o'sha funksiyaning Γ konturdagi $f(\zeta)$ qiymati orqali ifodalaydi. [6].

Koshi formulasi murakkab kontur uchun ham o'z kuchini saqlaydi.

Koshining (1.2.3) integral formulasini keltirib chiqarishda biz *f(z)* funksiyani $\overline{G}$ sohada analitik va Γ ni yopiq chiziq deb faraz qilgan edik. Agar bu ikki farazimizning birortasi buzilsa, Koshi formulasi o'rinli bo'lmaydi. Ma'lumki, o'sha (1.2.3) formulaning o'ng tomoni Koshi integrali deyilar edi. Biz endi Koshi integraliga qaraganda umumiyroq bir integralni tekshiramiz.

Tekislikda biror silliq Γ chiziq olaylik. Bu chiziq yopiq bo'lmasligi ham mumkin. Faraz qilaylik, $\varphi(\zeta)$ bir qiymatli funksiya mana shu Γ chiziqda uzluksiz bo'lsin. Agar biz Γ chiziqda yotmaydigan biror z nuqtani olsak, u vaqtda

$$\frac{\varphi(\zeta)}{\zeta - z}$$

kasr Γ chiziqning barcha nuqtalarida uzluksiz bo'ladi, chunki ζ nuqta Γ ustida yotuvchi ixtiyoriy nuqtadan iborat bo'lgani uchun $\zeta \neq z$. Shu sababdan

$$\frac{1}{2\pi i}\int \frac{\varphi(\zeta)d\zeta}{\zeta - z}$$

integral tekislikdagi har bir z (Γ da yotmaydigan) nuqta uchun aniq qiymatga ega. Demak, o'sha integral z ning funksiyasidir:

$$\Phi(z) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{\varphi(\zeta)d\zeta}{\zeta - z}. \qquad (1.2.4)$$

Bu integralning xususiy hollarini tekshiraylik. Agar Γ chiziq yopiq bo'lib, u bilan chegaralangan G sohada $\varphi(\zeta)$ funksiya analitik bo'lsa, u holda:

a) z nuqta G ning tashqarisida yotgan bo'lsa, (1.2.4) integral nolga teng bo'ladi;

b) z nuqta G ning ichida yotgan bo'lsa, (1.2.4) tenglik Koshinig (1.2.3) formulasiga aylanadi.

Shu sababli (1.2.4) ning o'ng tomoni Koshi tipidagi integral deb ataladi.

Soxotskiy formulalari. [10] **.**Ushbu

$$\Phi(z)=\frac{1}{2\pi i}\oint_{\Gamma}\frac{\varphi(\zeta)d\zeta}{\zeta-z}$$

Koshi tipidagi integralni tekshiramiz, bundagi $\varphi(\zeta)$ Gyol'der shartini qanoatlantiradigan funksiya bo'lsin. Γ ni yopiq egri chiziq deb hisoblaymiz, aks holda uni biror egri chiziq bilan yopiq egri chiziqqacha to'ldirib, to'ldiruvchi chiziqda $\varphi(\zeta)=0$ deb hisoblashimiz mumkin. $\Phi(z)$ analitik funksiyaning chegaraviy qiymatlarini mos ravishda $\Phi^{+}(\zeta_0)$ va $\Phi^{-}(\zeta_0)$ bilan belgilaymiz. Bundagi $\Phi^{+}(\zeta_0)$ miqdor $\Phi(z)$ ning z nuqta Γ ning ichida yotib konturdagi ζ_0 nuqtaga intilgandagi limiti bo'lib, $\Phi^{-}(\zeta_0)$ esa z nuqta kontur tashqarisidan ζ_0 nuqtaga intilgandagi limitidir (kontur ochiq bo'lsa, ular chap va o'ng chegaraviy qiymatlarga mos keladi).

Koshi tipidagi integralning ζ_0 nuqtadagi qiymatni $\Phi(\zeta_0)$ bilan belgilaymiz, ya'ni

$$\Phi(\zeta_0)=\frac{1}{2\pi i}\oint_{\Gamma}\frac{\varphi(\zeta)d\zeta}{\zeta-\zeta_0}.$$

$\Phi(z)$ funksiyani quyidagi ko'rinishda yozib olamiz:

$$\Phi(z)=\frac{1}{2\pi i}\oint_{\Gamma}\frac{\varphi(\zeta)d\zeta}{\zeta-z}=\frac{1}{2\pi i}\oint_{\Gamma}\frac{\varphi(\zeta)-\varphi(\zeta_0)}{\zeta-z}d\zeta+\frac{\varphi(\zeta_0)}{2\pi i}\oint_{\Gamma}\frac{d\zeta}{\zeta-z} \tag{1.2.5}$$

(1.2.5) ning o'ng tomonidagi birinchi integralning $z\to\zeta_0$ dagi limiti

$$\int_{\Gamma}\frac{\varphi(\zeta)-\varphi(\zeta_0)}{\zeta-\zeta_0}d\zeta,$$

ikkinchisi esa z nuqta konturning ichida yoki tashqarisida yotishiga qarab, $2\pi i$ yoki 0 ga teng. Bularni e'tiborga olib, (1.2.5) formulada $z\to\zeta_0$ deb limitga o'tsak, ushbu

$$\Phi^{+}(\zeta_0)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\varphi(\zeta)-\varphi(\zeta_0)}{\zeta-\zeta_0}d\zeta+\varphi(\zeta_0);\quad \Phi^{-}(\zeta_0)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\varphi(\zeta)-\varphi(\zeta_0)}{\zeta-\zeta_0}d\zeta$$

tengliklarga ega bo'lamiz. [20].

Bu formuladagi integralning o'rniga uning yuqoridagi qiymatini olib kelib qo'ysak, quyidagi formulalarni hosil qilamiz:

$$\Phi^{+}(\zeta_0)=\Phi(\zeta_0)+\frac{1}{2}\varphi(\zeta_0), \quad \Phi^{-}(\zeta_0)=\Phi(\zeta_0)-\frac{1}{2}\varphi(\zeta_0) \qquad (1.2.6)$$

(1.2.6) formulalarni birinchi marta 1873-yilda rus matematigi Yulian Vasilevich Soxotskiy isbot qilgan. Shuning uchun ham bu formulalar Soxotskiy nomi bilan yuritiladi. [10].

1.3-§. Koshi tipidagi integralning Koshi integral formulasiga aylanish shartlari.

1.3.1-teorema. Faraz qilaylik φ Gyolder shartini qanoatlantirsin. U holda Koshi tipidagi integral Koshi integraliga aylanishi uchun $\Phi^{-}(z)\equiv 0$ shartning bajarilishi zarur va yetarli [7]. [10] da φ funksiyaga Gyolder sharti Soxotskiy formulasini isbotlash uchun qo`yilgan. [22].

Teoremani isbotlashda quyidagi tengliklardan foydalanamiz:

$$\Phi^{+}(\zeta)-\Phi^{-}(\zeta)=\varphi(\zeta), \quad \zeta\in\Gamma \qquad (1.3.1)$$

$$\lim_{z\to\zeta}\Phi^{+}(z)=\varphi(\zeta), \qquad \zeta\in\Gamma \qquad (1.3.2)$$

Isbot. Zaruriyligi. φ Gyolder shartini va (1.3.2) shartni qanoatlantirsin. (1.3.2) shartdan (1.3.1) ga asosan $\Phi^{-}(\zeta)=0$ ekanligi kelib chiqadi. Analitik davom ettirishning yagonaligiga ko`ra esa $\Phi^{-}(z)\equiv 0$ ekanligi kelib chiqadi.

Yetarliligi. $\varphi(\zeta)$ Γ da Gyolder shartini va $\Phi^{-}(z)\equiv 0$ shartlarni qanoatlantiradi. Bundan darhol $\Phi^{-}(\zeta)\equiv 0$ ekanligini ko`rishimiz mumkin. U holda (1.3.1) dan (1.3.2) kelib chiqadi.

1.3.2-teorema. Faraz qilaylik φ - Gyolder shartini qanoatlantirsin. U holda Koshi tipidagi integral Koshi integraliga aylanishi uchun quyidagi momentlar shartining bajarilishi zarur va yetarli:[7].

$$\int_{\Gamma} \varphi(\zeta)\zeta^n d\zeta = 0, \qquad n = 0,1,2,...$$

Isbot. 1.3.1-teoremaga asosan Koshi tipidagi integralning Koshi integral formulasiga aylanishi uchun

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{\varphi(\zeta)d\zeta}{\zeta - z} \equiv 0, \qquad z \notin \overline{G}$$

shartning bajarilishi zarur va yetarli. $z \notin \overline{G}$ da kompleks sonning moduli yetarlicha katta bo`lganda ushbu tengliklarni yozish mumkin:

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{\varphi(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\Gamma} \frac{\varphi(\zeta)d\zeta}{z(1-\frac{\zeta}{z})} = \frac{1}{2\pi i z}\int_{\Gamma} \varphi(\zeta)(1+\frac{\zeta}{z}+\frac{\zeta^2}{z^2}+.....)d\zeta = \frac{1}{2\pi i z}\int_{\Gamma} \varphi(\zeta)d\zeta +$$

$$+\frac{1}{2\pi i z^2}\int_{\Gamma} \varphi(\zeta)\zeta \; d\zeta + ... + ...$$

darajali qator yoyilmasining yagonaligidan

$\int_{\Gamma} \varphi(\zeta)\zeta^n d\zeta = 0 \qquad n = 0,1,2,...$ tenglik kelib chiqadi. Teorema isbotlandi.

1.4-§. Koshi-Riman sistemasiga doir misollar.

[3] da berilgan Koshi-Riman tenglamasiga doir berilgan quyidagi misollarni yechamiz:

1-misol. Chegarasi bo'lakli silliq bo'lgan D sohada quyidagi elliptik sistemaning umumiy yechimini toping.

$$\begin{cases} u_x - v_y = g_1 \\ u_y + v_x = g_2 \end{cases}$$

Yechilishi:

$$\begin{cases} u_x - v_y = g_1 \\ u_y + v_x = g_2 \end{cases} \Rightarrow \begin{cases} u_x - v_y = g_1 \\ iu_y + iv_x = ig_2 \end{cases} \Rightarrow u_x + iv_x - v_y + iu_y = g_1 + ig_2 \Rightarrow$$

$$\frac{\partial W}{\partial x} + i\frac{\partial W}{\partial y} = g_1 + ig_2$$

Bundan

$$\frac{\partial W}{\partial \bar{z}} = G(z), \;\; 2G = g_1 + ig_2$$

Umumiy yechimni $W(z) = W_0(z) + W_1(z)$ ko`rinishda izlaymiz. Bu yerda $W_0(z) = \varphi(z)$ tenglamaning bir jinsli qismining umumiy yechimi bo`lib, $\varphi(z)$ D sohada z kompleks o`zgaruvchining analitik funksiyasi, $W_1(z)$ esa bir jinsli bo`lmagan tenglamaning bitta xususiy yechimi.

Bu tenglamani yechish uchun Puasson tenglamasini qaraymiz:

$$\Delta \mathrm{P(x, y) = q(x, y)}, \qquad \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = q(x, y), \qquad z = x + iy, \quad \zeta = \xi + i\eta$$

Bu Puasson tenglamasining yechimi quydagi logarifmik potensial ko'rinishida ifodalanadi.

$$P(x, y) = -\frac{1}{2\pi}\iint_D \ln|\zeta - z| q(\zeta) d\xi d\eta$$

$$W_1(z) = \frac{2}{\pi}\iint_D \frac{\partial}{\partial \mathrm{z}} \ln|\zeta - z| G(\zeta) d\xi d\eta =$$

$$= \frac{2}{\pi}\iint_D - \frac{1}{\sqrt{(\zeta - z)(\bar{\zeta} - \bar{z})}} \frac{\sqrt{\bar{\zeta} - \bar{z}}}{2\sqrt{\zeta - z}} G(\zeta) d\xi d\eta = -\frac{1}{\pi}\iint_D \frac{G(\zeta)}{\zeta - z} d\xi d\eta$$

Demak, umumiy yechim quyidagicha bo`ladi:

$$W = \varphi(z) - \frac{1}{\pi}\iint_D \frac{G(\zeta)}{\zeta - z} d\xi d\eta$$

2-misol. Chegarasi bo'lakli silliq bo'lgan D sohada quyidagi elliptik sistemaning umumiy yechimini toping.

$$\begin{cases} u_x + v_y - v = g_1 \\ u_y - v_x - u = g_2 \end{cases}$$

Yechilishi: $u = e^{ax+by}U$, $v = e^{ax+by}V$ bo'lsin

$$\begin{cases} (ae^{ax+by}U + e^{ax+by}U_x) + (be^{ax+by}V + e^{ax+by}V_y) - e^{ax+by}V = g_1 \\ (be^{ax+by}U + e^{ax+by}U_y) - (ae^{ax+by}V + e^{ax+by}V_x) - e^{ax+by}u = g_2 \end{cases}$$

$$\begin{cases} U_x + V_y + aU + (b+1)V = e^{-ax-by} g_1 \\ U_y + V_x - aV + (b-1)U = e^{-ax-by} g_2 \end{cases} \quad \Rightarrow \quad a=0,\ b=1,\ G_1 = e^{-y} g_1,$$

$G_2 = e^{-y} g_2$ olsak $\begin{cases} U_x + V_y = G_1 \\ U_y - V_x = G_2 \end{cases}$

hosil bo'ladi. Endi $W = U + iV$, $2G = G_1 + iG_2$, $z = x + iy$, $t = \xi + i\eta$

$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ belgilashlar kiritilsa, sistema $\frac{\partial W(z)}{\partial \bar{z}} = G(z)$

ko'rinishga keladi. Buning xususiy yechimi $W_0(z) = -\frac{1}{\pi}\int_D \frac{G(t)}{t-z} d\xi d\eta$ ekanligidan

umumiy yechimni $W = W_1 + W_0$ ko'rinishda izlaymiz. Bunda W_1 $\frac{\partial w}{\partial \bar{z}} = 0$

tenglamaning umumiy yechimi. Bu yerdan $W_1 = \bar{\varphi}(\bar{z})$ kelib chiqadi. Bunda φ ixtiyoriy analitik funksiya

$$W = U + iV = e^{-y}(u + iv) = e^{-y}\omega$$

$$2G = G_1 + iG_2 = e^{-y}(g_1 + ig_2) = e^{-y} 2g$$

ekanligidan

$$\omega(z) = e^{y}\left[\varphi(z) - \frac{1}{\pi}\int_D \frac{g(t)}{t-z} e^{-\eta} d\xi d\eta\right]$$

kelib chiqadi. U holda $U = \mathrm{Re}\,\omega(z)$, $V = \mathrm{Im}\,\omega(z)$ bo'ladi. [21].

