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Mavzu:Parabolik va giperbolik tipdagi tenglamalar uchun qo'yilgan chegaraviy masalalarni Fur'e usulida hamda Puasson formulasidan foydalanib yechish.

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Quyidagi tenglamalar uchun qo'yilgan Koshi masalasini Puasson formula-sidan foydalanib yeching.

1-misol

$$U_t = U_{xx} + 2t$$
$$U(x,0) = 1$$

Yechish: Puasson formulasidan foydalanib yechamiz:

Puasson formulasi: $U(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(\xi) 2e^{-\frac{|x-\xi|^2}{4a^2 t}} d\xi + \int_0^t \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi(t-\tau)}} \cdot e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} f(\xi,\tau) d\xi d\tau$

$a = 1;\ f(x,t) = 2t; \varphi(x) = 1$

$$U(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{|x-\xi|^2}{4t}} d\xi + \int_0^t \int_{-\infty}^{+\infty} \frac{2\tau e^{\frac{|x-\xi|^2}{4(t-\tau)}}}{2\sqrt{\pi(t-\tau)}} d\xi d\tau = \frac{1}{2\sqrt{\pi t}} 2\sqrt{t}\sqrt{\pi} + \int_0^t \frac{\tau}{\sqrt{\pi(t-\tau)}} \int_{-\infty}^{+\infty} e^{\frac{|x-\xi|^2}{4(t-\tau)}} d\xi d\tau =$$

$$= 1 + \int_0^t \frac{\tau}{\sqrt{\pi\ \ t-\tau}} 2\sqrt{t-\tau}\sqrt{\pi} d\tau = 1 + \int_0^t 2\tau d\tau = 1 + t^2$$

Javob: $U(x,t) = 1 + t^2$.

2-misol

$$U_t = U_{xx} + \sin t$$
$$U(x,t)|_{t=0} = 0$$
$$\varphi(x) = 0$$
$$f(x,t) = \sin t$$

Yechim:

$$U(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{\sin \tau e^{-\frac{|x-\xi|^2}{4a^2(t-\tau)}}}{2a\sqrt{\pi(t-\tau)}} d\xi d\tau = \int_0^t \frac{\sin \tau}{2a\sqrt{\pi(t-\tau)}} \int_{-\infty}^{\infty} e^{-\frac{|x-\xi|^2}{4a^2(t-\tau)}} d\xi d\tau$$

Bizga ma'lumki:

$$\int_{-\infty}^{\infty} e^{-\rho^2} d\rho = \sqrt{\pi}$$

$$\rho^2 = \frac{|x-\xi|^2}{4a^2(t-\tau)}$$

$$\rho = \frac{x-\xi}{2a\sqrt{t-\tau}}$$

$$d\rho = -\frac{d\xi}{2a\sqrt{t-\tau}}$$

$$\xi = -\infty \qquad \rho = \infty$$

$$\xi = \infty \qquad \rho = -\infty$$

$$\int_{-\infty}^{\infty} e^{-\rho^2} 2a\sqrt{t-\tau}\, d\rho = 2a\sqrt{t-\tau}\sqrt{\pi}$$

$$U(x,t) = \int_0^t \frac{\sin\tau}{2a\sqrt{t-\tau}} 2a\sqrt{t-\tau}\sqrt{\pi}\, d\tau = \int_0^t \sin\tau = -\cos t\,|_0^t =$$

$$= -(\cos t - 1) = 1 - \cos t$$

$$Javob:\ U(x,t) = 1 - \cos t$$

Quyidagi giperbolik tipdagi tenglamalar uchun qo'yilgan chegaraviy Masalalarni Fur'e usulida yeching:

1-misol.

$$U_{tt} = a^2 U_{xx} + f(x,t)$$
$$U(x,0) = 0; U_t(x,0) = 0$$
$$X'(x) + \lambda^2 X(x) = 0$$
$$X'(0) = 0 \ (1)$$
$$X'(l) = 0 \ (2)$$
$$X = C_1 \cos\lambda x + C_2 \sin\lambda x$$
$$X' = -C_1\lambda \sin\lambda x + C_2\lambda\cos\lambda x$$
(1)

$$\Rightarrow C_2 = 0$$

$$-C_1\lambda \sin\lambda l = 0, C_1 \neq 0 \Rightarrow \sin\lambda l = 0 \Rightarrow \lambda l = \pi n \Rightarrow \lambda_n = \frac{\pi n}{l} \Rightarrow X_n(x) = \cos\frac{\pi n}{l} x, n = 1,2,...$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x)$$

$$f_n(t) = \frac{1}{\|X_n(x)\|^2}\int_0^l f(x,t) X_n(x)dx; \|X_n(x)\|^2 = \int_0^l X_n^2(x)dx = \int_0^l \cos^2(\lambda_n x)dx = \frac{l}{2}$$

$$\Rightarrow f_n(t) = \frac{2}{l}\int_0^l f(x,t)\cos\lambda_n x dx$$

Yechimni $U(x,t) = \sum_{n=1}^{\infty} U_n(t) X_n(x)$ koinishda izlaymiz. $\lambda_n = 0$ hol uchun $U_0(x,t)$ ni alohida hisoblaymiz.

