O'ZBEKISTON RESPUBLIKASI XALQ TA'LIMI VAZIRLIGI

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TENGSIZLIKLAR-II. ISBOTLASHNING ZAMONAVIY USULLARI

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Fizika –matematika fanlari doktori, professor A. A’zamov umumiy tahriri ostida.

Qo’llanmada tengsizliklarni isbotlashning yangi samarali usullari va ularni qo’llanishiga doir turli matematik olimpiadalardagi masalalar keltirilgan.

Qo’llanma umumiy o’rta ta’lim maktablari, akademik litseylar va kasb–hunar kollejlarining iqtidorli o’quvchilari, matematika fani o’qituvchilari hamda pedagogika oliy o’quv yurtlari talabalari uchun mo’ljallangan.

Qo’llanmadan sinfdan tashqari mashg’ulotlarda, o’quvchilarni turli matematik musobaqalarga tayyorlash jarayonida foydalanish mumkin.

Taqrizchilar: TVDPI matematika kafedrasi mudiri, f.–m.f.n., dotsent Sh.B. Bekmatov
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Ushbu qo’llanma Respublika ta’lim markazi qoshidagi matematika fanidan ilmiy-metodik kengash tomonidan nashrga tavsiya etilgan. (15 iyun 2008 y., 8 -sonli bayyonnoma)

Qo’llanmaning yaratilishi Vazirlar Mahkamasi huzuridagi Fan va texnologiyalarni rivojlantirishni muvofiqlashtirish Q’omitasi tomonidan moliyalashtirilgan (ХИД 1-16 – sonli innovatsiya loyihasi**)**

1-BOB. FUNKSIYANING XOSSALARI YORDAMIDA TENGSIZLIKLARNI ISBOTLASH USULLARI

1-§. Funksiyaning monotonlik xossasi yordamida isbotlanadigan tengsizliklar

Ta'rif. $f(x)$ funksiya $(a;b)$ oraliqda aniqlangan bo'lsin. Agar ixtiyoriy $x_1 \le x_2$ tengsizlikni qanoatlantiradigan $x_1, x_2 \in (a,b)$ nuqtalar uchun $f(x_1) \le f(x_2)$ $(f(x_1) \ge f(x_2))$ tengsizlik bajarilsa, u holda f funksiya (a,b) oraliqda *o'suvchi (kamayuvchi)* funksiya deyiladi, (a,b) oraliq esa *monotonlik oralig'i* deb yuritiladi.

Ta'rif. $f(x)$ funksiya $(a;b)$ intervalda aniqlangan bo'lsin. Agar ixtiyoriy $x_1 < x_2$ tengsizlikni qanoatlantiradigan $x_1, x_2 \in (a,b)$ nuqtalar uchun $f(x_1) < f(x_2)$ $(f(x_1) > f(x_2))$ tengsizlik bajarilsa, u holda f funksiya (a,b) oraliqda *qat'iy o'suvchi (kamayuvchi) funksiya* deyiladi.

Teorema. $f(x)$ funksiya $(a;b)$ oraliqda aniqlangan va differentsiallanuvchi bo'lsin. $f(x)$ funksiya $(a;b)$ intervalda o'suvchi (kamayuvchi) bo'lishi uchun shu intervalda $f'(x) \ge 0$ $(f'(x) \le 0)$ tengsizlik bajarilishi zarur va etarli.

Agar $(a;b)$ intervalda $f'(x) > 0$ $(f'(x) < 0)$ tengsizlik bajarilsa, u holda $f(x)$ funksiya $(a;b)$ intervalda qat'iy o'suvchi (kamayuvchi) bo'ladi.

1-masala. e^{π} va π^{e} sonlarni taqqoslang.

Yechilishi. $f : [e; +\infty) \to \mathbf{R}$, $f(x) = \dfrac{\ln x}{x}$ funksiyani qaraymiz. Uning hosilasi barcha x, $x \in (e; +\infty)$ larda $f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ manfiy qiymat qabul qiladi va f funksiya $[e; +\infty)$ da uzluksiz, shunday qilib, f $[e; +\infty)$ da qat'iy kamayadi. Bu yerdan, $e < \pi$ ekanligini hisobga olib

$$f(e) > f(\pi) \Rightarrow \frac{\ln e}{e} > \frac{\ln \pi}{\pi} \Rightarrow \pi \ln e > e \ln \pi$$

ni olamiz. Demak, $e^{\pi} > \pi^{e}$.

2-masala. $x_n = 1 + \frac{1}{2} + \ldots \frac{1}{n}$, $n=1, 2, \ldots$ sonli ketma-ketlikni chegaralanganlikka tekshiring.

Yechilishi. Dastlab

$$\ln(1+x) \le x \quad (x \ge 0) \qquad (1)$$

tengsizlikni isbotlaymiz. Buning uchun $f : [0; +\infty) \to \mathbf{R}$; $f(x) = x - \ln(1+x)$ funksiyani qaraymiz. f funksiya aniqlanish sohasida uzluksiz va barcha x, $x \in (0; +\infty)$ lar uchun

$f'(x) = \frac{x}{x+1}$ tenglik o'rinli, bu yerdan

$f'(x) > 0$, $(x \in (0; +\infty))$ ekanligi kelib chiqadi. Shunday qilib f funksiya $D(f)$ aniqlanish sohasida qat'iy o'sadi va demak, $f(x) \ge f(0)$ $(x \ge 0)$ dan (1) tengsizlikning to'g'riligi kelib chiqadi.