3-misol. Chegaralangan va chegarasi ∂D bo'lakli silliq bo'lgan D sohada quyidagi elliptik tenglamalar sistemasining umumiy yechimini toping [3]:

$$\begin{cases} u_x + v_y - u = g_1 \\ u_y - v_x + v = g_2 \end{cases}$$

Yechilishi: $u(x,y) = e^{ax+by} U(x,y)$, $v(x,y) = e^{ax+by} V(x,y)$ deb olaylik. U holda

$$\begin{cases} ae^{ax+by}U + e^{ax+by}U_x + (be^{ax+by}V + e^{ax+by}V_y) - e^{ax+by}U = g_1 \\ (be^{ax+by}U + e^{ax+by}U_y) - (ae^{ax+by}V + e^{ax+by}V_x) + e^{ax+by}V = g_2 \end{cases}$$

Bu yerdan

$$\begin{cases} e^{ax+by}U_x + e^{ax+by}V_y + be^{ax+by}V + (a-1)e^{ax+by}U = g_1 \\ e^{ax+by}U_y - e^{ax+by}V_x + be^{ax+by}U - (a-1)e^{ax+by}V = g_2 \end{cases}$$

Endi, $b=0$, $a=1$ va $G_1(x,y)=e^{-x}\, g_1(x,y)$, $G_2(x,y)=-e^{-x}\, g_2(x,y)$ deb belgilasak, u holda quyidagi tenglamalar sistemasiga kelamiz:

$$\begin{cases} U_x + V_y = G_1 \\ U_y - V_x = -G_2 \end{cases}$$

Quyidagi almashtirishlarni kiritamiz:

$W=V+iU$, $2G=G_2+iG_1$, $z=x+iy$, $t=\xi + i\eta$, $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

Natijada berilgan tenglamalar sistemasi ushbu ko'rinishga keladi:

$$\frac{\partial W}{\partial \bar{z}} = G(z) \tag{1.4.1}$$

Ushbu tenglamaning xususiy yechimi $W_0(z) = -\frac{1}{\pi}\int_D \frac{G(t)}{t-z}\, d\xi\, d\eta$ ekanidan, umumiy yechimni $W = W_0 + W_1$ ko'rinishida izlash mumkin. Bu yerda W_1 bir jinsli yoki Koshi-Riman tenglamalar sistemasi $\frac{\partial W}{\partial \bar{z}} = 0$ ning umumiy yechimi. Bu yechim ixtiyoriy $\varphi(z)$ analitik funksiyadan iborat bo'lgani uchun (1.4.1) tenglamaning umumiy yechimi quyidagi ko'rinishda bo'ladi:

$$W = \varphi(z) - \frac{1}{\pi}\int_D \frac{G(z)}{t-z}\, d\xi\, d\eta$$

$W = e^x\left(u(x,y) + v(x,y)\right)$, $2G(x,y) = e^x\left(g_2(x,y) + ig_1(x,y)\right)$ ekanidan

$$W(z) = e^{-x}[\varphi(z) - \frac{1}{\pi}\int_D \frac{g(t)}{t-z} e^{-x}\, d\xi\, d\eta]$$

bu yerda $2g(x,y) = ig_1(x,y) - g_2(x,y)$ ekanligi kelib chiqadi. Bundan esa

$$v(x,y) = \mathrm{Re}\left(W(z)\right), \quad u(x,y) = \mathrm{Im}\left(W(z)\right)$$

Javob: $u(x,y)=\operatorname{Im}(W(z)),\ v(x,y)=\operatorname{Re}(W(z))$ bu yerda

$$W(z)=e^{-x}[\varphi(z)-\frac{1}{\pi}\int_D \frac{g(t)}{t-z}e^{-x}\,d\xi\,d\eta].$$

4-misol. Chegaralangan va chegarasi ∂D bo'lakli silliq bo'lgan D sohada quyidagi elliptik tenglamalar sistemasining umumiy yechimini toping [3]:

$$\begin{cases} u_x - v_y - 2v = g_1 \\ u_y + v_x + 2u = g_2 \end{cases}$$

Yechilishi: $u(x,y)=e^{ax+by}U(x,y)$, $v(x,y)=e^{ax+by}V(x,y)$ deb olaylik. U holda

$$\begin{cases} ae^{ax+by}U + e^{ax+by}U_x - (be^{ax+by}V + e^{ax+by}V_y) - 2e^{ax+by}V = g_1 \\ be^{ax+by}U + e^{ax+by}U_y + ae^{ax+by}V + e^{ax+by}V_x + 2e^{ax+by}U = g_2 \end{cases}$$

Bu yerdan

$$\begin{cases} e^{ax+by}U_x - e^{ax+by}V_y + ae^{ax+by}U - (b+2)e^{ax+by}V = g_1 \\ e^{ax+by}U_y + e^{ax+by}V_x + ae^{ax+by}V + (b+2)e^{ax+by}U = g_2 \end{cases}$$

Endi, $b=-2$, $a=0$ olsak va $G_1(x,y)=e^{-2y}\ g_1(x,y)$, $G_2(x,y)=e^{-2y}\ g_2(x,y)$ deb belgilasak, u holda quyidagi tenglamalar sistemasiga kelamiz:

$$\begin{cases} U_x - V_y = G_1 \\ U_y + V_x = G_2 \end{cases}$$

Quyidagi almashtirishlarni kiritamiz:

$W=U+iV$, $2G=G_1+G_2$, $z=x+iy$, $t=\xi+i\eta$, $\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y})$, $\frac{\partial}{\partial \bar{z}}=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})$.

Natijada berilgan tenglamalar sistemasi ushbu ko'rinishga keladi:

$$\frac{\partial W}{\partial \bar{z}} = G(z)$$

Ushbu tenglamaning xususiy yechimi $W_0(z)=-\frac{1}{\pi}\int_D \frac{G(t)}{t-z}\,d\xi\,d\eta$ ekanidan, umumiy yechimni $W=W_0+W_1$ ko'rinishida izlash mumkin. Bu yerda W_1 bir jinsli yoki Koshi-Riman tenglamalar sistemasi $\frac{\partial W}{\partial \bar{z}}=0$ ning umumiy yechimi. Bu yechim ixtiyoriy $\varphi(z)$ analitik funksiyadan iborat bo'lgani uchun (1.4.1) tenglamaning umumiy yechimi quyidagi ko'rinishda bo'ladi:

$$W=\varphi(z)-\frac{1}{\pi}\int_D \frac{G(z)}{t-z}\,d\xi\,d\eta$$

$W=e^{2y}(u(x,y)+v(x,y))=e^{2y}w(z)$, $2G(x,y)=e^{-2y}(g_1(x,y)+g_2(x,y))$ ekanidan

$$w(z)=e^{-2y}[\varphi(z)-\frac{1}{\pi}\int_D \frac{g(t)}{t-z}e^{2\eta}\,d\xi\,d\eta],$$

bu yerda $2g(x,y)=g_1(x,y)+g_2(x,y)$, ekanligi kelib chiqadi. Bundan esa

$$u(x,y)=Re(w(z)),\quad v(x,y)=Im(w(z)).$$

Javob: $u(x,y)=Re(w(z))$, $v(x,y)=Im(w(z))$, bu yerda

$$w(z)=e^{-2y}[\varphi(z)-\frac{1}{\pi}\int_D \frac{g(t)}{t-z}e^{2\eta}\,d\xi\,d\eta].$$

5-misol. Chegaralangan va chegarasi ∂D bo'lakli silliq bo'lgan D sohada quyidagi elliptik tenglamalar sistemasining umumiy yechimini toping [3].

$$\begin{cases} u_x-v_y+av=g_1 \\ u_y+v_x-au=g_2 \end{cases}$$

Yechilishi: $u(x,y)=e^{cx+by}U(x,y)$, $v(x,y)=e^{cx+by}V(x,y)$ deb olaylik. U holda

$$\begin{cases} ce^{cx+by}U+e^{cx+by}U_x-(be^{cx+by}V+e^{cx+by}V_y)+ae^{cx+by}V=g_1 \\ be^{cx+by}U+e^{cx+by}U_y+ce^{cx+by}V+e^{cx+by}V_x-ae^{cx+by}U=g_2 \end{cases}$$

Bu yerdan,

$$\begin{cases} e^{cx+by}U_x-e^{cx+by}V_y+ce^{cx+by}U-(b-a)e^{cx+by}V=g_1 \\ e^{cx+by}U_y+e^{cx+by}V_x+ce^{cx+by}V+(b-a)e^{cx+by}U=g_2 \end{cases}$$

Endi, $b=a$, $c=0$ olsak va $G_1(x,y)=e^{ay}g_1(x,y)$, $G_2(x,y)=e^{ay}g_2(x,y)$ deb belgilasak, u holda quyidagi tenglamalar sistemasiga kelamiz:

$$\begin{cases} U_x - V_y = G_1 \\ U_y + V_x = G_2 \end{cases}$$

Quyidagi almashtirishlarni kiritamiz:

$$W = U + iV,\ \ 2G = G_1 + iG_2,\ \ z = x + iy\, t = \xi + i\eta\,,$$

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}),\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

Natijada berilgan tenglamalar sistemasi ushbu ko’rinishga keladi:

$$\frac{\partial W}{\partial \bar{z}} = G(z)$$

Ushbu tenglamaning xususiy yechimi $W_0(z) = -\frac{1}{\pi}\int\limits_D \frac{G(t)}{t-z}\, d\xi\, d\eta$ ekanidan, umumiy yechimni $W = W_0 + W_1$ ko’rinishida izlash mumkin. Bu yerda W_1 bir jinsli yoki Koshi-Riman tenglamalar sistemasi $\frac{\partial W}{\partial \bar{z}} = 0$ ning umumiy yechimi. Bu yechim ixtiyoriy $\varphi(z)$ analitik funksiyadan iborat bo’lgani uchun (1.4.1) tenglamaning umumiy yechimi quyidagi ko’rinishda bo’ladi:

$$W = \varphi(z) - \frac{1}{\pi}\int\limits_D \frac{G(z)}{t-z}\, d\xi\, d\eta$$

$W = e^{-\alpha y}\left(u(x,y) + v(x,y)\right) = e^{-\alpha y} w(z)$ $\qquad 2G(x,y) = e^{\alpha y}\left(g_1(x,y) + ig_2(x,y)\right)$ ekanidan

$$w(z) = e^{ay}[\varphi(z) - \frac{1}{\pi}\int\limits_D \frac{g(t)}{t-z} e^{-a\eta}\, d\xi\, d\eta],$$

bu yerda $2g(x,y) = g_1(x,y) + ig_2(x,y)$, ekanligi kelib chiqadi. Bundan esa

$$u(x,y) = \mathrm{Re}\,(w(z)),\ \ v(x,y) = \mathrm{Im}\,(w(z))$$

Javob: $u(x,y) = \mathrm{Re}\,(w(z)),\ \ v(x,y) = \mathrm{Im}\,(w(z))$, bu yerda

$$w(z) = e^{ay}[\varphi(z) - \frac{1}{\pi}\int\limits_D \frac{g(t)}{t-z} e^{-a\eta}\, d\xi\, d\eta],$$

6-misol. Chegarasi bo'lakli silliq bo'lgan D chegaralangan tekis sohada quyidagi elliptik sistemaning umumiy yechimini toping [3].

$$\begin{cases} au_x + bv_y - acu = g_1 \\ au_y + bv_x - bcv = g_2 \end{cases}$$

$u = e^{\alpha x+\beta y}U,\ v = e^{\alpha x+\beta y}V$ bo'lsin.

$$a\left(\alpha e^{\alpha x+\beta y}U + e^{\alpha x+\beta y}U_x\right) + b\left(\beta e^{\alpha x+\beta y} + e^{\alpha x+\beta y}V_y\right) - ace^{\alpha x+\beta y}U = g_1$$
$$a\left(\beta e^{\alpha x+\beta y}U + e^{\alpha x+\beta y}U_x\right) - b\left(\alpha e^{\alpha x+\beta y} + e^{\alpha x+\beta y}V_x\right) + bce^{\alpha x+\beta y}V = g_2$$

U holda

$$aU_x + bV_y + a(\alpha - c)U + \beta bV = e^{-\alpha x-\beta y}g_1$$
$$aU_y - bV_x - b(\alpha - c)V + \beta bV = e^{-\alpha-\beta y}g_2$$

$\beta = 0,\ \ \alpha = 0$ deb olamiz.

$$G_1 = e^{-cx}g_1,\ G_2 = e^{-cx}g_2$$
$$aU_x + bV_y = G_1$$
$$aU_y - bV_x = G_2$$

$P = aU,\ Q = bV$ olsak,

$$\begin{cases} P_x + Q_y = G_1 \\ P_y - Q_x = G_2 \end{cases}$$

$W = P + iQ,\ \ 2G = G_1 + iG_2,\ \ z = x + iy,\ t = \xi + i\eta$ almashtirishlarni olsak,

$\dfrac{\partial W}{\partial z} = G(z)$ tenglama hosil bo'ladi. Buning xususiy yechimi:

$W_0 = -\dfrac{1}{\pi}\displaystyle\int_D \dfrac{\overline{G}(t)}{\bar{t} - \bar{z}}d\xi d\eta$. Berilgan sistemaning umumiy yechimini quyidagi

$W = W_1 + W_0$ ko'rinishda izlaymiz. Bunda W_1 $\dfrac{\partial W}{\partial z} = 0$ tenglamaning umumiy yechimi. Bu istalgan $W_1 = \overline{\varphi}(z)$ analitik funksiyadan iborat.

Demak

$$W=\bar{\varphi}(z)-\frac{1}{\pi}\iint_D \frac{\overline{G}(t)}{\bar{t}-\bar{z}}d\xi d\eta$$

$$W=P+iQ=U+ibV=e^{-cx}(u+ibV)=e^{-cx}\omega$$

$$2G=G_1+iG_2=e^{-cx}(g_1+ig_2)=e^{-cx}2g$$

$$\omega(z)=au+ibv=e^{cx}\left[\bar{\varphi}(z)-\frac{1}{\pi}\iint_D \frac{\overline{G}(t)}{\bar{t}-\bar{z}}e^{-cx}d\xi d\eta\right]$$

U holda $u(x,y)=\frac{1}{a}\operatorname{Re}(\omega(z)),\ v(x,y)=\frac{1}{b}\operatorname{Im}(\omega(z))$

II BOB. Tekislikda Koshi-Riman sistemasi uchun Koshi masalasi. Soha chegarasining qismida berilgan funksiyani soha ichiga golomorf davom ettirish masalasi.

2.1-§. Analitik davom ettirish masalasi yechimining yagonaligi haqida.

2.1.1-Теорема. $F(z)$ funksiya $|z|<1$ doirada golomorf va chegaralangan bo'lsin. Agar $F(z)$ funksiya qiymatlari $|z|=1$ aylana ustidagi E, $mesE>0$ to'plamda nolga intilsa, u holda $F(z)$ aynan nolga teng bo'ladi. [10].

Isbot. $|F(z)|<1$ va $F(z)$ aynan nolga teng emas deb olamiz. U holda $U(z)=\ln|F(z)|$ funksiya $|z|<1$ doirada $F(z)$ funksiyaning nollaridan tashqarida manfiy ishorali garmonik bo'lib, bu nuqtalarda $-\infty$ ga aylanadi.

$\rho<1$ radyusli $|z|=\rho$ aylanani shunday o'kazamazki, u funksiyaning nolini o'zida saqlamasin, ρ birga qancha yaqin bo'lmasin bunday funksiya topiladi, chunki $F(z)$ funksiyaning nolli to'plami birlik aylana ichida limitik nuqtaga ega emas. Bu aylana ustida U funksiya $U(\rho e^{iv})$ manfiy ishorali qiymatlarni qabul qiladi. Bu qiymatlar uchun Puasson integralini qaraymiz.

$$U_\rho=\frac{1}{2\pi}\int_0^{2\pi}U(\rho e^{i\theta})\frac{\rho^2-r^2}{\rho^2+r^2-2\rho rCos(\theta-\varphi)}d\theta$$

Bunda $z=re^{i\varphi}$, $r<\rho$. $U_\rho(z)$ funksiya

$|z|<\rho$ doirada garmonik bo'lib, $|z|\le\rho$ da uzluksiz bo'ladi.

Quyidagi ayirmani qaraymiz

$$D_r(z)=U(z)-U_\rho(z) \qquad (2.1.1)$$

$D_r(z)$ funksiya $|z|<\rho$ doira ichida chekli sondagi nuqtalardan tashqarida garmonik bo'lib, bu nuqtalarda $-\infty$ ga teng bo'ladi.

Maksimum prinspiga asosan $|z|<\rho$ bo'lganda $D_\rho(z)\le 0$ ni hosil qilamiz.

(1) formuladan $U(z)=U_\rho(z)+D_\rho(z)$ (1^1) ni olamiz.