$$U_{tt} = n\sum_{n=1}^{\infty} U_n''(t) X_n(x); U_{xx} = \sum_{n=1}^{\infty} U_n(t) X_n''(x) \sum_{n=1}^{\infty} U_n''(t)$$

$$\sum_{n=1}^{\infty} X_n(x)\left[U_n''(t) + a^2\lambda_n^2 U_n(t)\right] = \sum_{n=1}^{\infty} f_n(t) X_n(x) \Rightarrow$$

$$U_n''(t) + a^2\lambda_n^2 U_n(t) = f_n(t)$$
$$U_n(0) = 0$$
$$U_n'(0) = 0$$

$2-misol$

$$u_{tt}=a^2u_{xx}+f\quad x,t$$
$$u(x,0)=0\;\; u_l(x,l)=0$$
$$u_x(0,t)=u(l,t)=0$$
$$0<x<l,\quad t>0$$
$$f(x,t)=Ae^{-t}\cos\frac{\pi}{2l}x$$
$$\begin{cases} X''(x)+\lambda^2X(x)=0\\ X'(0)=X(l)=0\end{cases}$$
$$X(x)=e^{kx}$$
$$k^2+\lambda^2=0$$
$$k=\pm\lambda i$$
$$X(x)=c_1\cos\lambda x+c_2\sin\lambda x$$
$$X'\;\;x\;\;=-c_1\lambda\sin\lambda x+c_2\lambda\cos\lambda x$$
$$X'(0)=c_2\lambda=0\;\Rightarrow c_2=0$$
$$X(l)=c_1\cos\lambda l=0,\quad c_1\neq 0\Rightarrow\;\cos\lambda l=0$$
$$\lambda_n=\frac{\pi+2\pi n}{2l}\quad n=0,1,...$$
$$X_n(x)=\cos\lambda_n x=\cos\frac{\pi+2\pi n}{2l}x$$
$$u(x,t)=\sum_{n=0}^{\infty}u_n(t)X_n(x)$$
$$\sum_{n=0}^{\infty}(u_n''(t)X_n(x)-a^2X_n''(x)u_n(t))=f(x,t)$$
$$X_n''(x)=\left(\frac{\pi+2\pi n}{2l}\right)^2\cos\frac{\pi+2\pi n}{2l}x$$
$$\sum_{n=0}^{\infty}(u_n''(t)-a^2\left(\frac{\pi+2\pi n}{2l}\right)^2u_n(t))\cos\left(\frac{\pi+2\pi n}{2l}x\right)^2=Ae^{-t}\cos\frac{\pi}{2l}x$$
$$\Rightarrow\left(u_n''(t)-a^2\left(\frac{\pi+2\pi n}{2l}\right)^2u_n(t)\right)=Ae^{-t}$$

.3-misol

Tenglamaning umumiy yechimini toping.

$$U_{xxxy} - U_{xyyy} = 0$$

Yechish:

$U_{xy} = V$ deb belgilasak, uholda $V_{xx} - V_{yy} = 0$ ko’rinishidagi tenglamaga kelamiz.

Endi giperbolik tipdagi tenglamaning umumiy yechimini topamiz. Buning uchun uning harakteristik tenglamasini tuzamiz:

$$(dy)^2 - (dx)^2 = 0$$

Bu tenglamani yechib: $\begin{matrix} \xi = y + x \\ \eta = y - x \end{matrix}$ deb belgilaymiz va $V(x, y) = V(\xi, \eta)$ almashtirish bajaramiz.

U holda $\begin{cases} V_{xx} = V_{\eta\eta} - 2V_{\xi\eta} + V_{\xi\xi} \\ V_{yy} = V_{\eta\eta} + 2V_{\xi\eta} + V_{\xi\xi} \end{cases}$ tengliklarga ega bo’lamiz . Bularni yuqoridagi tenglamaga olib borib qo’ysak quyidagicha tenglama hosil bo’ladi:

$$V_{\xi\eta} = 0$$

Bu tenglamani ikki marta ξ va η lar bo’yicha integrallab umumiy yechimni topamiz: $V(\xi, \eta) = f(\xi) + g(\eta) \quad \Rightarrow V(x, y) = f(x + y) + g(y - x)$.