(1) tengsizlikdan $x = \frac{1}{n}$ deb olib $(n = 1,2,...)$,

$$\ln\left(1 + \frac{1}{n}\right) \le \frac{1}{n} (n = 1,2,...)$$

yoki

$$\ln(n+1) - \ln n \le \frac{1}{n} \quad (n = 1,2,...) \qquad (2)$$

ni hosil qilamiz.

(2) tengsizlikdan

$$\ln 2 - \ln 1 \le 1, \ \ln 3 - \ln 2 \le \frac{1}{2}, ..., \ \ln(n+1) - \ln n \le \frac{1}{n} \qquad (3)$$

kelib chiqadi. (3) tengsizliklarni hadma-had qo'shib,

$$\ln(n+1) \le 1 + ... + \frac{1}{n}$$

tengsizlikni olamiz.

Demak, $x_n = 1 + \frac{1}{2} + \ldots \frac{1}{n}$, n=1, 2, … sonli ketma-ketlik chegaralanmagan.

3-masala (Bernulli tengsizligi). Ixtiyoriy $x > -1$; $\alpha > 1$ uchun

$$(1 + x)^{\alpha} \geq 1 + \alpha x , \qquad (4)$$

tengsizlik o'rinli, shu bilan birga tenglik o'rinli faqat $x = 0$ da

Yechilishi. $f(x) = (1 + x)^{\alpha} - 1 - \alpha x$, $(x \in [-1; +\infty))$ funksiyani qaraymiz, bu yerda α – fiksirlangan 1 dan katta son. Bu funksiyaning hosilasini hisoblaymiz:

$$f'(x) = \alpha(1 + x)^{\alpha-1} - \alpha = \alpha((1 + x)^{\alpha-1} - 1) \quad (x > -1).$$

$\alpha > 1$ shartdan, $x \in (-1; 0)$ uchun $f'(x) < 0$ va $x \in (0; +\infty)$ uchun $f'(x) > 0$ ekanligi kelib chiqadi. Demak, f funksiya $[-1; 0]$ da kamayadi va $[0; +\infty)$ da o'sadi. Bundan barcha $x \in [-1;+\infty)\backslash\{0\}$ lar uchun $f(x) > f(0)$ tengsizlik o'rinli, ya'ni,

$$(1 + x)^{\alpha} - 1 - \alpha x > 1 - 1 \text{ va } (1 + x)^{\alpha} > 1 + \alpha x$$

$(x \in [-1; 0) \cup (0; +\infty), \alpha > 1)$ deb hulosa qilamiz.

$(1 + x)^{\alpha} = 1 + \alpha x$ tenglik $x = 0$ da bajarilishini eslatib o'tish qolyapti.

Izoh.

$$(1 + x)^{\alpha} \leq 1 + \alpha x \quad (x \geq -1; 0 < \alpha < 1),$$

$$(1 + x)^{\alpha} \geq 1 + \alpha x \quad (x \geq -1; \alpha < 0).$$

tengsizliklar shunga o'xshash isbotlanadi.

4-masala. (Yung tengsizligi) Agar $p, q \in \mathbf{R}\backslash\{0, 1\}$ sonlar $\frac{1}{p} + \frac{1}{q} = 1$ tenglikni qanoatlantirsa, u holda ixtiyoriy a, b musbat sonlar uchun

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab \quad (p>1), \qquad (5)$$

$$\frac{1}{p}a^p + \frac{1}{q}b^q \leq ab \quad (p<1) \qquad (6)$$

tengsizliklar bajariladi.

Bundan tashqari, tenglik bajariladi faqat va faqat shu holdaki, qachonki $a^p = b^q$ bo'lsa.

Yechilishi. $p > 1$ holni qaraymiz. Ixtiyoriy musbat a sonni fiksirlab, $f:(0,+\infty)\to R;\quad f(b)=\frac{1}{p}a^p+\frac{1}{q}b^q-ab$ funksiyani aniqlaymiz.

Bu funksiyaning hosilasi $f'(b) = b^{q-1} - a$ ga teng. Elementar hisoblashlar yordamida $a^{\frac{1}{q-1}}$ nuqtada f funksiya uzining eng kichik kiymatiga erishishini kurish mumkin, ya'ni

$$f(b) > f(a^{\frac{1}{q-1}}), b > 0. \qquad (7)$$

ko'rsatiladi.

(7) tengsizlikdan $\frac{1}{p}+\frac{1}{q}=1$ ekanligini hisobga olib,

$$\frac{1}{p}a^p+\frac{1}{q}b^q-ab\geq 0 \quad (a>0, b>0; p>1)$$

olinadi. (5) tengsizlik isbotlanadi. (7) dan tenglik $b=a^{\frac{1}{q-1}}$, ya'ni $a^p = b^q$ holda o'rinli ekanligi kelib chiqadi. (6) tengsizlik shunga o'xshash isbotlanadi.

5-masala.

$$|sinx| \leq |x| \qquad (x\in\mathbf{R}) \qquad (8)$$

tengsizlikni isbotlang

Yechilishi. Ikkala qismning juftligidan $x \geq 0$ holni qarash etarli. Bundan tashqari, $|sinx| \leq 1$ ligidan $0 \leq x \leq 1$ holni o'rganish etarli. Shu maqsadda f: $[0;1]\to \boldsymbol{R}$, $f(x) = x - \sin x$ funksiyani qaraymiz. f funksiyaning hosilasi

$$f'(x) = 1 - \cos x \qquad (x \in [0; 1]).$$

Kosinusning chegaralanganligidan ($|\cos x| \le 1;\ x \in \boldsymbol{R}$) $f'(x) \ge 0$ deb hulosalaymiz. Bu yerdan f funksiya o'zining aniqlanish sohasida monoton o'suvchi bo'lishi kelib chiqadi va shuning uchun

$$f(x) \ge f(0) \quad (x \in [0;\,1]) \quad \text{yoki} \quad x - \sin x \ge 0, \quad (x \in [0;\,1])$$

tengsizlik o'rinli bo'ladi. Bu yerdan esa berilgan tengsizlik kelib chiqadi.