$F(z)$ ning nollaridan farqli o'zgarmas z nuqtani olib $\rho \to 1$ da $U_\rho(z)$ funksiya $-\infty$ ga intilishini ko'rsatamiz. U holda $D_\rho(z) \le 0$ ekanligini etiborga olib (2.1.1) munosabatdan izlanayotgan qarama-qarshilikni hosil qilamiz. Puasson integrali $U_\rho(z)$ ni yuqoridan baholash uchun

$$U_\rho(z) < \frac{\rho - r}{\rho + r} \frac{1}{2\pi} \int_0^{2\pi} U(\rho e^{i\theta}) d\theta \qquad (2.1.2)$$

tengsizlikni yozib olamiz. E, $mesE > 0$ to'plamning θ nuqtalari uchun

$$\lim_{\rho \to 1} U(\rho e^{i\theta}) = -\infty$$

bo'lgani uchun, P, $mesP$ mukammal to'plam mavjud bo'lib, unda tekis ravishda $U(\rho e^{i\theta}) \to -\infty$ bo'ladi. (2.1.2) tengsizlikni

$$U_\rho(z) < \frac{\rho - r}{\rho + r} \times \frac{1}{2\pi} \left[\int_\rho U(\rho e^{i\theta}) d\theta + \int_{c\rho} U(\rho e^{i\theta}) d\theta \right]$$

ko'rinishda yozib olamiz, $\rho \to 1$ dan $\int_\rho U(\rho e^{i\theta}) d\theta \to -\infty$ va $\int_{c\rho} U(\rho e^{i\theta}) d\theta < 0$

bo'lgani uchun oxirgi tengsizlikdan $U(\rho e^{i\theta})$ ning $-\infty$ ga intilishi kelib chiqadi.

2.2-§. Tekislikda Koshi-Riman sistemasi uchun Koshi masalasini Karleman funksiyasi yordamida yechish.

Tekislikda chegaralangan D soha berilgan bo'lib Γ uning chegarasi Γ_1, Γ_2 to'plamlar Γ to'plamning qismlari bo'lsin. $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma = \Gamma_1$ to'plamda $(g(x,y)) = (h(x,y))$ funksiyar jufti berilgan bo'lsin. D sohada Koshi – Riman sistemasining $(U(x,y),V(x,y)) = (g(x,y),h(x,y))$ shartni qanoatlantiruvchi yechimini topish masalasini ya`ni Koshi masalasini qaraymiz. [12].

Agar $f(z) = U(x,y) + iV(x,y)$, $\varphi(z) = g(x,y) + ih(x,y)$, $z = x + iy$ belgilashlarni kiritsak, Koshi masalasi quyidagi analitik davom ettirish masalasiga ekvivalent bo'ladi. D sohada analitik bo'lgan hamda $f(z) = \varphi(z)$, $z \in \Gamma_1$ shartni qanoatlantiruvchi $f(z)$ funksiya topilsin. Agar $\Gamma_1 = \Gamma$ bo'lsa analitik davom ettirish masalasining yechimi Koshi integrali yordamida beriladi, yani

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)d\zeta}{\zeta - z}$$

$\Gamma_1 \neq \Gamma$ bo'lganda analitik davom ettirish masalasi Laplas uchun Koshi masalasiga ekvivalent bo'ladi.

I. Γ_1 to'plamda $U(z)$ garmonik funksiyalar va uning normal xosilasi $\frac{\partial U(z)}{\partial n}$ ning qiymatlari berilgan bo'lsin.

$$U(z) = g(z), \quad \frac{\partial U(z)}{\partial n} = h(z), \quad z \in \Gamma_1$$

Quyidagi funksiyani qaraymiz

$$\varphi(z) = g(z) + i\int_{z_0}^{z} h(s)\,dS + C_1, \quad z \in \Gamma_1,$$

bunda $z_0 - \Gamma_1$ ning chegarasidan biri. U holda $\varphi(z)$ funksiya D sohada $f(z) = U(x,y) + iV(x,y)$ analitik bo'lgan funksiyaning chegaraviy qiymatlaridan iborat bo'ladi. Shunday qilib Γ_1 da U, $\frac{\partial U}{\partial n}$ ma'lum bo'lsa Γ_1 da $\varphi(z)$ analitik funksiyaning qiymatlari ma'lum deb hisoblash mumkin.

II. Γ_1 da $f(z) = U(x, y) + iV(x, y)$ analitik funksiya qiymatlari berilgan bo'lsin.

U, V garmonik funksiyalar. Koshi – Riman shartlaridan $\frac{\partial U(z)}{\partial n} = \frac{\partial V(z)}{\partial S}$, $z \in \Gamma_1$ tenglik kelib chiqadi.

Bunda $\frac{\partial V}{\partial S}$ V funksiyadan Γ_1 bo'ylab hosila. Shuning uchun $f(z)$ va $\overline{f(z)}$ funksiyalardan Γ_1 bo'ylab hosila olib $\frac{\partial U(z)}{\partial n} = \frac{1}{r} \frac{\partial}{\partial S} \left(f(z) - \overline{f(z)}\right)$, $z \in \Gamma_1$ ni olamiz. Faraz qilaylik $U(x, y)$ funksiya uchun Koshi berilganlariga kelamiz. Shunday qilib $\Gamma_1 = \Gamma$ bo'lsa, u holda analitik davom ettirish masalasi Laplas tenglamasi uchun Koshi masalasiga teng kuchli va demak nokorrekt bo'ladi.

Chegaralangan D sohada analitik, $\overline{D}$ sohada uzluksiz bo'lgan, hamda

$$|f(z)| \le c \quad (z \in D) \tag{2.2.1}$$

shartni qanoatlantiruvchi funksiyalar to'plamini qaraymiz. Bu to'plamni M orqali belgilasak, Montel teoremasi (kompaktlik prinspi) ga ko'ra M kompakt to'plam bo'ladi. Teoremaga ko'ra qaralayotgan analitik davom ettirish masalasi yechimi yagona bo'ladi. Agar masalaning yechimi mavjud va M ga tegishli deb olsak, u holda A.N.Tixonov teoremasiga asosan masala turg'un bo'ladi. Demak M to'plamda analitik davom ettirish masalasi shartli korrekt bo'ladi. Qaralayotgan masala yechimi turg'unligi haqidagi teoremani isbotlashdan oldin, kompakt o'zgaruvchili funksiyalar nazariyasida ma'lum bo'lgan garmonik o'lchov tushunchasini ko'rib o'tamiz.

2.2.1-Ta'rif: Γ_1 chiziqning D sohadagi z nuqtasiga nisbatan garmonik o'lchovi deb, Γ_1 da birga teng, Γ_2 da nolga teng bo'lgan D sohada garmonik bo'lgan $\omega(z)$ funksiyaning z nuqtasidagi qiymatiga aytiladi. [12].

2.2.1-Teorema: Faraz qilaylik $f(z)$ da regulyar analitik, D ning yopig'ida uzluksiz hamda (3.1) shartni, shu bilan birga $f(z)$ funksiya

$$|f(z)| \le \varepsilon, \quad z \in \Gamma_1 \tag{2.2.2}$$

$$|f(z)| \le \varepsilon^{\omega(z)} c^{\varepsilon-\omega(z)} \tag{2.2.3}$$

tengsizlikni qanoatlantirsin. U holda (2.2.3) shart o'rinli.

Isbot: Quyidagi funksiyalarni qaraymiz.

$$\varphi(z) = \ln|f(z)| \tag{2.2.4}$$

ma'lumki $\varphi(z)$ funksiya D sohaning $f(z)=0$ shartni qanoatlantiruvchi nuqtalarda regulyar garmonik bo'ladi.

Agar $z_0 = D$ nuqtada $f(z_0)=0$ bo'lsa, u holda $z \to z_0$ da $\varphi(\tau) \to -\infty$ bo'ladi.

(2.2.1), (2.2.2) dan

$$\begin{cases} \varphi(z) \le \ln\varepsilon, & z \in \Gamma_1 \\ \varphi(z) \le \ln c, & z \in \Gamma_2 \end{cases} \tag{2.2.5}$$

tenglikning o'rinli ekanligi kelib chiqadi. (2.2.4) va (2.2.5) ga ko'ra $\psi(z) = \omega(z)\ln\varepsilon + \left[-\omega(z)\right]\ln c$ funksiya uchun $\varphi(z) \le \psi(z)$ tengsizlik o'rinli bo'ladi. Bundan isbotlanishi talab etilgan (2.2.3) tenglik kelib chiqadi. Korrekt masalalarni tekshirishda yechimning yagonaligi va turg'unligi o'rnatilgandan keyingi bosqich regulirizatsiyalovchi oilasini qurishning asosiy metodidan biri Karleman formulasi metodi hisoblanadi. [12].

2.2.2-Ta'rif: D soha va Γ_1 egri chiziqning Karleman funksiyasi deb, ikki kompleks o'zgaruvchili va bitta haqiqiy o'zgaruvchining quyidagi ikki xossalarga ega bo'lgan $G(z,\xi,\gamma)$ funksiyaga aytiladi.

1. $G(z,\zeta,\gamma) = \dfrac{1}{\zeta - z} + \overline{G}(z,\zeta,\gamma)$ bunda $\overline{G}(z,\zeta,\gamma)$ ζ o'zgaruvchining analitik funksiyasi bo'lib, D sohada analitik va chegaralangan,

2. $G(z,\zeta,\gamma)$ funksiya $\int_\Gamma |G(z,\zeta,\gamma)||d\zeta| \le \alpha$ tengsizlikni qanoatlantiradi.

Karleman funksiyasi yordamida analitik davom ettirish masalasining regulirizatsiyasini quramiz. Quyida operatorlar oilasini qaraymiz.

Γ_1 da aniqlangan har bir uzluksiz $\varphi(z)$ funksiyaga, D sohada

$$\varphi_\alpha(z) = \frac{1}{2\pi i}\int_\Gamma G(z,\zeta,\gamma)\varphi(\zeta)d\zeta$$

formula bilan aniqlangan $\varphi_\alpha(z)$ funksiya mos qo'yiladi. Bunday yo'l bilan aniqlangan operatorlar oilasi qaralayotgan analitik davom ettirish masalasi uchun regulirizatsiyalovchi oila bo'lishini ko'rsatamiz. Har bir $\alpha > 0$ uchun operatorlar oilasi uzluksiz bo'ladi. Bundan tashqari

$$\int_\Gamma \overline{G}(x,\zeta,\gamma) f(\zeta)d\zeta = 0$$

tenglikdan

$$f(z) = \frac{1}{2\pi i}\int_\Gamma G(z,\zeta,\gamma) f(\zeta)d\zeta + \int_\Gamma G(z,\zeta,\gamma)d\zeta \qquad (2.2.6)$$

tenglikni hosil qilamiz. Karleman funksiyasi ta'rifidan (2.2.6) ning o'ng tomonidagi ikkinchi qo'shiluvchi $\alpha \to 0$ da nolga intilishi kelib chiqadi. Buni esa regulirizatsiya masala shartida berilgan funksiya taqribiy berilganda taqribiy yechimni toppish masalasiga qo'llaymiz. Bizga Γ_1 to'plamda $f_\varepsilon(z)$ funksiya berilgan bo'lsin:

$$|f_\varepsilon(z) - \varphi(z)| \le \varepsilon, \qquad z \in \Gamma_1 \qquad (2.2.7)$$

$f_{\alpha\varepsilon}(z)$ orqali $f_{\alpha\varepsilon}(z) = \frac{1}{2\pi i}\int_{\Gamma_1} G(z,\zeta,\alpha) f_\varepsilon(\zeta)d\zeta$ funksiyani belgilab

$$f(z) - f_{\alpha\varepsilon}(z) = \frac{1}{2\pi i}\int_{\Gamma_1} G(z,\zeta,\alpha)[f(\zeta) - f_\varepsilon(\zeta)]d\zeta + \frac{1}{2\pi i}\int_{\Gamma_2} G(z,\zeta,\alpha) f(\zeta)d\zeta$$

ayirmani baholaymiz. (2.2.1) tengsizlik va Karleman funksiyasi ta'rifidan

$$\left|\int_{\Gamma_2} G(z,\zeta,\alpha) f(\zeta)d\zeta\right| \le C\cdot\alpha$$

tengsizlik o’rinli ekanligi kelib chiqadi.

$$\mu(z,\alpha)=\int_{\Gamma_1}|G(z,\zeta,\alpha)||d\zeta|$$

belgilash kiritsak, (2.2.7) dan

$$|f(z)-f_{\alpha\varepsilon}(z)|\le \varepsilon\cdot\mu(z,\alpha)+C\cdot\alpha$$

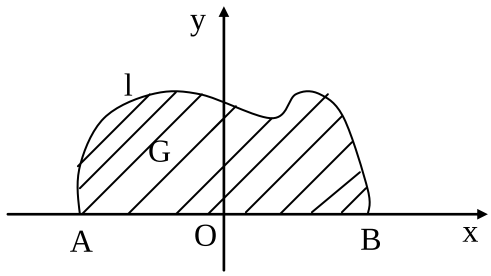
bunda $\varepsilon\cdot\mu(z,\alpha)=C\cdot\alpha$ desak,

$$|f(z)-f_{\alpha\varepsilon}(z)|\le 2\varepsilon\cdot\mu(z,\alpha).$$

bo’ladi. [3].

2.3-§. Bir o’lchovli Koshi-Riman tenglamasi uchun Koshi masalasi yechimining mavjudligi. Fok-Kuni teoremasi.

Z kompleks tekisligidagi chegarasi L konturdan iborat bo’lgan sohani qaraymiz. Bunda L haqiqiy o’qning AB kesmasi va haqiqiy o’qdan yuqorida joylashgan l konturdan iborat. G sohada Koshi-Riman tenglamasini qaraymiz. [15].



$$\frac{\partial g(z)}{\partial \bar z}=0$$

(2.3.1-rasm)

Bu tenglama yechimining C soha ichidagi qiymatlarini uning l konturdagi limitik qiymatlari orqali ifodalash masalasini qaraymiz. Bu masala Koshi – Riman tenglamalar sistemasi uchun Koshi masalasidan iborat bo’lib, kompleks o’zgaruvchining funksiyasi G analitik sohasi chegarasi qismidagi qiymatlari bo’yicha shu sohaga davom ettirish masalasiga teng kuchlidir.

2.3.1-Teorema: $g(z)$ funksiya G ning ichida analitik va l gacha uzluksiz bo’lsin. U holda istalgan $z\in G$ uchun

$$g(z) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_l \frac{g(\zeta)\exp[-i\tau(\zeta - z)]}{\zeta - z} d\zeta \qquad (2.3.1)$$

$$g(z) = \frac{1}{2\pi i} \int_l \frac{g(\zeta)}{\zeta - z} d\zeta + (-2\pi)^{-1} \int_0^\infty d\tau \exp(i\tau z) \int_l g(\zeta)\exp(-i\tau\zeta) d\zeta \qquad (2.3.1a)$$

((2.3.1) va (2.3.1a) formula Karleman formulasi.) o'rinli.

Isbot: (2.3.1a) formula (2.3.1) ning boshqacha shaklda yozilishidan iborat. Shuning uchun faqat (2.3.1) ni isbotlashimiz yetarli.

Koshi integral formulasiga asosan

$$g(z) = \frac{1}{2\pi i} \int_{l+AB} \frac{g(\zeta)\exp[-i\tau(\zeta - z)]}{\zeta - z} d\zeta \qquad z \in C \qquad (2.3.2)$$

AB bo'yicha olingan integral $\tau > 0$ bo'lganda moduli bo'yicha $SM(2\pi \operatorname{Im} z)^{-1} \exp(-\tau \operatorname{Im} z)$ dan oshmaydi, Bunda $M = \max_{z\in AB} |g(z)|$ va $S = |AB|$, $\operatorname{Im} z > 0$ bo'lgani uchun $\tau \to \infty$ da bu integral nolga intiladi. Istalgan $\varepsilon > 0$ uchun C ning ichida yotuvchi $\operatorname{Im} z \geq \varepsilon$ tengsizlikni qanoatlantiruvchi barcha z uchun bu integralning nolga intilishi tekis bo'ladi. Bundan (2.3.1) kelib chiqadi. (2.3.1) va (2.3.1a) formulalar yordamida l da berilgan funksiyani C sohada analitik davom ettirish shartlarini toppish mumkin.