Yuqoridagi belgilashga ko’ra $U_{xy} = f(x + y) + g(y - x)$ tenglamani olamiz. Bu tenglamani x va y bo’yicha integrallaymiz. Natijada tenglamaning umumiy yechimi quyidagicha bo’ladi:

$$U(x, y) = F(x + y) - G(y - x) + \varphi(x) + \psi(y) .$$

4-misol .Quyidagi Koshi masalasining umumiy yechimi topilsin.

$U_{yy} = 9U_{xx} + \sin x \qquad U(x,0) = 1 \qquad U_y(x,0) = 1$

Yechish:

Bu tenglamani ham birinchi navbatta xarakteristik tenglamasini tuzib olamiz:

$(dx)^2 - 9(dy)^2 = 0$ tenglamani yechib $\begin{cases} \xi = x + 3y \\ \eta = x - 3y \end{cases}$ larni topamiz va

$U(x,y) = U(\xi,\eta)$ almashtirishni bajaramiz.

$\begin{cases} U_{xx} = U_{\eta\eta} + 2U_{\xi\eta} + U_{\xi\xi} \\ U_{yy} = 9U_{\eta\eta} - 18U_{\xi\eta} + 9U_{\xi\xi} \end{cases}$ tengliklarni tenglamaga olib borib qo'ysak quyidagi tenglamaga kelamiz:

$$-36U_{\xi\eta} = Sin\frac{\xi + \eta}{2}$$

Bu tenglamani ξ va η lar bo'yicha integrallab quyidagi umumiy yechimni olamiz:

$U = \frac{1}{9}Sinx - f(x+3y) - g(x-3y)$

Endi yuqoridagi boshlang'ich shartlarni qanoatlantiruvchi yechimini topamiz:

$\begin{cases} U(x,0) = \frac{1}{9}Sinx - f(x) - g(x) = 1 \\ U_y(x,0) = 3g'(x) - 3f(x) = 1 \end{cases}$ birinchi tenglamadan x bo'yicha hosila olib uchga ko'paytiramiz va ikkinchi tenglamaga qo'shamiz. Natijada quyidagi $\frac{1}{3}Cosx - 6f'(x) = 1$ tenglamaga kelamiz. Bu yerdan

$f(x) = -\frac{1}{18}Sinx - \frac{1}{6}x + c$ bo'lishi kelib chiqadi. Bundan foydalanib $g(x)$ funksiyani ham topamiz:

$$g(x) = \frac{1}{3}Sinx + \frac{1}{6}x - 1 - c$$

Endi topilgan funksiyalarni umumiy yechimga olib borib qo'yamiz.

$U(x,y) = \frac{1}{9}Sinx + y + 1 + \frac{1}{18}Sin(x+3y) - \frac{1}{3}Sin(x-3y)$

5-misol. Tenglamaning umumiy yechimini toping.

$$U_{xx}-U_{yy}+\frac{2}{x}U_{x}=2$$

Yechish:

Bu tenglamani yechish uchun $U=\frac{V}{x}$ almashtirish bajaramiz. Undan kerakli hosilalarni olamiz : $U_{x}=-\frac{V}{x^{2}}+\frac{V}{x}$ $U_{xx}=-2\frac{V_{x}}{x^{2}}+\frac{V_{xx}}{x}+2\frac{V}{x^{3}}$ $U_{yy}=\frac{V_{yy}}{x}$ Topilganlarni borib soddalashtirsak quyidagi tenglamaga kelamiz:

$V_{xx}-V_{yy}=2x$ Endi bu tenglamaning xarakteristikasini tuzamiz: $dy^{2}-dx^{2}=0$

$\begin{matrix}\xi=y+x\\ \eta=y-x\end{matrix}$ deb belgilasak $\begin{cases}V_{xx}=V_{\eta\eta}-2V_{\xi\eta}+V_{\xi\xi}\\ V_{yy}=V_{\eta\eta}+2V_{\xi\eta}+V_{\xi\xi}\end{cases}$ tenglikka ega bo'lamiz.

Ularni tenglamaga qo'yamiz $V_{\xi\eta}=\frac{\eta-\xi}{4}$ $\Rightarrow$ $V=\frac{\xi\eta^{2}}{8}-\frac{\xi^{2}\eta}{8}+f\ \xi\ +g\ \eta$ u holda

$U=\frac{1}{4x}\ x\ x^{2}-y^{2}\ +f\ \xi\ +g\ \eta$ umumiy yechimga ega bo'lamiz.