6-masala. Agar $a > b > c$ bo'lsa, u holda

$$a^2(b - c) + b^2(c - a) + c^2(a - b) > 0$$

tengsizlik o'rinli bo'lishini isbotlang.

Yechilishi. $f(t) = (b + t)^2(b - c) + b^2(c - (b + t)) + c^2((b + t) - b)$ ko'rinishdagi $f:[0;+\infty) \to \boldsymbol{R}$ funksiyani qaraymiz, bu yerda a, b, c lar $a > b > c$ tengsizlikni qanoatlantiruvchi haqiqiy parametrlar. f funksiyaning $[0;\,+\infty)$ da qat'iy o'suvchi bo'lishi avvalgi masalalardagidek isbotlanadi va shunday qilib

$$f(a - b) > f(0)$$

tengsizlik o'rinli. Ohirgi tengsizlik berilgan tengsizlikka tengkuchli.

2-§. Funksiyaning qavariqlik xossasi yordamida isbotlanadigan tengsizliklar

$(a;b)$ – haqiqiy sonlar o'qidagi oraliq berilgan bo'lsin.

Ta'rif: $f:(a;b) \to \boldsymbol{R}$ funksiya $(a;b)$ da quyidan qavariq deyiladi, agar barcha x_1, $x_2 \in (a;b)$ shunday $\lambda_1 \ge 0$, $\lambda_2 \ge 0$ va $\lambda_1 + \lambda_2 = 1$ shartlarni qanoatlantiruvchi λ_1, λ_2 sonlar uchun

$$f(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_1 f(x_1) + \lambda_2 f(x_2) \qquad (1)$$

tengsizlik o'rinli bo'lsa.

Yuqoridan qavariq funksiyaning ta'rifi esa yuqorida keltirilgan (1) tengsizlik belgisini qarama-qarshisiga almashtirishdan olinadi.

Yensen tengsizligi. $f:(a;b)\to \boldsymbol{R}$ – quyidan (yuqoridan) qavariq funksiya bo'lsin. U holda barcha $x_j \in (a;b)$ $(j = 1,...,n)$ lar va

$$\lambda_1 + ... + \lambda_n = 1$$

tenglikni qanoatlantiruvchi ixtiyoriy $\lambda_j \geq 0$ $(j = 1,...,n)$ sonlar uchun

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \ldots + f(\lambda_n x_n)$$

$$(f(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) \geq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \ldots + f(\lambda_n x_n)$$

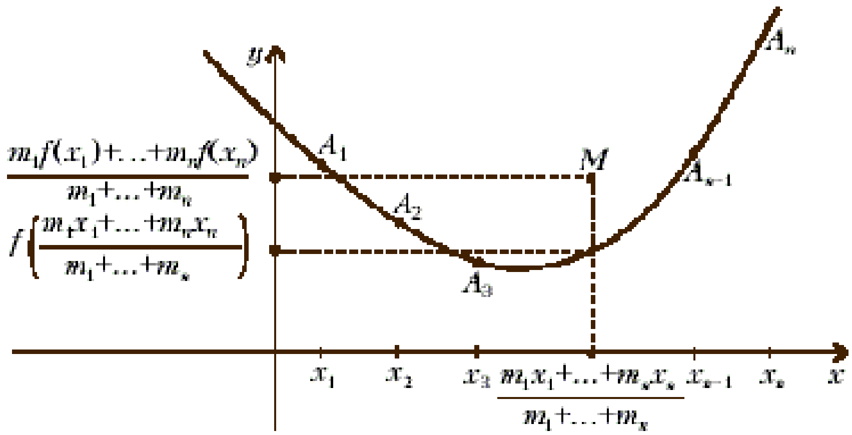
tengsizlik o'rinli.

Isboti: $y = f(x)$ funksiyaning grafigida abtsissalari $x_1, x_2, ..., x_n$ bo'lgan $A_1, A_2, ..., A_n$ nuqtalarni qaraymiz. Bu nuqtalarda $m_1, m_2, ..., m_n$ massali yuklarni joylashtiramiz. Bu nuqtalar massalari markazi

$$\left(\frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n}; \frac{m_1 f(x_1) + m_2 f(x_2) + \ldots + m_n f(x_n)}{m_1 + m_2 + \ldots + m_n} \right)$$

$A_1, A_2, ..., A_n$ nuqtalar qavariq funksiyaning grafigi ustida yotganligidan, ularning massalar markazi ham grafik ustida yotadi. Bu esa massalar markazi M ning ordinatasi shu abstsissaga ega bo'lgan nuqtaning ordinatasidan kichik emasligini bildiradi, ya'ni (1-chiz.),

$$f\left(\frac{m_1 x_1 + \ldots + m_n x_n}{m_1 + \ldots + m_n} \right) \leq \frac{m_1 f(x_1) + \ldots + m_n f(x_n)}{m_1 + \ldots + m_n} . \qquad (2)$$



1-chiz.

Isbotni tugatish uchun $m_1 = a_1, \ldots, m_n = a_n$ olish qolyapti. Biroq, ikkita muhim izoh mavjud. Birinchidan, Yensenning (1) tengsizligini isbotlash jarayonida biz (2) tengsizlikni isbotladik. Aslida bu tengsizliklar teng kuchli.