Fok-Kuni ning [15] ishlarida quyidagi teorema isbotlangan:

2.3.2-Teorema: (Fok – Kuni) l egri chiziqda Lipshist shartini qanoatlantiruvchi $\varphi(z)$ funksiya berilgan bo'lsin. U holda

1. Agar G ning ichida analitik, $\overline{G} = G \cup L$ da uzluksiz bo'lgan hamda l da $\varphi(\zeta)$ ga teng bo'lgan $g(z)$ funksiya mavjud bo'lsa, u holda quyidagi shart bajariladi

$$\lim_{\tau\to\infty} I_\tau[\varphi] = 0 \qquad (2.3.3)$$

bunda $$I_\tau[\varphi] = \left| \int_l \varphi(z)\exp(-i\tau\zeta) d\zeta \right| .$$

2. Agar $\varphi(\zeta)$ funksiya uchun (2.3.3) shart o'rinli bo'lsa, u holda G ning ichida analitik, l da $\varphi(\zeta)$ ga teng bo'lgan $g(z)$ funksiya mavjud bo'ladi.

Isbot: 1) Koshi teoremasiga asosan istalgan τ uchun

$$\int_l g(\zeta)\exp(-i\tau\zeta)d\zeta = -\int_A^B g(x)\exp(-i\tau x)dx \qquad (2.3.4)$$

ga ega bo'lamiz. Bu tenglikning o'ng tomonidagi integral Riman-Lebeg teoremasiga asosan nolga intilgani uchun (2.3.4) - tenglikdan (2.3.3) - tenglik kelib chiqadi.

2) $\varphi(\zeta)$ funksiya (2.3.3) shartni qanoatlantirsin. (2.3.1a) tenglikning o'ng tomonidagi $g(\zeta)$ ni $\varphi(\zeta)$ ga almashtirib

$$g(z) = \frac{1}{2\pi i}\int_l \frac{\varphi(\zeta)}{\zeta - z}d\zeta + (-2\pi)^{-1}\int_0^\infty d\tau \exp(i\tau z)\int_l \varphi(\zeta)\exp(-i\tau\zeta)d\zeta \qquad (*)$$

ni hosil qilamiz. (*) dagi birinchi qo'shiluvchi l egri chiziqdan tashqarida analitik funksiya bo'lib, uning l da pastdan va yuqoridan limit qiymatlari ayirmasi $\varphi(\zeta)$ ga teng. (*) dagi ikkinchi qo'shiluvchi (2.3.3) ga asosan butun yuqori yarim tekislikda analitik funksiyani ifodalaydi. Demak (*) ifoda G ning ichida biror $g_1(z)$ analitik funksiyani va C ning tashqarisida, yani yuqori yarim tekislikda biror $g_2(z)$ analitik funksiyani ifodalaydi. Bu funksiyalar Soxotskiy formulasiga asosan $g_1(\zeta) - g_2(\zeta) = \varphi(z)$, $\zeta \in l$ shartni qanoatlantiradi. Ikkinchi tomondan (*) ifoda (2.1) ning o'ng tomonidan $g(\zeta)$ ni $\varphi(\zeta)$ ga almashtirish natijasida hosil bo'lgan ifodaga teng. $\operatorname{Im} z > \max_{\xi\in l}(\operatorname{Im}\zeta)$ bo'lganda $g_2(z) = 0$ bo'ladi. Analitik davom ettirish yagonaligiga asosan $g_2(z) \equiv 0$ bo'ladi. U holda $g(\zeta) = \varphi(\zeta)$ va demak $g_1(z)$ izlanayotgan $g(z)$ funksiya bo'ladi.

2.4-§. Soha chegarasining qismida berilgan funksiyani soha ichiga golomorf davom ettirish mumkinligi haqida.

[1] da quyidagi teorema isbotlangan: ($\Omega \subset C^1$ sohaning yoylarida berilgan funksiyani sohaga golomorf davom ettirish mumkinligi haqida.)

Ω_1 - $\gamma_1 = \{z : |z| = 1\}$ birlik aylana va γ_1 ning ichida yotuvchi hamda γ_1 ni ikki nuqtasini tutashtiruvchi Γ ochiq yoy bilan chegaralangan soha bo'lsin. Bundan tashqari nol $\overline{\Omega}_1$ dan tashqarida yotadi deb faraz qilamiz. Γ $\partial\Omega_1$ chegaraning bir qismi deb xisoblaymiz. Quyidagi belgilashni kiritamiz. [1].

$$a_k = \int_{\Gamma} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta, \qquad (k = 0,1,2,...)$$

2.4.1-Teorema: Agar $f \in C(\Gamma) \cap L^1(\Gamma)$ bo'lsa, u holda $F|_{\Gamma} = f$ tenglikni qanoatlantiruvchi $F \in A(\Omega_1) \cap C(\Omega_1 \cup \Gamma)$ ning mavjud bo'lishi uchun

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} \le 1 \qquad (2.4.1)$$

shartning bajarilishi zarur va yetarlidir.

Isbot: (Zarurligi) $\Gamma_{\varepsilon} = \Gamma \cap \{z : |z| < 1 - \varepsilon\}$ belgilashni kiritamiz, bu yerda $0 < \varepsilon < 1$ da $a_k^{\varepsilon} = \int_{\Gamma_{\varepsilon}} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta$ bo'lsin. Faraz qilaylik, teoremada keltirilgan F funksiya mavjud bo'lsin. U holda a_k^{ε} lar $F\zeta^{-k-1}$ dan $\gamma_{1-\varepsilon}$ aylananing qismi bo'yicha olingan tegishli integrallarga teng bo'ladi. Shuning uchun $|a_k^{\varepsilon}| \le C(\varepsilon)(1-\varepsilon)^{-k-1}$, bundan tashqari $a_k = a_k^{\varepsilon} + \int_{\Gamma \backslash \Gamma_{\varepsilon}} \frac{f(\zeta)d\zeta}{\zeta^{k+1}}$, natijada

$$|a_k| \le \frac{C(\varepsilon)}{(1-\varepsilon)^{k+1}} + \frac{C_1(\varepsilon)}{(1-\varepsilon)^{k+1}}$$ tengsizlik hosil bo'ladi.

Endi quyidagini hasil qilamiz.

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} \le \frac{1}{1-\varepsilon}$$

Bu erda $\varepsilon \to +0$ da (3) – tenglik hosil bo'ladi.

(Yetarliligi) Koshi tipidagi

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta - z}d\varsigma \qquad (2.4.2)$$

integralni qaraymiz. Bu integral $\Omega_1$da golomorf bo'lgan F_+ va birlik doiraning qolgan qismida ($\overline{\Omega}_1$ chiqarib tashlangandan keyin) golomorf bo'lgan F_- funksiyalarni beradiki, Γ da ularning normal bo'yicha limitik qiymatlarining farqi $f(\zeta)$ ga teng bo'ladi.

$$F_+(\zeta) - F_-(\zeta) = f(\zeta), \quad \zeta \in \Gamma \qquad (2.4.3)$$

Agar F_+ va F_-funksiylardan biri mos sohada Γ gacha uzluksiz bo'lsa, u holda ikkinchisi ham shu hususiyatga ega bo'ladi. (2.4.3) ni z bo'yicha nolning atrofida darajali qatorga yoyib, koiffisentlar $a_k/(2\pi i)$ ga tengligini hosil qilamiz. Bundan va (2.4.1) dan F_-ning birlik doiraning hamma joyida golomorfligi kelib chiqadi. U holda $F_+(\varsigma) - F_-(\varsigma) \in A(\Omega_1) \cap C(\Omega_1 \cup \Gamma)$ va (2.4.3) ga asosan $F = F_+ - F_-$ deb hisoblash mumkin.

2.4.1-Natija. $f \in C(\Gamma)$ bo'lsin $F|_{\Gamma} = f$ tenglikni qanoatlantiradigan $F \in A(\Omega_1) \cap C(\Omega_1 \cup \Gamma)$ funksiya mavjud bo'lishi uchun ixtiyoriy ε, $0 < \varepsilon < \varepsilon_0 < 1$ da

$$\overline{\lim_{k\to\infty}}\sqrt[k]{|a_k|} \le \frac{1}{1-\varepsilon} \qquad (2.4.4)$$

tengsizlikning o'rinli bo'lishi zarur va yetarlidir. [1].

Olingan natijani o'zida silliq ochiq Γ yoyni saqlovchi Ω Jordan chegaraga ega bo'lgan bir bog'lamli sohaga ko'chiramiz. Γ ning uchlarini $\overline{\Omega}_1$ dan tashqarida yotuvchi l egri chiziq bilan tutashtiramiz Ω^1 orqali $l \cup (\partial\Omega \setminus \Gamma)$ chegaraga ega bo'lgan sohani belgilaymiz. $\varphi(z)$ Ω^1 ni birlik aylanaga shunday komplanar akslantirsinki, nolning asli $\Omega^1 \setminus \overline{\Omega}$ da yotadigan nuqta bo'lsin. $\varphi(\Gamma)$ ning asli $\varphi(\Gamma)$ da yotuvchi qismini Γ_ε orqali belgilaymiz.

$$A_k^\varepsilon = \int_{\Gamma_\varepsilon} \frac{f(\zeta)d\varphi(\zeta)}{\varphi^{k+1}(\zeta)} \tag{2.4.5}$$

bo'lsin. 2.4.1-natijadan quyidagi natija kelib chiqadi.

2.4.1$^{/}$-natija. Agar $f \in C(\Gamma)$ bo'lsa, u holda $F \setminus_\Gamma = f$ ni qanoatlantiruvchi $F \in A(\Omega_1) \cap C(\Omega_1 \cup \Gamma)$ funksiya mavjud bo'lishi uchun ixtiyoriy ε, $0 < \varepsilon < \varepsilon_0 < 1$ da quyidagi tengsizlikning bajarilishi zarur va yetarlidir.

$$\overline{\lim_{k\to\infty}} \sqrt[k]{\left|A_k^\varepsilon\right|} \le \frac{1}{1-\varepsilon} \tag{2.4.6}$$

(2.4.6) ning o'rniga (2.4.1) ning analogini olish mumkin, lekin (2.4.5) dagi formani butun Γ egri chiziqda integrallanuvchi bo'lishini talab qilishga to'g'ri keladi.

2.4.1^1-natijani quyidagi masalasiga qo'llaymiz. Jordan chegarasiga ega bo'lgan bir bog'lamli Ω soha va shu sohani ikkiga bo'luvchi silliq ochiq. $\Gamma \subset \Omega$ yoy berilgan bo'lsin. $\Omega = \Omega^1 \cup \Omega^2 \cup \Gamma$. Shunday $f \in C(\Gamma)$ funksiyani topish talab qilamizki, uni Ω dagi funksiyaga golomorf davom ettirish mumkin bo'lsin. [1].

Shunday qilib gap endi sohaning chegarasida emas, balki sohaning ichida yotuvchi funksiyani yoydan golomorf davom ettirish mumkinligini o'rganish haqida bormoqda.

Ω ni birlik doiraga ikkita φ_1 va φ_2 konform akslantirishlarni qaraymiz. Agar (2.4.6) tipidagi ikkala shart ham bajarilsa, u holda f ham Ω^1 va ham Ω^2 da golomorf davom etadi. Bu holda, turgan gap mos davom ettirishlar butun Ω sohada ham golomorf bo'ladi. Shunday qilib quyidagi tasdiqlarga ega bo'lamiz.

2.4.2-natija: Silliq ochiq Γ yoy Jordan chegarasiga ega bo'lgan bir bog'lamli Ω sohani ikki sohaga ajratsin va $f \in C(\Gamma)$ bo'lsin. U holda $F|_\Gamma = f$ ni qanoatlan-tiruvchi $F \in A(\Omega)$ funksiya mavjud bo'lishi uchun (2.4.6) tipdagi ikki shartning bajarishi zarur va yetarlidir.

Misol: Ω butun C^1 komoleks tekislikdan iborat bo'lsin. Bu holda birlik doiraga konform akslantirish mavjud bo'lmaydi. Γ G^1 ni ikki sohaga ajratuvchi oddiy silliq chiziq bo'lsin. Faraz qilaylik $\pm i$ nuqtalar har xil sohalarda yotsin. Markazi $\pm i$ nuqtalarda bo'lgan doiralardan foydalanamiz va aytaylik

$$a_k^{\pm} = \int_{\Gamma} \frac{f(\zeta)\,d\zeta}{(\zeta \pm i)^{k+1}}, \qquad k = 0,1,...$$

bo'lsin, bu yerda $f \in C(\Gamma) \cap L^1(\Gamma)$ Koshi – Adamar formulasining isbotidagi kabi f butun funksiyaga golomorf davom ettirish uchun

$$\lim_{k\to\infty} \sqrt[k]{|a_k^+|} = \lim_{k\to\infty} \sqrt[k]{|a_k^-|} = 0$$

Shartning bajarilishi zarur va yetarli ekanligini hosil qilamiz. $f \in L^1(\Gamma)$ shartning bajarilmasligini talab qilmaslik ham mumkin. Lekin bu holda masalaning javobi ancha murakkab bo'ladi.

III BOB. Umumlashgan analitik funksiyalarni davom ettirish masalasi

3.1-§. Koshi-Riman tenglamalar sistemasining kompleks o'zgaruvchili shakli.

Biror D yopiq sohada $w = u(x,y) + iv(x,y)$ funksiya analitik deyiladi agar

$$\partial_{\bar z} w = 0 \qquad (3.1.1)$$

Koshi-Riman tenglamasini qanoatlantirsa. Bunda

$$\bar z = x - iy, \;\; \partial_{\bar z} = \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$$

(3.1.1) tenglamani unga ekvivalent bo'lgan haqiqiy o'zgaruvchili

$$\begin{cases} u_x - v_y = 0 \\ u_y + v_x = 0 \end{cases} \qquad (3.1.2)$$

tenglamalar sistema bilan almashtirish masalasini qaraymiz. Biz bilamizki

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right), \;\; \bar w = u(x,y) - iv(x,y).$$

Quyidagi sistemalarni tuzib olamiz: $\begin{cases} \bar w = u(x,y) - iv(x,y) \\ w = u(x,y) + iv(x,y) \end{cases}$ (3.1.3)

$$\begin{cases} \partial_{\bar z} = \dfrac{\partial}{\partial \bar z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right) \\ \partial_z = \dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right) \end{cases} \qquad (3.1.4)$$

(3.1.3) sistemani $u(x,y)$, $v(x,y)$ funksiyalarga nisbatan, (3.1.4) sistemani esa $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ larga nisbatan yechamiz:

$$u = \frac{1}{2}(w + \bar w) \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar z}$$
$$v = \frac{1}{2i}(w - \bar w) \qquad \frac{\partial}{\partial y} = \frac{1}{i}\left(\frac{\partial}{\partial \bar z} - \frac{\partial}{\partial z}\right) \qquad (3.1.5)$$

Hosil qilingan (3.1.5) ifodalarni (3.1.2) sistemaga qo'yamiz:

$$\begin{cases} \dfrac{1}{2}\left[(\partial_{\bar z} + \partial_z)(w + \bar w) - \dfrac{1}{i}(\partial_{\bar z} - \partial_z)\dfrac{1}{i}(w - \bar w)\right] = 0 \\ \dfrac{1}{2}\left[\dfrac{1}{i}(\partial_{\bar z} - \partial_z)(w + \bar w) + \dfrac{1}{i}(\partial_{\bar z} + \partial_z)(w - \bar w)\right] = 0 \end{cases}$$

Bu sistemani soddalashtirib,

$$\begin{cases} \partial_{\bar z} w + \partial_z \overline{w} = 0 \\ \partial_{\bar z} w - \partial_z \overline{w} = 0 \end{cases} \quad (3.1.6)$$

sistemaga ega bo'lamiz. Hosil qilingan sistemaning tenglamalarini qo'shib yuborib isbotlanishi talab etilgan tenglamaga ega bo'lamiz:

$$\partial_{\bar z} w = 0 \quad (3.1.7)$$

Shuningdek (3.1.6) sistema tenglamalarini ayirib yuborib

$$\partial_z \overline{w} = 0 \quad (3.1.8)$$

tenglamaga ega bo'lamiz. (3.1.8) tenglamani qanoatlantiruvchi funksiyalar antigolomorf yoki antianalitik funksiyalar deyiladi.