Javob: $U=\frac{1}{4x}\ x\ x^{2}-y^{2}\ +f\ \xi\ +g\ \eta$.

Quyidagi parabolik tenglamalar uchun qo'yilgan chegaraviy masalalarni Fur'e usulida yeching.

1-misol

$$U_{t}=a^{2}U_{xx}$$
$$U(0,t)=0$$
$$U_{x}(l,t)=0$$
$$U(x,0)=\varphi(x)$$

Xos funksiyani topamiz

$$X'' + \lambda^2 X = 0$$

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$

$$\begin{cases} X(0) = 0 \\ X'(l) = 0 \end{cases}$$

$$C_1 = 0$$

: $X'(l) = C_2 \lambda \cos \lambda l = 0$

$$\lambda_n = \frac{\pi + 2\pi n}{2l}; \; n = 0, 1, 2, \ldots$$

$$X_n(x) = \sin \lambda_n (x)$$

Yechim

$U(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x)$ ko'rinishda izlanadi. Endi $T_n(t)$ ni topamiz:

$$T'(t) + \lambda^2 a^2 T(t) = 0 \Rightarrow T_n(t) = A_n e^{-\lambda_n^2 a^2 t} \Rightarrow U(x,t) = \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 a^2 t} \sin \lambda_n x$$

$$U(x,0) = \sum_{n=0}^{\infty} A_n \sin \lambda_n x = \varphi(x)$$

$$A_n = \frac{2}{l} \int_0^l \varphi(x) \sin \lambda_n x dx$$

$$U(x,t) = \sum_{n=0}^{\infty} e^{-\lambda_n^2 a^2 t} \sin \lambda_n x \int_0^l \varphi(x) \sin \lambda_n x dx$$

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2-misol. $0 < x < l,\ t > 0$ yarim yo'lda quyidagi aralash masalani yeching :

$$u_t = a^2 u_{xx},\ u_x(0,t) = u_x(l,t) = q,\ u(x,0) = Ax.$$

Yechish : Masalada qo'yilgan chegaraviy shartlar bir jinsli bo'lmagani uchun, yechimni $u(x,t) = v(x,t) + w(x,t)$ ko'rinishda qidiramiz. Undan quyidagi munosabatlarni hosil qilamiz :

$$\begin{cases} u_x(0,t) = v_x(0,t) + w_x(0,t) = q \\ u_x(l,t) = v_x(l,t) + w_x(l,t) = q \end{cases} \Rightarrow \begin{cases} w_x(0,t) = q, v_x(0,t) = 0 \\ w_x(l,t) = q,\ v_x(l,t) = 0 \end{cases}$$

$w(x,t)$ funksiyani $w(x,t) = (Ax + B)q$ ko'rinishda qidiramiz. $w_x(x,t) = Aq$ va yuqoridagi chegaraviy shartlardan $A = 1, B = 0$ desak, $w(x,t) = qx$ funksiyaga ega bo'lamiz.

$v(x,t)$ funksiyaga nisbatan esa quyidagi tenglamalar sistemasini hosil qilamiz :

$$\begin{cases} v_t = a^2 v_{xx} \\ v_x(0,t) = v_x(l,t) = 0 \\ v(x,0) = x(A - q) \end{cases}$$

Endi yuqoridagi masalani Furye usulida yechamiz : $v(x,t) = X(x)T(t)$.

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -л^2 \Rightarrow \begin{cases} X(x) = c_1 \cos лx + c_2 \sin лx \\ X'(0) = X'(l) = 0 \end{cases} \Rightarrow \begin{cases} X'(0) = c_2 = 0 \\ л_n l = pn, n \in N \end{cases}$$

$$X_n(x) = \cos л_n x = \cos\frac{pn}{l}x, n \in N,\ \frac{T'(t)}{a^2 T(t)} = -л^2 \Rightarrow T_n(t) = A_n e^{-a^2 л^2 t}$$

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-a^2 л^2 t} \cos\frac{pn}{l}x, v(x,0) = \sum_{n=1}^{\infty} A_n \cos\frac{pn}{l}x = x(A - q)$$

$$A_n = \frac{2}{l}\int_0^l x(A - q)\cos\frac{pn}{l}x\,dx = \frac{2l}{p^2 n^2}(A - q)(\cos pn - 1)$$

$$v(x,t) = \sum_{n=1}^{\infty} \frac{2l}{p^2 n^2}(A - q)((-1)^n - 1)e^{-a^2 л^2 t}\cos\frac{pn}{l}x$$

Foydalanilgan adabiyotlar:

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