(1) tengsizlikda

$$a_i = \frac{m_1}{m_1 + \ldots + m_n} \quad (i = 1, 2, \ldots, n)$$

deb olib, biz (2) tengsizlikni olamiz. Shuning uchun tabiiy ravishda bu tengsizliklar Yensen tengsizliklari deb ataladi. (1) tengsizlik ancha ixcham ko'rinadi, biroq tatbiq qilish uchun (2) tengsizlikdan foydalanish qulayroq. Ikkinchidan, agar esli funksiya $f(x)$ funksiya qavariq bo'lsa, u holda uning uchun (1) va (2) Yensen tengsizliklaring ishoralari qarama-qarshisiga o'zgaradi. Buni isbotlash uchun $-f(x)$ qavariq funksiyani qarash etarli.

Teorema. $(a;b)$ oraliqda uzluksiz va ikkinchi tartibli hosilaga ega bo'lgan $f : (a;b) \to \mathbf{R}$ funksiya shu intervalda quyidan (yuqoridan) qavariq bo'lishi uchun $(a;b)$ da $f''(x) \geq 0$ $(f''(x) \leq 0)$ tengsizlikning bajarilishi zarur va etarli.

1-masala (O'rta qiymatlar haqidagi Koshi tengsizligi). Ixtiyoriy nomanfiy $a_1, a_2, ..., a_n$ sonlar uchun

$$\sqrt[n]{a_1 a_2 ... a_n} \le \frac{a_1 + a_2 + ... + a_n}{n} \qquad (2)$$

tengnsizlik o'rinli, ya'ni o'rta geometrik qiymat o'rta arifmetik qiymatdan katta emas.

Yechilishi. Agar a_j sonlardan biri 0 ga teng bo'lsa, u holda (2) tengsizlikning bajarilishi ravshan, shuning uchun barcha a_j sonlar musbat deb hisoblaymiz.

$f(x) = \ln x$ ($x > 0$) funksiyani qaraymiz. f funksiya $(0; +\infty)$ da yuqoridan qavariq ekanligi ravshan. Yensen tengsizligiga asoslanib

$$\ln\left(\sum_{i=1}^{n} \frac{1}{n} a_i\right) \ge \sum_{i=1}^{n} \frac{1}{n} \ln a_i$$

tengsizlikni hosil qilamiz.

2-masala. $x_1, ..., x_n$ – nomanfiy sonlar bo'lsin.

$$f : [0, +\infty) \to R; \quad f(\alpha) = \left(\frac{x_1^\alpha + \ldots + x_n^\alpha}{n}\right)^{\frac{1}{\alpha}}$$

funksiya monoton o'suvchi ekanligini isbotlang.

Yechilishi. $0 < \alpha < \beta$ bo'lsin. $h(x) = x^{\frac{\beta}{\alpha}}, x \ge 0$ $(x \ge 0)$ funksiyani qaraymiz. $h''(x) = \frac{\beta}{\alpha}\left(\frac{\beta}{\alpha} - 1\right) x^{\frac{\beta}{\alpha}} > 0 \quad (x > 0)$, shunday qilib h funksiya $[0; +\infty)$ da quyidan qavariq. Yensen tengsizligiga ko'ra

$$h\left(\sum_{i=1}^{n} \frac{1}{n} x_i^\alpha\right) \le \sum_{i=1}^{n} \frac{1}{n} h(x_i^\alpha) \text{ yoki } \left(\sum_{i=1}^{n} \frac{1}{n} x_i^\alpha\right)^{\frac{\beta}{\alpha}} \le \sum_{i=1}^{n} \frac{1}{n} x_i^\beta,$$

bu yerdan $f(\alpha) \le f(\beta)$ kelib chiqadi.

3-masala. $\sin\alpha + \sin\beta + \sin\gamma \le \frac{3\sqrt{3}}{2}$ tengsizlikni isbotlang, bu yerda

α, β, γ - biror uchburchakning ichki burchaklari.

Yechilishi. $f : [0; \pi] \to \mathbf{R}$; $f(x) = sinx$ funksiyani qaraymiz. $x \in (0; \pi)$ lar uchun $f''(x) = -\sin x$ va $f''(x) < 0$, shuning uchun f funksiya $[0; \pi]$ da yuqoridan qavariq. Yensen tengsizligiga ko'ra

$$f\left(\frac{\alpha}{3}+\frac{\beta}{3}+\frac{\gamma}{3}\right) \geq \frac{f(\alpha)}{3}+\frac{f(\beta)}{3}+\frac{f(\gamma)}{3} \text{ yoki } sin\frac{\pi}{3} \geq \frac{1}{3}(sin\alpha + sin\beta + sin\gamma),$$

bu yerdan $sin\alpha + sin\beta + sin\gamma \leq \frac{3\sqrt{3}}{2}$ ni olamiz.

4-masala. Ixtiyoriy musbat a_j, b_j ($j = 1,.., n$) sonlar uchun

$$\left(\frac{a_1+\ldots+a_n}{b_1+\ldots+b_n}\right)^{b_1+\ldots+b_n} \geq \left(\frac{a_1}{b_1}\right)^{b_1} \ldots \left(\frac{a_n}{b_n}\right)^{b_n}$$

tengsizlik o'rinli bo'lishini isbotlang

Yechilishi. $f : [0; +\infty) \to \mathbf{R}$, $f(x) = lnx$ funksiyani qaraymiz. Bu funksiya yuqoridan qavariq. Shunday qilib, Yensen tengsizligiga ko'ra

$$f\left(\sum_{i=1}^{n} \frac{b_i}{b_1+\ldots+b_n} \cdot \frac{a_i}{b_i}\right) \leq \sum_{i=1}^{n} \frac{b_i}{b_1+\ldots+b_n} \cdot f\left(\frac{a_i}{b_i}\right)$$

yoki

$$(b_1+\ldots+b_n)\ln\frac{a_1+\ldots+a_n}{b_1+\ldots+b_n} \geq \sum_{i=1}^{n} b_i \ln\left(\frac{a_i}{b_i}\right)$$

demak,

$$\left(\frac{a_1+\ldots+a_n}{b_1+\ldots+b_n}\right)^{b_1+\ldots+b_n} \geq \left(\frac{a_1}{b_1}\right)^{b_1} \ldots \left(\frac{a_n}{b_n}\right)^{b_n}.$$