Endi umumlashgan Koshi-Riman tenglamasi

$$\partial_{\bar z} w + A(z) w + B(z) \overline{w} = 0 \quad (3.1.9)$$

va

$$\begin{cases} u_x - v_y + a(x,y)u + b(x,y)v = 0 \\ u_y + v_x + c(x,y)u + d(x,y)v = 0 \end{cases} \quad (3.1.10)$$

haqiqiy o'zgaruvchili sistemaning ekvivalent bo'lishini isbotlaymiz.

Buning uchun yuqorida biz qaragan (3.1.5) ifodalarni (3.1.10) sistemaga qo'yamiz:

$$\begin{cases} \frac{1}{2}\left[(\partial_{\bar z} + \partial_z)(w + \overline{w}) - \frac{1}{i}(\partial_z - \partial_{\bar z})\frac{1}{i}(w - \overline{w}) \right] + \frac{a(x,y)}{2}(w + \overline{w}) + \frac{b(x,y)}{2i}(w - \overline{w}) = 0 \\ \frac{1}{2}\left[\frac{1}{i}(\partial_{\bar z} - \partial_z)(w + \overline{w}) + \frac{1}{i}(\partial_{\bar z} + \partial_z)(w - \overline{w}) \right] + \frac{c(x,y)}{2}(w + \overline{w}) + \frac{d(x,y)}{2i}(w - \overline{w}) = 0 \end{cases}$$

Bu sistemani soddalashtirib

$$\begin{cases} 2\partial_{\bar z} w + 2\partial_z \overline{w} + (a(x,y) - ib(x,y))w + (a(x,y) + ib(x,y))\overline{w} = 0 \\ 2\partial_{\bar z} w - 2\partial_z \overline{w} + (c(x,y) - id(x,y))w + (c(x,y) + id(x,y))\overline{w} = 0 \end{cases} \quad (3.1.11)$$

Hosil qilingan (3.1.8) sistemaning tenglamalarini mos ravishda qo'shib quyidagini olamiz:

$$4\partial_{\bar z} w + [a(x,y) + c(x,y) - i(b(x,y) + d(x,y))]w + [a(x,y) + c(x,y) + i(b(x,y) + d(x,y))]\overline{w} = 0$$

Oxirgi tenglamani to'rtga bo'lib va

$$A(z) = \frac{1}{4}[a(x,y) + c(x,y) - i(b(x,y) + d(x,y))] \qquad B(z) = \frac{1}{4}[a(x,y) + c(x,y) + i(b(x,y) + d(x,y))]$$

belgilashlarni kiritib, (3.1.9) tenglamani hosil qilamiz.

3.2-§. Umumlashgan analitik funksiyalar to'la sistemalari. Umumlashgan darajali qatorlar.

$$\partial_{\bar{Z}}\omega + A\omega + B\overline{\omega} = 0, \quad (A, B \in L_{P,2}(E)) \tag{3.2.1}$$

tenglamaning biror $\omega_n (n = 1,2,...)$ xususiy yechimlar sistemasi berilgan bo'lsin. U G sohaga nisbatan yechimlar to'liq sistamasidan iborat deb olamiz, agar bu tenglamaning G da regulyar ixtiyoriy ω yechimini G ichida haqiqiy koiffisintli

$$C_1\omega_1 + C_2\omega_2 + C_3\omega_3 + C_4\omega_4 + ... + C_n\omega_n$$

ko'rinishdagi chiziqli argumentlar yordamida tekis yaqinlashtirish mumkin bo'lsa. G sohaga nisbatan to'liq ixtiyoriy $G + \Gamma$ da uzluksiz Φ_n analitik funksiyalar sistemasiga u (3.2.1) tenglamaning $\omega_n = k^n(\Phi_n, G)$ yechimlar to'liq sistemasini mos qo'yadi.

Masalan: Quyidagi z ga bog'liq ratsional funksiyalar sistemasini qaraymiz.

$$(z - z_0)^{n-1}, \ (z - z_j)^{-k} \quad (j = 1,..m, \ n, h = 1,2,...) \tag{3.2.2}$$

Bu yerda $z_1,...,z_n$ tayinlangan nuqtalar. Ma'lumki bu sistema $\Gamma_0, \Gamma_1,...\Gamma_n$ nuqtalar $\Gamma_j \ (j = 1,2,...,n)$ lar ichida joylashgan bilan chegaralangan ixtiyoriy G sohaga nisbatan to'liq (golomorf funksiyalar sistemasida) (3.2.2) funksiyalar sistemasiga formula yordamida quyidagi (3.2.1) tenglamaning G ga nisbatan to'liq xususiy yechimlari sistemasi mos qo'yiladi.

$$\begin{aligned}
&\omega_{2n}(z, z_0) = K((z - z_0)^n, G) \\
&\omega_{2n+1}(z, z_0) = K(i(z - z_0)^n, G) \quad (n = 0,1,2,...) \\
&\omega_{2n+1}(z, z_j) = \frac{\partial^{n-1}\Omega_1(z, z_j, G)}{\partial z_j^{n-1}} + \frac{\partial^{n-1}\Omega_2(z, z_j, G)}{\partial \overline{z_j^{n-1}}} \\
&\omega_{-2n}(z, z_j) = i\frac{\partial^{n-1}\Omega_1(z, z_j, G)}{\partial z_j^{n-1}} - i\frac{\partial^{n-1}\Omega_2(z, z_j, G)}{\partial \overline{z_j^{n-1}}} \qquad (n = 0,1,2,...)
\end{aligned} \tag{3.2.3}$$

Biz bu yerda $z \in G$ da $\Omega_j(z, \zeta, G)$ va $\Omega_2(z, \zeta, G)$ yadrolar $G + \Gamma$ dan tashqarida ζ ga nisbatan golomorf. Osonlik bilan ko'rish mumkinki $\omega_n(z, z_0)$ $[n/2]$ darajali umumlashgan polinom, $\omega_{-m}(z, z_o)$ cheksizlikda nolga aylanuvchi $[n+1/2]$ tartibli z_j

yagona qutubli umumlashgan ratsional funksiya. Quyida biz ko'ramizki, $\omega_n(z,z_0)$ cheksiz uzoqlashgan nuqta atrofida ζ ga nisbatan $\Omega_j(z,\zeta,G)$ funksiyalarning yoyilmalari koiffisentlaridan iborat.

Haqiqatan agar, $x \in G$, ζ esa $G+\Gamma$ dan tashqarida yotsa, u holda Koshi formulasiga asosan

$$\left.\begin{aligned} \Omega_1(z,\zeta,G) &= -\frac{1}{2\pi i}\int_\Gamma \frac{\Omega_1(z,t,G)}{t-\zeta}dt \\ \Omega_2(z,\zeta,G) &= \frac{1}{2\pi i}\int_\Gamma \frac{\Omega_2(z,t,G)}{t-\zeta}d\bar{t} \end{aligned}\right\} \qquad (3.2.4)$$

kelib chiqadi. Endi bu tenglikning o'ng qismlarini yetarlicha katta ζ larda $\zeta - z_0$ va $\bar{\zeta} - \bar{z}$ larning manfiy darajalari bo'yicha yoyib, formulaga ko'ra

$$\left.\begin{aligned} \Omega_1(z,\zeta,G) &= \frac{1}{2}\sum_{k=0}^{\infty}[\omega_{2k}(z,z_0) - i\omega_{2k+1}(z,z_0)](\zeta - z_0)^{-k-1}, \\ \Omega_2(z,\zeta,G) &= \frac{1}{2}\sum_{k=0}^{\infty}[\omega_{2k}(z,z_0) + i\omega_{2k+1}(z,z_0)](\bar{\zeta} - \bar{z}_0)^{-k-1}, \end{aligned}\right\} \qquad (3.2.5)$$

ni olamiz. G - $|z - z_0| < \rho$ doira bo'lsin. U holda bu qator bu doiraning ichida va tashqarisida z va ζ ga nisbatan tekis yaqinlashadi.

Agar $|z - z_0| > \rho$, $|\zeta - z_0| < \rho$, shartlar bajarilgan bo'lsa,

$$\left.\begin{aligned} \Omega_1(z,\zeta,G) &= -\frac{1}{2}\sum_{k=0}^{\infty}[\omega'_{2k}(\zeta,z_0) - i\omega'_{2k+1}(\zeta,z_0)](z - z_0)^{-k-1} \\ \Omega_2(z,\zeta,G) &= -\frac{1}{2}\sum_{k=0}^{\infty}[\overline{\omega_{2k}(\zeta,z_0)} + i\overline{\omega_{2k+1}(\zeta,z_0)}](z - z_0)^{-k-1} \end{aligned}\right\} \qquad (3.2.6)$$

yoyilmalarga ega bo'lamiz, bu yerda $\omega_n'(z,z_0)$ $[\frac{n}{2}]$ darajali

$$\partial_{\bar{z}}\omega' + A\omega' + \bar{B}\bar{\omega}' = 0, \qquad (3.2.7)$$

$$\left.\begin{aligned} \omega'_{2n}(z,z_0) &= K'[(z - z_0)^k, G] \\ \omega'_{2n+1}(z,z_0) &= K[i(z - z_0), G] \end{aligned}\right\} \qquad (3.2.8)$$

qo'shimcha shartlarni qanoatlantiruvchi umumlashgan polinomlar. (3.2.6) yoyilmalar

$$\Omega_j(z,\zeta,G) = -\frac{1}{2\pi i}\int_\Gamma \frac{\Omega_j(z,\zeta,G)}{t-\zeta}dt \qquad (j=1,2,...) \tag{3.2.9}$$

formulalardan olinadi. Ular agar $\zeta \in G$ va z $G+\Gamma$ dan tashqarida esa o'rinli bo'ladi.

II. $\omega_n(z,z_0)$ $(n=0,\pm1,\pm2,...)$ umumlashgan ratsional funksiyalar sistemasi yordamida analitik funksiyalar uchun Teylor va Loran qatorlari umumlashmasi hisoblanadi. (3.2.1) tenglamaning ixtiyoriy yechimlari yoyilmalarini olish mumkin. $G - |z-z_0| < \rho$ doirada $\Gamma - |z-z_0| = \rho$ aylana bo'lsin. Bunday holda shar $\omega_n(z,z_0)$ umumlashgan polinomlarni $\omega_n(z,z_0,\rho)$ deb belgilaymiz. Agar $\omega(z)$ G ning ichida (3.2.1) tenglamani qanoatlantirsa va $G+\Gamma$ da uzluksiz bo'lsa, u holda u

$$\omega(z) = K(\omega,G) = \frac{1}{2\pi i}\int_\Gamma \Omega_1(z,\zeta,G)\omega(\zeta)d\zeta - \Omega_2(z,\zeta,G)\overline{\omega(\zeta)}d\overline{\zeta} \tag{3.2.10}$$

yoki $$\omega(z) = K(\Phi,G) = \frac{1}{2\pi i}\int_\Gamma \Omega_1(z,\zeta,G)\Phi(\zeta)d\zeta - \Omega_2(z,\zeta,G)\overline{\Phi}(\zeta)d\overline{\zeta} \tag{3.2.11}$$

formula bilan ifodalanadi. Bu yerda

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)}{\zeta - z}d\zeta \tag{3.2.12}$$

oxirgi tenglik o'ng qismini $(z-z_0)^n$ darajalar bo'yicha qatorga yoyib,

$$\Phi(z) = \sum_{k=0}^{\infty}(C_{2k} + iC_{2k+1})(z-z_0)^k \tag{3.2.13}$$

ni olamiz, bu yerda G_k – haqiqiy o'zgarmaslar bo'lib

$$C_{2k} + C_{2k+1} = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)}{\zeta^{k+1}}d\zeta \qquad (k=0,1,...) \tag{3.2.14}$$

tenglamalardan aniqlanadi. Endi (3.2.1) tenglamaning quyidagi yechimlari sistemasini qaraymiz.

$$\omega_n(z) = \sum_{k=0}^{n} C_n\omega_k(z,z_0,\rho) = K(\Phi_m,G) \tag{3.2.15}$$

bu yerda

$$\Phi_n(z) = \sum_{k=0}^{[n/2]} (C_{2k} + iC_{2k+1})(z - z_0)^k$$

$$\omega(z) - \omega_n(z) = K(\Phi - \Phi_m, G) = K(\omega - \Phi_n, G)$$

bo'lgani uchun

$$|\omega(z) - \omega_n(z)| \le \frac{1}{2\pi} \int_{\Gamma} \left[|\Omega_1(z,\zeta,G)| + |\Omega_2(z,\zeta,G)| \right] |\varphi(\zeta) - \Phi_n(\zeta)| ds$$

tengsizlik o'rinli bo'ladi.

Agar z G doiraning biror G' yopiq qism to'plamiga tegishli bo'lsa, u holda

$$|\omega(z) - \omega_n(z)| \le M(G') L_1(\varphi - \Phi_n, \Gamma) \tag{3.2.16}$$

ga ega bo'lamiz. Γ aylanada (3.2.13) qator (3.2.14) formulaga ko'ra $\omega(G)$ uzluksiz funksiyaning Fure qatori bo'lgani uchun u Γ da ω ga ixtiyoriy $L_P(\Gamma)$, $p \ge 0$ metrkasida yaqinlashadi. Shuning uchun $n \to \infty$ da $L_p(\omega - \Phi_n, G) \to 0$ bunga ko'ra (3.2.16) dan $\omega_n(z)$ ketma-ketlik $\omega(z)$ ga G ichida tekis yaqinlashishga ega bo'lamiz. Shunday qilib $\omega(z)$ G doira ichida tekis yaqinlashuvchi

$$\omega(z) = \sum_{k=0}^{\infty} C_k \omega_k(z, z_0, \rho) \tag{3.2.17}$$

qatorga yoyilishi isbotlandi. Bu qator koiffisentlari uchun ma'lum integral formulalar bilan ustma – ust tushuvchi (3.2.14) formulalar bilan ifodalanadi. Endi G ; $|z - z_0| = \rho_0$ aylanalar bilan chegaralangan. $0 \le \rho_0 \le |z - z_0| < \rho$ halqa bo'lsin. Aylanalarni Γ_0, Γ_1 deb belgilaymiz. Bu sohaga mos keluvchi umumlashgan $\omega_n(z, z_0)$ $(n = 0,1,2,\ldots)$ ratsional funksiyalarni qarab, ularni $\omega_n(z, z_0, \rho_0, \rho_1)$ dab belgilaymiz , agar $\omega(z)$ G ichida (3.2.1) tenglamani qanoatlantirsa, va $\overline{G}$ da uzluksiz bo'lsa, u holda u quyidagi qatorga yoyiladi.

$$\omega(z) = \sum_{-\infty}^{+\infty} C_k \omega_k(z, z_0, \rho_0, \rho_1) \tag{3.2.18}$$

Bu yerda C_k haqiqiy o'zgarmaslar, ular quyidagi formulalar bilan hisoblanadi.

$$C_{2k} + iC_{2k=1} = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\omega(\zeta)}{\zeta^{k+1}} d\zeta \qquad (k = 0,1,2,\ldots)$$
$$C_{2k} + iC_{2k=1} = \frac{1}{2\pi i}\int_{\Gamma_0} \omega(\zeta)\zeta^{k} d\zeta \qquad (k = -1,-2,\ldots) \tag{3.2.19}$$

(3.2.18) qator $0 \le \rho_0 \le |z - z_0| < \rho$ halqa ichida tekis yaqinlashadi. Bu tasdiq (3.2.17) qator uchun yuqoridagi mulohazani deyarli so'zma-so'z takrorlash bilan isbotlanadi.

3.3-§. Chegaraning qismida berilgan funksiyani shu sohaga umumlashgan analitik funksiya sifatida davom ettirish masalasi

$D \subset C$ soha $\gamma_1 = \{z : |z| = 1\}$ birlik doira va γ_1 ning ikki nuqtasini tutashtiruvchi, to'lasincha γ_1 ning ichida yotuvchi egri chiziq bilan chegaralangan soha bo'lsin. Bunda z nuqta $\overline{D}$ sohadan tashqarida yotadi. M egri chiziq ustida berilganfunksiyani D sohaga

$$\frac{\partial w}{\partial \bar{z}} + A(z)w + B(z)\overline{w} = 0 \tag{3.3.1}$$

tenglamaning yechimi sifatida davom ettirish masalasini qaraymiz. Boshqacha aytganda $\varphi \in C_\alpha(M)$ funksiya qanday shartlarni qanoatlantirsa, shunday $w \in U_{p,2}(A, B, D)$ umumlashgan analitik funksiya mavjud bo'ladiki, uning M egri chiziqdagi chegaraviy qiymatlari φ bilan ustma-ust tushadi

$$w(z) = \varphi(z), \; z \in M \tag{3.3.2}$$

Quydagi belgilashni olamiz

$$a_k = a_{2k} + ia_{2k+1} = \int_M \frac{\varphi(\zeta)d\zeta}{\zeta^{k+1}}, \quad k = 0,1,2,, \ldots$$

bu yerda a_k- haqiqiy o'zgarmaslar.

Teorema: $\varphi \in C(M) \cap L(M)$ bo'lsin. U holda $w|_M = \varphi$ shartni qanoatlantiruvchi $w \in U_{p,2}(A, B, D) \cap C(D \cup M)$ funksiya mavjud bo'lishi uchun

$$\overline{\lim}_{k\to\infty} \sqrt[k]{|\alpha_k|} \le 1 \tag{3.3.3}$$

tengsizlikning bajarilishi zarur va yetarli.

Isbot: Zarurligi. Quydagi belgilashni olamiz

$$a_{2k}^{\varepsilon}+ia_{2k+1}^{\varepsilon}=\int_{M_{\varepsilon}}\frac{\varphi(\zeta)d\zeta}{\zeta^{k+1}},.$$

$$M_{\varepsilon}=M\cap\{z:|z|<1-\varepsilon\}.$$

Teorema shartidagi w funksiya mavjud bo'lsin. U holda umumlashgan Koshi integral fo'rmulasiga asosan,

$$\frac{1}{2\pi i}\int_{M_{\varepsilon}\cup\gamma_{1-\varepsilon}}\Omega_1(\zeta,z)w(\zeta)-\Omega_2(\zeta,z)\overline{w}(\zeta)d\vec{\zeta}=0\,,\ \ z\notin\overline{D} \qquad (3.3.4)$$

tenglikni hosil qilamiz . (3.3.4) integralni umumlashgan darajali qatorga yoyib,

$$\sum_{k=0}^{\infty}c_k^{\varepsilon}w_k(z,0,\rho)=0$$

ni olamiz, bu yerda

$$c_{2k}^{\varepsilon}+ic_{2k+1}^{\varepsilon}=\int\limits_{M_{\varepsilon}\cup\gamma_{1-\varepsilon}}\frac{w(\zeta)}{\zeta^{k+1}}d\zeta.$$

Umumlashgan darajali qatorga yoyilmaning yagonaligidan

$$c_{2k}^{\varepsilon}+ic_{2k+1}^{\varepsilon}=0,\qquad k=0,1,2\ldots$$

Tenglikni olamiz. Bu tenglikdan $a_{2k+1}^{\varepsilon}+ia_{2k+1}^{\varepsilon}$ sonlar $w\zeta^{-k-1}$ funksiyalardan $\gamma_{1-\varepsilon}$ aylana qismi bo'yicha olingan mos integralga tengligi kelib chiqadi. Bularga asosan

$$\alpha_k=a_{2k}^{\varepsilon}+ia_{2k+1}^{\varepsilon}+\int_{M/M_{\varepsilon}}\frac{\varphi(\zeta)}{\zeta^{k+1}}d\zeta$$

tenglikdan

$$|\alpha_k|\leq\frac{c(\varepsilon)}{(1-\varepsilon)^{k+1}}+\frac{c_1}{(1-\varepsilon)^{k+1}}$$

tengsizlikni hosil qilamiz. Demak,

$$\sqrt{a_{2k}^2+a_{2k+1}^2}\leq C^{\mathrm{I}}(\varepsilon)(1-\varepsilon)^{-k-1}$$

tengsizlik o'rinli bo'ladi. Bu yerdan

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|\alpha_k|} \leq \frac{1}{1-\varepsilon}$$

ni hosil qilamiz. Oxirgi tengsizlikdan $\varepsilon \to 0$ da $(3.3.3)$ ga kelamiz.

Yetarliligi. Umumlashgan Koshi tipidagi integralni qaraymiz.

$$\frac{1}{2\pi i}\int_M \Omega_1(\zeta,z)w(\zeta) - \Omega_2(\zeta,z)\overline{w}(\zeta)d\zeta \qquad (3.3.5)$$

Bu integral $w_+ \in U_{p,2}(A,B,D)$ va $w_- \in U_{p,2}(A,B,D_1/D)$ funksiyalarni aniqlaydi va bu funksiyalarning M dagi limitik qiymatlari ayirmasi $\varphi(\zeta)$ ga teng, ya'ni

$$w_+ - w_- = \varphi(\zeta),\ \ \zeta \in M \qquad (3.3.6)$$

Bu yerda w_+ yoki w_- funksiyalardan biri o'zining aniqlanish sohasida M to'plamgacha uzluksiz bo'lsa, ikkinchisi ham shu hossaga ega bo'ladi, $(3.3.5)$ integralni nol nuqta atrofida umumlashgan darajali qatorga yoyib, bu qatorning koyffisentlari $a_k/2\pi i$ ga tengligini ko'ramiz. Bu yerdan va $(3.3.3)$ shartdan w_- funksiyani D_1 sohaga $(3.3.1)$ tenglama yechimi sifatida davom ettirish mumkin degan xulosaga kelamiz, ya'ni $w_- \in U_{p,2}(A,B,D_1)$. U holda $(w_+ - w_-) \in U_{p,2}(A,B,D) \cap C(D \cup M)$ va $(3.3.6)$ ga ko'ra

$$w = w_+ - w_-$$

deb olish mumkin. Bu $w(z)$ funksiya φ funksiyaning D sohaga umumlashgan analitik davomi bo'ladi. Teorema isbotlandi.

Natija: $\varphi \in C(M)$ bo'lsin. $w|_M = \varphi$ shartni qanoatlantiruvchi $w \in U_{p,2}(A,B,D) \cap C(D \cup M)$ funksiya mavjud bo'lishi uchun istalgan $\varepsilon(0 < \varepsilon < \varepsilon_0 < 1)$ uchun

$$\overline{\lim_{k\to\infty}} \sqrt[2k]{(a^{\varepsilon}_{2k})^2 + (a^{\varepsilon}_{2k+1})^2} \leq \frac{1}{1-\varepsilon}$$

tengsizlikning bajarilishi zarur.

3.4-§. Umumlashgan analitik funksiyalarni davom ettirish

D_ρ- $G_\rho = \left\{ \left| \arg z - \frac{\pi}{2} \right| < \frac{\pi}{2\rho} \right\}$ burchak va G_ρ ichida yotuvchi S egri chiziqdan iborat bo'lakli-silliq chegarali $z = x + iy$ kompleks tekisligidagi chegaralangan bir bog'lamli soha bo'lsin. $C_\alpha(E)$- E tekislikda Gyol'der shartini qanoatlantiruvchi funksiyalar to'plami, $L_{p,2}$- orqali

$$f(z) \in L_P(E_1), \quad |z|^{-2} f(\frac{1}{z}) \in L_P(E_1), \quad E_1 = \{z : |z| < 1\},$$

shartlarni qanoatlantiruvchi $f(z)$ funksiyalar to'plamni, $U_{P,2}(A, B, D_\rho)$ - orqali D_ρ sohada

$$\frac{\partial W}{\partial \overline{z}} + A(z)W + B(z)\overline{W} = 0, \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}), \qquad (3.4.1)$$

tenglamaning yechimlar to'plamni belgilaymiz, bu yerda $A, B \in L_{P,2}(E) \cap C_\alpha(E)$, $P > 2$. $A(z) \equiv 0$, $B(z) \equiv 0$ da $U_{P,2}(A, B, D_\rho)$ to'plam D_ρ- sohada analitik funksiyalar to'plami $U(D_\rho)$ bilan ustma –ust tushadi.

Masalaning qo'yilishi. $\varphi(z) \in C(S)$ funksiyaga qanday shartlar qo'yilganda shunday $W(z) \in U_{P,2}(A, B, D_\rho) \cap C(D_\rho \cup S)$ funksiya mavjud bo'ladiki, uning S dagi chegaraviy qiymatlari $\varphi(z)$ bilan ustma-ust tushadi, yani $W(z) = \varphi(z)$, $z \in S$ bo'ladi. Bu masalani yechish uchun dastlab (3.4.1) tenglama uchun Karleman formulasini quramiz. D_ρ sohaga nisbatan S egri chiziqning kompleks o'zgaruvchili analitik funksiyasi uchun Karleman funksiyasini qaraymiz:

$$\Phi_\sigma(z, \zeta) = (\zeta - z)^{-1} \exp\left[\sigma\left((-i\zeta)^\rho - (-iz)^\rho\right)\right],$$

bu yerda σ- haqiqiy musbat sonli parametr, $\zeta = \xi + i\eta$, $(-iz)^\rho$ sifatida D_ρ sohada bu funksiyaning $z = i$ da birga teng bir qiymatli tarmog'ini tushunamiz.

I.N.Vekua [4] ning kitobida(176-bet) quyidagi moslik teoremasi isbotlangan.

Teorema-1. $\Phi(z)$ D sohada kompleks o'zgaruvchili analitik funksiya, ζ -D sohaning tayinlangan nuqtasi bo'lsin. U holda $z \in D$ nuqtaning quyidagi shartlarni qanoatlantiruvchi $W(z)$ funksiyasi mavjud: $W_0(z)=W(z)\Phi(z)$ funksiya D sohada uzluksiz va butun tekislikda uzluksiz davom ettiriladi, hamda

$$1)\ W_0\in C_{\frac{P-2}{P}}(E),\quad 2)\ W_0(z)\neq 0,\quad 3)\ W_0(\infty)=1$$

Bundan tashqari $\Phi(z)$ ning regulyarlik nuqtalarida W funksiya (3.4.1) tenglamani va chiziqli bo'lmagan

$$W(z)=\Phi(z)\exp\left(\frac{z-\zeta}{\pi}\iint_D \frac{A(t)+B(t)\frac{\overline{W(t)}}{W(t)}}{(t-\zeta)(t-z)}dS_t\right)$$

integral tenglamani qanoatlantiradi.

Bu teoremada D soha sifatida butun E tekislikni olamiz. U holda $\frac{1}{2}\Phi_\sigma$ va $\frac{1}{2i}\Phi_\sigma$ funksiyalarga $z=\zeta$ qutblarga ega bo'lgan va bu nuqtadan tashqari butun E tekislikda z o'zgaruvchi bo'yicha (3.4.1) tenglamaning yechimlari $X_j^\sigma(z,\zeta)$ funksiya- larga mos keladi, bunda $j=1,2$. quyidagi funksiyalarni qaraymiz:

$$\Omega_1^\sigma(z,\zeta)=X_1^\sigma(z,\zeta)+iX_2^\sigma(z,\zeta),\qquad -\Omega_1^\sigma(z,\zeta)=X_1^\sigma(z,\zeta)-iX_2^\sigma(z,\zeta)$$

$\sigma=0$ da $\Omega_1(z,\zeta)=\Omega_1^0(z,\zeta)$, $\Omega_2(z,\zeta)=\Omega_2^0(z,\zeta)$ funksiyalar $U_{P,2}(A,B,E)$ sinfning asosiy yadrolari deb ataladi.

Teorema-2. $W\in U_{P,2}(A,B,D_P)\cap C(\bar{D}_P)$ va $W(z)=\varphi(z)$, $z\in S$ bo'lsin. U holda davom ettirishning quyidagi teng kuchli formulalari o'rinli

$$W(z)=\lim_{\sigma\to\infty}\frac{1}{2\pi i}\int_S \Omega_1^\sigma(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2^\sigma(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta},\quad z\in D_\rho, \tag{3.4.2}$$

$$W(z)=\frac{1}{2\pi i}\int_S \Omega_1(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta}+\int_0^\infty J(z,\sigma)d\sigma,\quad z\in D_\rho, \tag{3.4.3}$$

bu yerda $J(z,\sigma)=\frac{1}{2\pi i}\int_S \gamma_1^\sigma(z,\zeta)\varphi(\zeta)d\zeta-\gamma_2^\sigma(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta},\quad \gamma_j^\sigma(z,\zeta)=\frac{\partial}{\partial\sigma}\Omega_j^\sigma(z,\zeta)$.

Isbot. Dastlab (3.4.2) va (3.3.4) formulalar teng kuchliligini isbotlaymiz. (3.4.3) formuladan

$$W(z)=\frac{1}{2\pi i}\int_S \left[\Omega_1(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta}\right]+\int_0^\infty J(z,\sigma)d\sigma=$$

$$=\frac{1}{2\pi i}\int_S \left[\Omega_1(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta}\right]+\frac{1}{2\pi i}\int_0^\infty\left[\int_S \left[\gamma_1^\sigma(z,\zeta)\varphi(\zeta)d\zeta-\gamma_2^\sigma(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta}\right]\right]d\sigma=$$

$$=\frac{1}{2\pi i}\int_S \left[\Omega_1(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta}\right]+\frac{1}{2\pi i}\int_0^\infty\frac{\partial}{\partial\sigma}\left[\int_S \left[\Omega_1^\sigma(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2^\sigma(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta}\right]\right]dS$$

$$=\frac{1}{2\pi i}\int_S \left[\Omega_1(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta}\right]+\frac{1}{2\pi i}\int_S \left[\Omega_1^\sigma(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2^\sigma(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta}\right]\Bigg|_{\sigma=0}^{\sigma=\infty}=$$

$$=\lim_{\sigma\to\infty}\frac{1}{2\pi i}\int_S \left[\Omega_1^\sigma(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2^\sigma(z,\zeta)\overline{\varphi(\zeta)}d\bar{\zeta}\right].$$

ga ega bo'lamiz. Shunday qilib, biz (3.4.2) formulaga kelamiz. (3.4.2) va (3.4.3) formulalar teng kuchliligi isbotlandi.