5-masala. (Gyuygens tengsizligi). Ixtiyoriy nomanfiy a_j ($j = 1,...,n$) sonlar uchun

$$(1+a_1)\ldots(1+a_n) \geq (1+\sqrt[n]{a_1\ldots a_n})^n$$

tengsizlik o'rinli bo'lishini isbotlang.

Yechilishi. $f : \mathbf{R} \to \mathbf{R}$, $f(x) = ln(1 + e^x)$ funksiyani qaraymiz. Barcha $x \in \mathbf{R}$ lar uchun $f''(x) > 0$ o'rinli. Shunday qilib, funksiya yuqoridan qavariq. Yensen tengsizligiga ko'ra

$$f\left(\sum_{i=1}^{n}\frac{1}{n}\ln a_i\right) \le \sum_{i=1}^{n}\frac{1}{n}f(\ln a_i) \Leftrightarrow \ln\left(1+\exp\sum_{i=1}^{n}\frac{1}{n}\ln a_i\right) \le \sum_{i=1}^{n}\frac{1}{n}\ln(1+a_i) \Leftrightarrow$$

$$\Leftrightarrow \ln((1+a_1)\ldots(1+a_n)) \ge n\ln(1+\sqrt[n]{a_1\ldots a_n})^n \Leftrightarrow$$

$$\Leftrightarrow (1+a_1)\ldots(1+a_n) \ge (1+\sqrt[n]{a_1\ldots a_n})^n$$

ni olamiz.

6-masala.

$$\sqrt{(\sum a_i)^2 + (\sum b_i)^2} \le \sum \sqrt{a_i^2 + b_i^2} \quad (a_i, b_i > 0)$$

tengsizlikni isbotlang.

Yechilishi. $f(x) = \sqrt{1+x^2}$ funksiyani qaraymiz. Barcha musbat $x \in \mathbf{R}$ lar uchun
$f''(x) > 0$ o'rinli. Shunday qilib, f funksiya o'zining aniqlanish sohasida quyidan qavariq.

$$x_i = \frac{a_i}{b_i}, \qquad \alpha_i = \frac{b_i}{\sum b_i}$$

deb olamiz. Bu funksiya uchun Yensen tengsizligini yozamiz.

$$\frac{b_1}{\sum b_i}\sqrt{1+(\frac{a_1}{b_1})^2} + .. + \frac{b_n}{\sum b_i}\sqrt{1+(\frac{a_n}{b_n})^2} \ge f(\frac{b_1}{\sum b_i}\ \frac{a_1}{b_1} + ..+ \frac{b_n}{\sum b_i}\ \frac{a_n}{b_n}) = f(\frac{\sum a_i}{\sum b_i}),$$

$$\frac{1}{\sum b_i}(\sqrt{b_1^2+a_1^2} + .. + \sqrt{b_n^2+a_n^2}) \ge \sqrt{1+(\frac{\sum a_i}{\sum b_i})^2}$$

Ohirgi tengsizlikning ikkala qismini $\sum b_i$ ga ko'paytiramiz va kerakli tengsizlikni olamiz.

7-masala. Tengsizlikni isbotlang:

$$\sum_{i=1}^{n}\frac{a_i}{S-a_i}\geq\frac{n}{n-1},$$

bu yerda

$$S=a_1+a_2+..+a_n,\ a_i>0.$$

Yechilishi.

$$f(x)=\frac{x}{S-x}$$

funksiyani qaraymiz. Barcha musbat $x\in\mathbf{R}$ lar uchun $f''(x)>0$ o'rinli. Shunday qilib, f funksiya o'zining aniqlanish sohasida quyidan qavariq.

Bu funksiya uchun Yensen tengsizligini yozamiz.

$$\frac{1}{n}\cdot\frac{a_1}{S-a_1}+...+\frac{1}{n}\cdot\frac{a_n}{S-a_n}\geq f\left(\frac{a_1}{n}+...+\frac{a_n}{n}\right)=f\left(\frac{S}{n}\right)=\frac{S/n}{S-S/n}=\frac{1}{n-1}.$$

Ikkala qismini n ga ko'paytiramiz va kerakli tengsizlikni olamiz.

8-masala. Tengsizlikni isbotlang:

$$\left(\sum_{i=1}^{n}x_i\right)^p<n^{p-1}\cdot\sum_{i=1}^{n}x_i^p,\quad p>1, x_i>0.$$

Isbot.

$$f(x) = x^p$$

funksiya uchun Yensen tengsizligini yozamiz.

$$f\left(\frac{x_1}{n} + \frac{x_2}{n} + \ldots + \frac{x_n}{n}\right) < \frac{1}{n}\left(x_1^p + x_2^p + \quad + x_n^p\right),$$

$$\frac{(x_1 + x_2 + \ldots + x_n)^p}{n^p} < \frac{1}{n}\left(x_1^p + x_2^p + \ldots + x_n^p\right).$$

Ikkala qismini n ga ko’paytiramiz va kerakli tengsizlikni olamiz.

$$(x_1 + x_2 + \ldots + x_n)^p < n^{p-1}\left(x_1^p + x_2^p + \ldots + x_n^p\right).$$

2-BOB. TRANS-TENGSIZLIK VA UNING TADBIQLARI .