Endi (3.4.2) formulani isbotlaymiz. Buning uchun Koshi formulasining o'rinliligini ko'rsatamiz.

$$W(z)=\frac{1}{2\pi i}\int_{\partial D_\rho}\left[\Omega_1^\sigma(z,\zeta)W(\zeta)d\zeta-\Omega_2^\sigma(z,\zeta)\overline{W(\zeta)}d\bar{\zeta}\right],\qquad z\in D_\rho. \tag{3.4.4}$$

1-teoremaga ko'ra $\frac{1}{2}\Phi_\sigma(z,\zeta)$ va $\frac{1}{2i}\Phi_\sigma(z,\zeta)$ funksiyalarga $\zeta\neq z$ da (3.4.1) ga qo'shma bo'lgan tenglamaning ζ o'zgaruvchi bo'yicha yechimlari $X_1'^\sigma(\zeta,z)$ va $X_2'^\sigma(\zeta,z)$ funksiyalar mos keladi.

$$\frac{\partial}{\partial\zeta}X_j'^\sigma(\zeta,z)-A(\zeta)X_j'^\sigma(\zeta,z)-\overline{B}(\zeta)\overline{X_j'^\sigma(\zeta,z)}=0$$

Agar $W(z)\in U_{P,2}(A,B,D_\rho)$, $W'(z)\in U_{P,2}(-A,-\overline{B},D_\rho)$ bo'lsa, u holda Grin ayniyati o'rinli:

$$\operatorname{Re}\left(\frac{1}{2i}\int_{\partial D_\rho}W(\zeta)W'(\zeta)d\zeta\right)=0. \tag{3.4.5}$$

Γ_ε- markazi $z\in D_\rho$ nuqtada bo'lgan $|\zeta-z|=\varepsilon$ aylana bo'lsin, bu yerda ε- yetarli kichik musbat son. W' sifatida $X_1'^\sigma(\zeta,z)$ va $X_2'^\sigma(\zeta,z)$ funksiyalarni olib, ∂D_ρ va Γ_ε bilan chegaralangan sohaga (3.4.5) formulani qo'llab

$$\int\limits_{\partial D_\rho} W(\zeta)X_k^{/\sigma}(\zeta,z)d\zeta-\overline{W(\zeta)}\overline{X_k^{/\sigma}(\zeta,z)}d\overline{\zeta}=$$

$$=\int\limits_{\Gamma_\varepsilon} W(\zeta)X_k^{/\sigma}(\zeta,z)d\zeta-\overline{W(\zeta)}\overline{X_k^{/\sigma}(\zeta,z)}d\overline{\zeta},\qquad (k=1,2.)$$

ga ega bo'lamiz. Bu tengliklardan ikkinchisini $k=2i$ ga ko'paytirib va birinchisi bilan qo'shib

$$\int\limits_{\partial D_\rho} W(\zeta)\Omega_1^{/\sigma}(\zeta,z)d\zeta-\overline{W(\zeta)}\overline{\Omega_2^{/\partial}(\zeta,z)}d\overline{\zeta}=\int\limits_{\Gamma_\varepsilon} W(\zeta)\Omega_1^{/\sigma}(\zeta,z)d\zeta-\overline{W(\zeta)}\overline{\Omega_2^{/\sigma}(\zeta,z)}d\overline{\zeta},\qquad(3.4.6)$$

ni olamiz, bu yerda

$$\Omega_1^{/\sigma}(\zeta,z)=X_1^{/\sigma}(\zeta,z)+iX_2^{/\sigma}(\zeta,z),\quad \Omega_2^{/\sigma}(\zeta,z)=X_1^{/\sigma}(\zeta,z)-iX_2^{/\sigma}(\zeta,z).$$

Quyidagi tengliklar o'rinli:

$$\Omega_1^{\prime\sigma}(\zeta,z)-\frac{1}{\zeta-z}=O\left(|\zeta-z|^{-\frac{2}{p}}\right),\quad \Omega_2^{\prime\sigma}(\zeta,z)=O\left(|\zeta-z|^{-\frac{2}{p}}\right),$$

(3.4.6) va oxirgi tengliklardan $\varepsilon\to 0$ da quyidagini olamiz

$$\int\limits_{\partial D_\rho} W(\zeta)\Omega_1^{/\sigma}(\zeta,z)d\zeta-\overline{W(\zeta)}\overline{\Omega_2^{/\sigma}(\zeta,z)}d\overline{\zeta}=2\pi i W(z),\ z\in D_\rho.\qquad(3.4.7)$$

ni olamiz.

$$\Omega_1^{\sigma}(z,\zeta)=-\Omega_1^{/\sigma}(\zeta,z),\quad \Omega_2^{\sigma}(z,\zeta)=-\overline{\Omega_1^{/}}(\zeta,z)$$

tengliklarga ko'ra (3.4.7) formuladan (3.4.4) ni olamiz. (3.4.4) formulani

$$W(z)=\frac{1}{2\pi i}\int\limits_S \Omega_1^{\sigma}(z,\zeta)\varphi(\zeta)d\zeta-\Omega_2^{\sigma}(z,\zeta)\overline{\varphi(\zeta)}d\overline{\zeta}+$$
$$\frac{1}{2\pi i}\int\limits_{\partial D_\rho\setminus S}\Omega_1^{\sigma}(z,\zeta)W(\zeta)d\zeta-\Omega_2^{\sigma}(z,\zeta)\overline{W(\zeta)}d\overline{\zeta},\quad z\in D_\rho\qquad(3.4.4')$$

ko'rinishida yozish mumkin. U holda (3.4.2) formulani isbotlash uchun (3.4.4') tenglikning o'ng qismidagi ikkinchi integral $\sigma\to\infty$ da nolga intilishini ko'rsatishimiz yetarli. Quyidagi tengsizlikga

$$\left|\frac{1}{2\pi i}\int\limits_{\partial D_\rho\setminus S}\Omega_1^{\sigma}(z,\zeta)W(\zeta)d\zeta-\Omega_2^{\sigma}(z,\zeta)\overline{W(\zeta)}d\overline{\zeta}\right|\le\frac{1}{2\pi}\int\limits_{\partial D_\rho\setminus S}\left(|\Omega_1^{\sigma}|+|\Omega_2^{\sigma}|\right)|W(\zeta)||d\zeta|\le$$
$$\le\frac{1}{\pi}\int\limits_{\partial D_\rho\setminus S}\left(|X_1^{\sigma}(z,\zeta)|+|X_2^{\sigma}(z,\zeta)|\right)|W(\zeta)||d\zeta|\qquad(3.4.8)$$

ga ega bo'lamiz. Oxirgi integralni baholash uchun $X_k^{\sigma}(z,\zeta)\ (k=1,2)$ funksiyalar uchun quyidagi ifodalardan foydalanamiz:

$$X_1^{\sigma}(z,\zeta)=\frac{1}{2}\Phi_{\sigma}(z,\zeta)\exp\left(\frac{z-\zeta}{\pi}\iint_{E}\frac{A(t)+B(t)\dfrac{\overline{X_2^{\sigma}(t,\zeta)}}{X_2^{\sigma}(t,\zeta)}}{(t-\zeta)(t-z)}dS_t\right),$$

$$X_2^{\sigma}(z,\zeta)=\frac{1}{2i}\Phi_{\sigma}(z,\zeta)\exp\left(\frac{z-\zeta}{\pi}\iint_{E}\frac{A(t)+B(t)\dfrac{\overline{X_2^{\sigma}(t,\zeta)}}{X_2^{\sigma}(t,\zeta)}}{(t-\zeta)(t-z)}dS_t\right).$$

Bu ifodalarning o'rinliligi 1- teoremadan kelib chiqadi. (3.4.8) tengsizliklardan foydalanib

$$\left|X_j^{\sigma}(z,\zeta)\right|_{\zeta\in\partial D_{\rho}\setminus S}\le\frac{1}{2}\left|e^{\sigma\left((-i\zeta)^{\rho}-(-iz)^{\rho}\right)}\right|_{\left|\arg\zeta-\frac{\pi}{2}\right|=\frac{\pi}{2\rho}}\times\left|\exp\left(\frac{z-\zeta}{\pi}\iint_{E}\frac{A(t)+B(t)\dfrac{\overline{X_j^{\sigma}(t,\zeta)}}{X_j^{\sigma}(t,\zeta)}}{(t-\zeta)(t-z)}dS_t\right)\right|\le \tag{3.4.9}$$

$$\le\frac{1}{2|x-z|}e^{-|z|^{\rho}Cos\left(\rho\varphi-\frac{\pi\rho}{2}\right)}\times e^{M_P L_{P,2}(|A|+|B|)},$$

ga ega bo'lamiz, bu yerda M_P o'zgarmas faqat p ga bog'liq, $L_{P,2}(f)=L_P(f,E_1)+L_p\left(|z|^{-2}f\left(\frac{1}{z}\right),E_1\right)$

(3.4.8) va (3.4.9) tengsizliklardan $z\in D$ da

$$\left|\frac{1}{2\pi i}\int_{\partial D_{\rho}\setminus S}\Omega_1^{\sigma}(z,\zeta)W(\zeta)d\zeta-\Omega_2^{\sigma}(z,\zeta)\overline{W(\zeta)}d\overline{\zeta}\right|\le\frac{1}{\pi}Me^{M_P L_{P,2}(|A|+|B|)}e^{-\sigma|z|^{\rho}Cos\left(\rho\varphi-\frac{\pi\rho}{2}\right)}\int_{\partial D_{\rho}\setminus S}\frac{dx}{|x-z|}, \tag{3.4.10}$$

ga ega bo'lamiz, bu yerda $M=\max_{\overline{D}_{\rho}}|W(z)|$, (3.4.4') tenglikda $\sigma\to\infty$ da limitga o'tib (3.4.10) tengsizlikga ko'ra (3.4.2) formulani olamiz. Teorema isbotlandi.

Teorema-3. $\varphi(z)\in L(S)\cap C\left(\overset{0}{S}\right)$ bo'lsin. U holda $W(z)=\varphi(z),\ z\in S$ shartni qanoatlantiruvchi $W\in U_{P,2}(A,B,D_{\rho})\cap C\left(D_{\rho}\cup\overset{0}{S}\right)$ funksiya mavjud bo'lishi uchun

$$\int_0^\infty \left|J(z,\sigma)d\sigma\right| < \infty$$

integralning har bir $K \subset G_\rho$ kompaktda tekis yaqinlashuvchi bo'lishi zarur va yetarli. Agar bu shart bajarilsa, u holda davom ettirish (3.4.2) va (3.4.3) teng kuchli formulalar bilan amalga oshiriladi.

Isbot. (Zarurligi) $W(z) = \varphi(z),\ z \in S$ shartni qanoatlantiruvchi $W \in U_{P,2}(A,B,D_\rho) \cap C\left(D_\rho \cup \overset{0}{S}\right)$ funksiya mavjud bo'lsin. Δ_ρ^τ orqali $G_\rho$dan ordinata o'qi bo'ylab yuqoriga $\tau > 0$ siljish bilan hosil qilinadigan sohani belgilaymiz, $S_\tau = S \cap \Delta_\rho^\tau$, $D_\rho^\tau = D_\rho \cap \Delta_\rho^\tau$. soha chegarasi S_τ egri chiziqdan va $\left|\arg z - \frac{\pi}{2}\right| = \frac{\pi}{2\rho}$ nurlarga parallel $P_\tau : \left|\arg(z - i\tau) - \frac{\pi}{2}\right| = \frac{\pi}{2\rho}$ nurlarning kesmalaridan iborat. $S_{1,\tau}$ va $S_{2,\tau}$ lar S egri chiziqning S_τ ga kirmagan qismlari.

$X_1^\sigma(z,\zeta)$ va $X_2^\sigma(z,\zeta)$ funksiyalar

$$\begin{aligned} X_1^\sigma(z,\zeta) - \frac{1}{\pi}\iint_E \frac{A(t)X_1^\sigma(t,\zeta) + B(t)\overline{X_1^\sigma(t,\zeta)}}{t-z}dS_t &= \frac{1}{2}\Phi_\sigma(z,\zeta), \\ X_2^\sigma(z,\zeta) - \frac{1}{\pi}\iint_E \frac{A(t)X_2^\sigma(t,\zeta) + B(t)\overline{X_2^\sigma(t,\zeta)}}{t-z}dS_t &= \frac{1}{2i}\Phi_\sigma(z,\zeta). \end{aligned} \qquad (3.4.11)$$

integral tenglamalarning yechimlari. (3.4.11) tenglamalarda asosan

$$\frac{1}{2}\tilde{\Phi}_\sigma(z,\zeta) = \frac{1}{2}\frac{\partial}{\partial\sigma}\Phi_\sigma(z,\zeta) = \frac{(-i\zeta)^\rho - (-iz)^\rho}{2(\zeta - z)}\exp\left[\sigma\left((-i\zeta)^\rho - (-iz)^\rho\right)\right],$$

$$\frac{1}{2i}\tilde{\Phi}_\sigma(z,\zeta) = \frac{1}{2i}\frac{\partial}{\partial\sigma}\Phi_\sigma(z,\zeta) = \frac{(-i\zeta)^\rho - (-iz)^\rho}{2i(\zeta - z)}\exp\left[\sigma\left((-i\zeta)^\rho - (-iz)^\rho\right)\right]$$

analitik funksiyalarga $\tilde{X}_1^\sigma(z,\zeta) = \frac{\partial}{\partial\sigma}X_1^\sigma(z,\zeta)$ va $\tilde{X}_2^\sigma(z,\zeta) = \frac{\partial}{\partial\sigma}X_2^\sigma(z,\zeta)$ funksiyalar mos keladi. Bu funksiyalar butun E tekislikda z o'zgaruvchi bo'yicha (3.4.1) tenglamani va quyidagi chiziqli bo'lmagan integral tenglamalarni qanoatlantiradi:

$$\tilde{X}_1^\sigma(z,\zeta) = \frac{1}{2}\tilde{\Phi}_\sigma(z,\zeta)e^{\tilde{\omega}_1^\sigma(z,\zeta)}, \quad \tilde{X}_2^\sigma(z,\zeta) = \frac{1}{2i}\tilde{\Phi}_\sigma(z,\zeta)e^{\tilde{\omega}_2^\sigma(z,\zeta)}, \qquad (3.4.12)$$

bu yerda

$$\tilde{\omega}_j(t,\zeta)=\frac{z-\zeta}{\pi}\iint_E \frac{A(t)+B(t)\dfrac{\overline{\tilde{X}_j^{\sigma}(t,\zeta)}}{\tilde{X}_j^{\sigma}(t,\zeta)}}{(t-\zeta)(t-z)}dS_t, \qquad j=1,2.$$

$\tilde{X}_1^{\sigma}(z,\zeta)$ funksiyalar E tekislikda z bo'yicha (3.4.1) tenglamani qanoatlantirgani uchun $\tilde{\Omega}_1^{\sigma}(z,\zeta)=\tilde{X}_1^{\sigma}(z,\zeta)+i\,\tilde{X}_2^{\sigma}(z,\zeta)$, $\tilde{\Omega}_2^{\sigma}(z,\xi)=\tilde{X}_1^{\sigma}(z,\zeta)-\tilde{X}_2^{\sigma}(z,\zeta)$ funksiyalar

$$\begin{aligned}&\frac{\partial}{\partial \bar z}\tilde{\Omega}_1^{\sigma}(z,\zeta)+A(z)\tilde{\Omega}_1^{\sigma}(z,\zeta)+B(z)\overline{\tilde{\Omega}_2^{\sigma}(z,\zeta)}=0,\\ &\frac{\partial}{\partial \bar z}\tilde{\Omega}_2^{\sigma}(z,\zeta)+A(z)\tilde{\Omega}_2^{\sigma}(z,\zeta)+B(z)\overline{\tilde{\Omega}_1^{\sigma}(z,\zeta)}=0\end{aligned} \tag{3.4.13}$$

tenglamalar sistemasini qanoatlantiradi.1-teoremaga ko'ra $\frac{1}{2}\tilde{\Phi}_{\sigma}(z,\zeta)$ va $\frac{1}{2i}\tilde{\Phi}_{\sigma}(z,\zeta)$ funksiyalarga $\tilde{X}_1^{/\sigma}(\zeta,z)$, $\tilde{X}_2^{/\sigma}(\zeta,z)\in U_{P,2}(-A,-\overline{B},E)$ funksiyalar mos keladi. $\tilde{X}_1^{/\sigma}(\zeta,z)$ $\tilde{X}_2^{/\sigma}(\zeta,z)$ funksiyalar ζ o'zgaruvchi bo'yicha (3.4.1) ga qo'shma tenglamani

$$\frac{\partial}{\partial\bar\zeta}\tilde{X}_k^{/\sigma}(\zeta,z)+A(\zeta)\tilde{X}_k^{/\sigma}(\zeta,z)-\overline{B}(\zeta)\overline{\tilde{X}_k^{/\sigma}}(\zeta,z)=0,\quad \zeta\neq z,\ \zeta\in E,\ k=1,2.$$

qanoatlantiradi. $W^{/}$ sifatida $\tilde{X}_1^{/\sigma}(\zeta,z)$, $\tilde{X}_2^{/\sigma}(\zeta,z)$ funksiyalarni olib (3.4.5) formulani D_{ρ}^{τ} sohaga qo'llaymiz:

$$\int_{\partial D_{\rho}} w(\zeta)\tilde{X}_k^{\prime\sigma}(\zeta,z)d\zeta-\overline{w}(\zeta)\overline{\tilde{X}_k^{\prime\sigma}(\zeta,z)}d\bar\zeta=0,\quad k=1,2,\quad z\in G_{\rho}$$