1-§. Trans-tengsizlik haqida .

Teorema. $a_1, ..., a_n$ sonli ketma-ketlik $a_1 \geq a_2 \geq ... \geq a_n$ shartni qanoatlantirsin. $a_1b_1 + a_2b_2 + ... + a_nb_n$ yig'indi eng katta qiymatiga $b_1 \geq b_2 \geq ... \geq b_n$ bo'lganda , eng kichik qiymatiga esa $b_1 \leq b_2 \leq ... \leq b_n$ bo'lganda yerishadi.

Isbot. r, s natural sonlar $r < s \leq n$ shartni qanoatlantirsin .
Isbot qilish uchun
$S = a_1c_1 + a_2c_2 + ... + a_rc_r + ... + a_sc_s + ... + a_nc_n$,
$S' = a_1c_1 + a_2c_2 + ... + a_rc_s + ... + a_sc_r + ... + a_nc_n$ yig'indilarni solishtirish etarli. Ular uchun

$$S' - S = a_rc_s + a_sc_r - a_rc_r - a_sc_s = (a_r - a_s)(c_s - c_r)$$

munosabatga ega bo'lamiz.

Demak, agar $c_s \leq c_r$ bo'lsa $S \leq S'$ va $c_s \geq c_r$ bo'lsa $S \geq S'$. Bu esa talab qilingan tasdiqni isbotlaydi.

Natija. Teoremadan ko'rinib turibdiki, agar

$$a_1 \geq a_2 \geq ... \geq a_n, \quad b_1 \geq b_2 \geq ... \geq b_n$$

bo'lsa, u holda $b_1, ..., b_n$ sonlarning ixtiyoriy $(x_1, ..., x_n)$ o'rin almashtirishi uchun

$$a_1b_1 + a_2b_2 + ... + a_nb_n \geq a_1x_1 + a_2x_2 + ... + a_nx_n \geq a_1b_n + a_2b_{n-1} + ... + a_nb_1 \qquad (1)$$

qo'shtengsizlik o'rinli.

Izoh. Xorijiy adabiyotlarida (1) ko'shtengsizlik "rearrangement inequality" deb yuritiladi. Qizig'i shundaki, hozirgacha (1) ko'shtengsizlik xattoki rus tilida aniq nomga ega emas. Uni nomlash uchun "trans-tengsizlik" terminini qo'llash mualliflardan biriga Xalqaro matematika olimpiadalarida ishtirok etuvchi Rossiya o'quvchilari terma jamoasining ilmiy rahbari dotsent N. Agaxanov taklif qildi.

Ta'rif. $(a_1,\ a_2,\ a_3)$ va (b_1, b_2, b_3) uchliklar *bir xil tartiblangan* deyiladi, agar a_1, a_2, a_3 va b_1, b_2, b_3 ketma-ketliklarlardan ikkalasi kamaymaydigan (ya'ni $a_1 \le a_2 \le a_3$ va $b_1 \le b_2 \le b_3$), yoki ikkalasi ortmaydigan (ya'ni $a_1 \ge a_2 \ge a_3$ va $b_1 \ge b_2 \ge b_3$) bo'lsa.

Agar a_1, a_2, a_3 va b_1, b_2, b_3 ketma-ketliklarlardan bittasi kamaymay-digan, boshqasi esa ortmaydigan bo'lsa, u holda $(a_1,\ a_2,\ a_3)$ va (b_1, b_2, b_3) uchliklar *turlicha tartiblangan* deyiladi.

Masalan, (-1,1,3) va (2,5,7) uchliklar bir xil tartiblangan, (-1,1,3) va (7,5,2) uchliklar esa turlicha tartiblangan.

a, b, c musbat sonlar uchun $a \ge b \ge c$ yoki $a \le b \le c$ bo'lsa, u holda (a,b,c) va $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ uchliklar turlicha, (a,b,c) va (a^n,b^n,c^n) uchliklar esa bir xil tartiblangan , bu yerda n - ixtiyoriy natural son.

$(a_1,\ a_2,\ a_3)$ va (b_1, b_2, b_3) uchliklar berilgan bo'lsin.

$(x_1,\ x_2,\ x_3)$ uchlik b_1, b_2, b_3 sonlarning o'rin almashtirishi bo'lsin.

U holda yuqoridagi teoremaning quyidagi muhim bo'lgan natijalarini qayd etamiz.

1) Agar $(a_1,\ a_2,\ a_3)$ va (b_1, b_2, b_3) uchliklar bir xil tartiblangan bo'lsa, u holda

$$a_1b_1 + a_2b_2 + a_3b_3 \ge a_1x_1 + a_2x_2 + a_3x_3 \qquad (2)$$

tengsizlik o'rinli.

2) Agar $(a_1,\ a_2,\ a_3)$ va (b_1, b_2, b_3) uchliklar turlicha tartiblangan bo'lsa, u holda

$$a_1b_1 + a_2b_2 + a_3b_3 \le a_1x_1 + a_2x_2 + a_3x_3 \qquad (3)$$

tengsizlik o'rinli.

2-§. Trans-tengsizlikni masalalar yechishga tadbiqlari.

1-masala. Ixtiyoriy a,b,c haqiqiy sonlar va n natural son uchun

a) $a^2+b^2+c^2\geq ab+bc+ca$

b) $a^n+b^n+c^n\geq a^{n-1}b+b^{n-1}c+c^{n-1}a$

tengsizliklarni isbotlang.

Yechilishi. $a^2+b^2+c^2\geq ab+bc+ca$ tengsizlik $a^n+b^n+c^n\geq a^{n-1}b+b^{n-1}c+c^{n-1}a$ tengsizlikning xususiy holi bo'lgani uchun $a^n+b^n+c^n\geq a^{n-1}b+b^{n-1}c+c^{n-1}a$ tengsizlikni isbotlaymiz.

$a\geq b\geq c$ deb faraz qilamiz. U holda $a^{n-1}\geq b^{n-1}\geq c^{n-1}$, ya'ni (a,b,c) va $(a^{n-1},b^{n-1},c^{n-1})$ uchliklar bir hil tartiblangan bo'ladi.

(2) tengsizlikda

$(a_1,\ a_2,\ a_3)=(a^{n-1},b^{n-1},c^{n-1})$, $(b_1,\ b_2,\ b_3)=(a,b,c)$, $(x_1,\ x_2,\ x_3)=(b,c,a)$ deb olsak,

$a^n+b^n+c^n=a^{n-1}a+b^{n-1}b+c^{n-1}c\geq a^{n-1}b+b^{n-1}c+c^{n-1}a$ tengsizlikka ega bo'lamiz.

2-masala. Ixtiyoriy a,b,c musbat sonlar uchun

a) $\dfrac{a+b+c}{abc}\leq\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}$

b) $\dfrac{a^2}{b^2}+\dfrac{b^2}{c^2}+\dfrac{c^2}{a^2}\geq\dfrac{b}{a}+\dfrac{c}{b}+\dfrac{a}{c}$

c) $\dfrac{a^2}{b}+\dfrac{b^2}{c}+\dfrac{c^2}{a}\geq a+b+c$

tengsizliklarni isbotlang.

Yechilishi. $a\geq b\geq c$ deb faraz qilamiz.

a) Ravshanki, $\left(\dfrac{1}{a},\dfrac{1}{b},\dfrac{1}{c}\right)$ va $\left(\dfrac{1}{a},\dfrac{1}{b},\dfrac{1}{c}\right)$ uchliklar bir xil tartiblangan.

(2) tengsizlikda

$(a_1, a_2, a_3)=(b_1, b_2, b_3)=\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$, $(x_1, x_2, x_3)=\left(\frac{1}{b},\frac{1}{c},\frac{1}{a}\right)$ deb olsak,

$$\frac{1}{a}\cdot\frac{1}{a}+\frac{1}{b}\cdot\frac{1}{b}+\frac{1}{c}\cdot\frac{1}{c}\geq\frac{1}{a}\cdot\frac{1}{b}+\frac{1}{b}\cdot\frac{1}{c}+\frac{1}{c}\cdot\frac{1}{a}=\frac{a+b+c}{abc}$$

tengsizlikni hosil qilamiz.

b) Ravshanki, $\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right)$ va $\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right)$ uchliklar bir xil tartiblangan.

(2) tengsizlikda

$(a_1, a_2, a_3)=(b_1, b_2, b_3)=\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right)$, $(x_1, x_2, x_3)=\left(\frac{b}{c},\frac{c}{a},\frac{a}{b}\right)$ deb olsak,

$$\frac{a}{b}\cdot\frac{a}{b}+\frac{b}{c}\cdot\frac{b}{c}+\frac{c}{a}\cdot\frac{c}{a}\geq\frac{a}{b}\cdot\frac{b}{c}+\frac{b}{c}\cdot\frac{c}{a}+\frac{c}{a}\cdot\frac{a}{b}=\frac{a}{c}+\frac{b}{a}+\frac{c}{b}$$

tengsizlikni hosil qilamiz.

c) Ravshanki,

$$a^2\geq b^2\geq c^2 \quad \text{va} \quad \frac{1}{c}\geq\frac{1}{b}\geq\frac{1}{a},$$

ya’ni (a^2,b^2,c^2) va $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ uchliklar turlicha tartiblangan.

(3) tengsizlikda

$(a_1, a_2, a_3)=(a^2,b^2,c^2)$, $(b_1, b_2, b_3)=\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$, $(x_1, x_2, x_3)=\left(\frac{1}{b},\frac{1}{c},\frac{1}{a}\right)$ deb olsak,

berilgan tengsizlikka tengkuchli bo’lgan

$$a^2\frac{1}{a}+b^2\frac{1}{b}+c^2\frac{1}{c}\leq a^2\frac{1}{b}+b^2\frac{1}{c}+c^2\frac{1}{a}$$

tengsizlikni hosil qilamiz.

Izoh. Ko’rinib turibdiki, barcha tengsiziklarda tenglik $a=b=c$ bo’lgandagina o’rinlidir.

3-masala. Ixtiyoriy *a*, *b*, *c* musbat sonlar uchun

$$a^3b + b^3c + c^3a \geq a^2bc + b^2ca + c^2ab$$

tengsizlik o’rinli bo’lishini isbotlang.

Yechilishi. Umumiylikka putur etkazmasdan $a \ge b \ge c$ deb faraz qilamiz. U holda

$$a^2 \ge b^2 \ge c^2 \quad \text{va} \quad \frac{1}{c} \ge \frac{1}{b} \ge \frac{1}{a},$$

ya’ni (a^2, b^2, c^2) va $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ uchliklar turlicha tartiblangan.

(3) da

$(a_1,\ a_2,\ a_3) = (a^2, b^2, c^2)$, $(b_1,\ b_2,\ b_3) = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, $(x_1,\ x_2,\ x_3) = \left(\frac{1}{c}, \frac{1}{a}, \frac{1}{b}\right)$

deb olsak, berilgan tengsizlikka tengkuchli bo’lgan

$$a^2\frac{1}{a} + b^2\frac{1}{b} + c^2\frac{1}{c} \le a^2\frac{1}{c} + b^2\frac{1}{a} + c^2\frac{1}{b}$$

tengsizlikni hosil qilamiz.