Bu tengliklardan $k=2$ ikkinchisini i ga ko'paytirib va birinchisi bilan qo'shib

$$\int_{\partial D_{\rho}} W(\zeta)\tilde{\Omega}_1^{/\sigma}(\zeta,z)d\zeta-\overline{W}(\zeta)\overline{\tilde{\Omega}_2^{/\sigma}(\zeta,z)}d\bar\zeta=0, \tag{3.4.14}$$

ni olamiz, bu yerda

$$\tilde{\Omega}_1^{\prime\sigma}(\zeta,z)=\tilde{X}_1^{\prime\sigma}(\zeta,z)+i\tilde{X}_2^{\prime\sigma}(\zeta,z),\quad \tilde{\Omega}_2^{\prime\sigma}(\zeta,z)=\tilde{X}_1^{\prime\sigma}(\zeta,z)-i\tilde{X}_2^{\prime\sigma}(\zeta,z).$$

[1] dagi

$$\tilde{\Omega}_1^{\sigma}=-\tilde{\Omega}_1^{\prime\sigma}(\zeta,z)\quad \tilde{\Omega}_2^{\sigma}=-\overline{\tilde{\Omega}}{}_2^{/\sigma}(\zeta,z)$$

tengliklarga ko'ra (3.4.14) tenglikni

$$\int\limits_{S+\left|\arg z-\frac{1}{2}\right|=\frac{\pi}{2\rho}} W(\zeta)\,\tilde{\Omega}_1^{/\sigma}(z,\zeta)d\zeta-\overline{W}(\zeta)\tilde{\Omega}_2^{/\sigma}(z,\zeta)d\overline{\zeta}=0,\quad z\in G_\rho .$$

ko'rinishda yozib olamiz. Bundan

$$\int\limits_S \tilde{\Omega}_1^{\sigma}(z,\zeta)\varphi(\zeta)d\zeta-\tilde{\Omega}_2^{\sigma}(z,\zeta)\overline{\varphi}(\zeta)d\overline{\zeta}=\int\limits_{\left|\arg z-\frac{\pi}{2}\right|=\frac{\pi}{2\rho}} \tilde{\Omega}_1^{\sigma}(z,\zeta)W(\zeta)d\zeta-\tilde{\Omega}_2^{\sigma}(z,\zeta)\overline{W}(\zeta)d\overline{\zeta}. \tag{3.4.15}$$

(3.4.15) tenglik o'ng qismida turuvchi integralni baholaymiz. (3.4.12) formuladan va (3.4.8) tengsizlikdan foydalanib

$$\left|\int\limits_{\left|\arg\zeta-\frac{1}{2}\right|=\frac{\pi}{2\rho}} \tilde{\Omega}_1^{\sigma}(z,\zeta)W(\zeta)d\zeta-\tilde{\Omega}_2^{\sigma}(z,\zeta)\overline{W}(\zeta)d\overline{\zeta}\right|\le \int\limits_{\left|\arg z-\frac{\pi}{2}\right|=\frac{\pi}{2\rho}}\left(\left|\tilde{\Omega}_1^{\sigma}(z,\zeta)\right|+\left|\tilde{\Omega}_2^{\sigma}(z,\zeta)\right|\right)\left|W(\zeta)\right|\left|d\zeta\right|\le$$

$$2e^{M_P L_{P.2}(|A|+|B|)}\int\limits_{\left|\arg\zeta-\frac{\pi}{2}\right|=\frac{\pi}{2\rho}}\left|W(\zeta)\right|\left|\frac{(-i\zeta)^{\rho}-(z)^{\rho}}{\zeta-z}\right|\left|\exp\left[\sigma\left((-i\zeta)^{\rho}-(-iz)^{\rho}\right)\right]\right|\left|d\zeta\right|\le$$

$$\le 2MC_\rho e^{M_P L_{P.2}(|A|+|B|)}\int\limits_{\left|\arg\zeta-\frac{\pi}{2}\right|=\frac{\pi}{2\rho}}\exp\left[\sigma\left(|\zeta|^{\rho}\cos\left(\rho\psi-\frac{\pi\rho}{2}\right)-|z|^{\rho}\cos\left(\rho\varphi-\frac{\pi\rho}{2}\right)\right)\right]\left|d\zeta\right|\le$$

$$\le 4dMC_\rho e^{M_P L_{P,2}(|A|+|B|)}e^{-\sigma|z|^{\rho}\cos\left(\rho\varphi-\frac{\pi\rho}{2}\right)},$$

ga ega bo'lamiz, bu yerda

$$d=\max_{S}|\zeta|,\quad C_\rho=\max_{\zeta\in\overline{D}_\rho}\left|\frac{(-i\zeta)^{\rho}-(z)^{\rho}}{\zeta-z}\right|,\quad M=\max_{\overline{D}_\rho}\left|W(\zeta)\right|.$$

Shunday qilib

$$\left|\int\limits_{\partial D_\rho/S}\tilde{\Omega}_1^{\sigma}(z,\zeta)W(\zeta)d\zeta-\tilde{\Omega}_2^{\sigma}(z,\zeta)\overline{W}(\zeta)d\overline{\zeta}\right|\le$$
$$\le 4dMC_\rho e^{M_P L_{P,2}(|A|+|B|)}e^{-\sigma|z|^{\rho}\cos(\rho\varphi-\frac{\pi\rho}{2})}. \tag{3.4.16}$$

(3.4.15) va (3.4.16) dan

$$\left|\int\limits_S\tilde{\Omega}_1^{\sigma}(z,\zeta)\varphi(\zeta)d\zeta-\tilde{\Omega}_2^{\sigma}(z,\zeta)\overline{\varphi}(\zeta)d\overline{\zeta}\right|\le$$
$$\le 4dC_\rho Me^{M_P L_{P,2}(|A|+|B|)}e^{-\sigma|z|^{\rho}\cos\left(\rho\varphi-\frac{\pi\rho}{2}\right)}. \tag{3.4.17}$$

ni olamiz. $z\in D_\rho$ da

$$-\frac{\pi}{2} < \rho\varphi - \frac{\pi\rho}{2} < \frac{\pi}{2}, \ (\arg z = \varphi) \text{ va } \cos\left(\rho\varphi - \frac{\pi\rho}{2}\right) > 0$$

bo'lgani uchun (3.4.17) tengsizlikdan

$$\int_0^\infty J(z,\sigma)d\sigma \qquad (3.4.18)$$

integralning har bir $K \subset G_\rho$ kompaktda tekis yaqinlashuvchiligi kelib chiqadi.

Yetarliligi. $\varphi(z)$ funksiya teorema shartini qanoatlantirsin. $W(z) = \varphi(z), z \in \overset{0}{S}$ shartni qanoatlantiradigan $W \in U_{P,2}(A, B, D_\rho) \cap C\left(D_\rho \cup \overset{0}{S}\right)$ funksiya mavjudligini ko'rsatamiz. (3.4.2) va (3.4.3) teng kuchli formulalar bilan berilgan $W(z)$ funksiyani qaraymiz. (3.4.3) formuladagi birinchi qo'shiluvchi Koshi tipidagi umumlashgan integral hisoblanadi. Bu integral mos ravishda D_ρ va $G_\rho \setminus \overline{D}_\rho$ sohalarda (3.4.1) tenglamani qanoatlantiruvchi shunday ikkita funksiyani beradiki, ularning normallar (yoki chekli burchaklar bo'yicha, mos $z^+ \in D_\rho$ va $z^- \in G_\rho \setminus \overline{D}_\rho$ nuqtalar ζ dan teng masofalarda yotib $\zeta \in \overset{0}{S}$ nuqtaga intilganda) bo'yicha limitik qiymatlari $\overset{0}{S}$ da $\varphi(\zeta)$ ga teng , bunda agar bu funksiyalardan biri mos sohada $\overset{0}{S}$ gacha uzluksiz bo'lsa, u holda ikkinchisi ham bu xossaga ega bo'ladi. (3.4.3) formuladagi ikkinchi qo'shiluvchi G_ρ sohada (3.4.1) tenglamani qanoatlantirishini ko'rsatamiz. (3.4.18) integralning tekis yaqinlashuvchiligiga ko'ra

$$\int_0^\infty \left[\frac{\partial J(z,\sigma)}{\partial \bar z} + A(z) J(z,\sigma) + B(z) \overline{J(z,\sigma)} \right] d\sigma =$$

$$= \int_0^\infty \left[\frac{1}{2\pi i} \int_S \left[\frac{\partial \tilde\Omega_1^\sigma(z,\zeta)}{\partial \bar z} \varphi(\zeta) d\zeta + A(z) \tilde\Omega_1^\sigma(z,\zeta) \varphi(\zeta) d\zeta - B(z) \overline{\tilde\Omega_1^\sigma(z,\zeta)}\, \overline{\varphi(\zeta)}\, d\bar\zeta - \right.\right.$$

$$\left.\left. - \frac{\partial \tilde\Omega_2^\sigma(z,\zeta)}{\partial \bar z} \overline{\varphi(\zeta)} d\bar\zeta - A(z) \tilde\Omega_2^\sigma(z,\zeta) \overline{\varphi(\zeta)} d\bar\zeta + B(z) \overline{\tilde\Omega_2^\sigma(z,\zeta)}\, \varphi(\zeta)\, d\zeta \right] \right] d\sigma =$$

$$= \frac{1}{2\pi i} \int_0^\infty \left\{ \int_S \left[\left(\frac{\partial \tilde\Omega_1^\sigma(z,\zeta)}{\partial \bar z} + A(z) \tilde\Omega_1^\sigma(z,\zeta) + B(z) \overline{\tilde\Omega_2^\sigma(z,\zeta)} \right) \varphi(\zeta) d\zeta - \right.\right.$$

$$\left.\left. - \left(\frac{\partial \tilde\Omega_2^\sigma(z,\zeta)}{\partial \bar z} + A(z) \tilde\Omega_2^\sigma(z,\zeta) + B(z) \overline{\tilde\Omega_1^\sigma(z,\zeta)} \right) \cdot \overline{\varphi(\zeta)}\, d\bar\zeta \right] \right\} d\sigma, \quad z \in G_\rho$$

ga ega bo'lamiz.

Bundan (3.4.13) tengliklarga ko'ra (3.4.3) formuladagi ikkinchi qo'shiluvchi G_ρ sohada (3.4.1) tenglamani qanoatlantiradi. Shunday qilib (3.4.3) formulaning o'ng qismi shunday ikkita $W_1 \in U_{P,2}(A,B,D_\rho)$ va $W_2 \in U_{P,2}(A,B,G_\rho \setminus \overline{D}_\rho)$ funksiyalarni beradiki, ixtiyoriy $\zeta \in \overset{0}{S}$ nuqta uchun (ko'rsatilgan ma'noda)

$$W_1(\zeta) - W_2(\zeta) = \varphi(\zeta) \tag{3.4.19}$$

tenglik o'rinli, bunda agar funksiyalardan biri $\overset{0}{S}$ gacha mos sohada uzluksiz bo'lsa, ikkinchisi ham bu xossaga ega bo'ladi. (3.4.2) formuladan $|z| > \max_S |\zeta|, \quad \arg z = \frac{\pi}{2}$ da $W_2(z) = 0$ kelib chiqadi. Umumlashgan analitik funksiyalar uchun yagonalik teoremasiga ko'ra [1], $G_p \setminus \overline{D}_\rho$ sohada $W_2(z) = 0$. Ravshanki $W_2(z)$ funksiya $G_p \setminus \overline{D}_\rho \cup \overset{0}{S}$ da uzluksiz davom ettiriladi. U holda $W_1(z)$ ham $C(D_\rho \cup \overset{0}{S})$ sinfning funksiyasi sifatida uzluksiz davom ettiriuladi. (3.4.19) tenglikdan

$$W_1(\zeta) = \varphi(\zeta), \quad \zeta \in \overset{0}{S}$$

kelib chiqadi. Bundan teorema tasdig'i kelib chiqadi, $W(z) \in U_{P,2}(A,B,D_\rho) \cap C(D_\rho \cup \overset{0}{S})$ funksiya sifatida $W_1(z)$ funksiyani olish mumkin.3-teorema isbotlandi.

3-teoremadan $A(z) \equiv 0, B(z) \equiv 0$ va $\rho = 1$ bo'lganda quyidagi teorema kelib chiqadi.

Teorema. (Fok-Kuni). $\varphi(z) \in L(S) \cap C(\overset{0}{S})$ bo'lsin. S da $W(\zeta) = \varphi(\zeta),\ \zeta \in \overset{0}{S}$ shartni qanoatlantiruvchi $W(z) \in U(D_1) \cap C(D_1 \cup \overset{0}{S})$ funksiya mavjud bo'lishi uchun

$$\left| \int_0^\infty \left[\int_S \varphi(\zeta) \exp[-i\sigma(\zeta - z)] d\zeta \right] d\sigma \right| < \infty$$

integralning har bir $K \subset C\{\operatorname{Im} z > 0\}$ kompaktda tekis yaqinlashuvchi bo'lishi zarur va yetarli.

Foydalanilgan adabiyotlar

[1]. Айзенберг Л.А., Кытманов А.М. О возможности голоморфного продолжения в область функций, заданных на связном куске ее границы.- Кросноярск: Институт физики СО АН СССР, 1990. Т.44, №50. С. 6-8.

[2]. Бицадзе А.В. Уравнение математической физики, Москва, «Наука» 1982.

[3]. Бицадзе А.В., Д.Ф.Колиниченко. Сборник задач по уравнением математической физики.- Москва.: Наука,-1977.

[4]. Векуа И.Н Обобщенные аналитические функции. – Москва: Физматгиз, 1959. – 628 с.

[5]. Ишанкулов Т., Хатамов И. Аналитическое продолжение функции в область функции, заданной на дуге границы этой области. Международная конференция "Обратные и некорректные задачи математической физики" Новосибирск, Россия, 5-12 августа 2012 г.

[6]. Ишанкулов Т., Хатамов И. О возможности продолжения функции заданной на части границы в область как решение системы уравнений Коши-Римана. Научной конференции. Некорректные и неклассические задачи математической физики и анализа. СамГУ 5-6 июля 2012 г.

[7]. Ишанкулов Т. Аналог теоремы Фока-Куни для эллиптической системы первого порядка на плоскости. СамДУ, Ахборотномаси, 1999. - С. 18 - 20.

[8]. Лавреньтев М.М. О некоторых некорректных задачах математической физики – Новосибирск. Изд-во ВЦ СЩАН СССР 1962. 92 с

[9]. Маркушевич А.И. Теория аналитических функций. Том II.

[10].Maqsudov Sh., M. Salohiddinov, S. Sirojjiddinov. Kompleks o`zgaruvchilarning funksiyalari nazariyasi.-Toshkent.O`qituvchi,1976.

[11]. Романов В.Г. Устойчивость в обратных задачах. М.: Научный мир, 2005. с.305.

[12]. Carleman T. Sure un problem dunicite pour les Systems equations aux derves partitielles a beaux variables indepents //Arkiv for Mat.Astr.och.Fys. 1939. Bb 26 №17 s. 1-9.

[13]. Teshaboyeva N. Matematik fizika metodlari.-Toshkent. O`qituvchi. 1966.

[14]. Тихонов А.Н., Самарский А.А. Уравнения математической физики.-Москва.: Московского университета, 1999.-409 с.

[15]. Фок В.А. Куни Ф.М. О введении «гасяшей» функции в дисперсионные соотношения.// Докл. АН СССР. 1959. Т. 127.

[16]. Шубин М.А. Лекции об уравнениях математической физики.-Москва.: МЦНМО, 2003.-303 с.

[17]. Abdiev T.B. Umumlashgan analitik funksiyalarni davom ettirish. Magistrlarning XIII-ilmiy amaliy anjumani materiallari. Samarqand 2013.

[18]. Ishankulov T, Abdiev T. Chegaraning qismida berilgan funksiyani sohaga umumlashgan analitik funksiya sifatida davom ettirish masalasi. Samarqand TATU filiali 2013-yil.

[18]. www.sernam.ru

[19]. www.ziyonet.uz

[20]. www.xplusy.isnet.ru

[21]. www.dxdy.ru

[22]. www.karsu.uz