4-masala. a, b, c –musbat sonlar bo’lsin.

$$(a^a b^b c^c)^2 \ge a^{b+c} b^{c+a} c^{a+b}$$

bo’lishini isbotlang

Yechilishi. Umumiylikka putur etkazmasdan $a \ge b \ge c$ deb faraz qilamiz. U holda

$$\ln a \ge \ln b \ge \ln c,$$

ya’ni (a, b, c) va $(\ln a, \ln b, \ln c)$ uchliklar bir xil tartiblangan.

(2) tengsizlikda $(a_1,\ a_2,\ a_3) = (\ln a, \ln b, \ln c)$, $(b_1,\ b_2,\ b_3) = (a, b, c)$, $(x_1,\ x_2,\ x_3) = (b, c, a)$

deb olsak,

$$a\ln a + b\ln b + c\ln c \ge b\ln a + c\ln b + a\ln c$$

tengsizlikka,

$(x_1,\ x_2,\ x_3) = (c, a, b)$ deb olsak,

$$a\ln a + b\ln b + c\ln c \ge c\ln a + a\ln b + b\ln c$$

tengsizlikka ega bo’lamiz.

Ularni hadma-had qo’shib

$$2(a\ln a + b\ln b + c\ln c) \geq (b + c)\ln a + (c + a)\ln a + (c + a)\ln b + (a + b)\ln c$$

yoki

$$\ln(a^a b^b c^c)^2 \geq \ln(a^{b+c} b^{c+a} c^{a+b})$$

ga ega bo'lamiz. Bu yerdan

$$(a^a b^b c^c)^2 \geq a^{b+c} b^{c+a} c^{a+b}$$

tengsizlik kelib chiqadi.

5-masala. (Moskva olimpiadasi –1963**)**. Ixtiyoriy $a,\ b,\ c$ musbat sonlar uchun

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

tengsizlik to'g'ri bo'lishini isbotlang.

Yechilishi. Tengsizlik simmetrik bo'lgani uchun umumiylikni chegaralamagan holda $a \geq b \geq c$ deb faraz qilamiz. U holda

$$\frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b}$$

tengsizlik o'rinli, ya'ni (a,b,c) va $\left(\frac{1}{b+c},\frac{1}{c+a},\frac{1}{a+b}\right)$ uchliklar bir xil tartiblangan bo'ladi.

(2) tengsizlikda $(a_1,\ a_2,\ a_3)=\left(\frac{1}{b+c},\frac{1}{c+a},\frac{1}{a+b}\right)$, $(b_1,\ b_2,\ b_3)=(a,b,c)$, $(x_1,\ x_2,\ x_3)=(b,c,a)$ deb olsak,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{b}{b+c}+\frac{c}{c+a}+\frac{a}{a+b}$$

tengsizlikka,

$(x_1,\ x_2,\ x_3)=(c,a,b)$ deb olsak,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{c}{b+c}+\frac{a}{c+a}+\frac{b}{a+b}$$

tengsizlikka ega bo'lamiz.

Oxirgi ikkita tengsizlikni hadma-had qo'shib va 2 ga bo'lib

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

ni hosil qilamiz.

6-masala. (XMO–1975**)**. Har biri n ta sondan iborat ikkita $a_1, a_2, \dots, a_n,\quad b_1, b_2, \dots, b_n$ ketma-ketlik berilgan bo'lib, ular $a_1\leq a_2\leq\dots\leq a_n,\quad b_1\leq b_2\leq\dots\leq b_n$ shartni qanoatlantirsin.

$$(a_1-b_1)^2+(a_2-b_2)^2+\dots+(a_n-b_n)^2\leq(a_1-c_1)^2+(a_2-c_2)^2+\dots+(a_n-c_n)^2$$

tengsizlikni isbotlang, bu yerda $(c_1, c_2, \dots, c_n)$ – $(b_1, b_2, \dots, b_n)$ ning o'rin almashtirishi.

Yechilishi. Sodda hisob-kitoblardan so'ng berilgan tengsizlikni

$$a_1b_1+a_2b_2+\dots+a_nb_n\geq a_1c_1+a_2c_2+\dots+a_nc_n$$

ko'rinishga keltiramiz. Bu tengsizlik esa (1) qo'sh tengsizlikdagi chap tengsizlikning o'zi.

7-masala. (XMO–1995**)**. $abc=1$ shartni qanoatlantiruvchi ixtiyoriy $a,\ b,\ c$ musbat sonlar uchun

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$$

tengsizlikni isbotlang.

Yechilishi. $a\geq b\geq c$ deb faraz qilamiz. U holda $\frac{1}{c}\geq\frac{1}{b}\geq\frac{1}{a}$ munosabatdan

$$\frac{1}{ac+bc}\geq\frac{1}{ab+bc}\geq\frac{1}{ab+ac}$$

tengsizliklarga ega bo'lamiz.

Shuning uchun

$$\frac{1}{c(ac+bc)} \geq \frac{1}{b(ab+bc)} \geq \frac{1}{a(ab+ac)}.$$

$ab \geq ac \geq bc$ bo'lgani bois (ab,ac,bc) va $\left(\frac{1}{c(ac+bc)},\frac{1}{b(ab+bc)},\frac{1}{a(ab+ac)}\right)$ uchliklar bir xil tartiblangan bo'ladi.

(2) tengsizlikda

$$(a_1,\ a_2,\ a_3)=\left(\frac{1}{c(ac+bc)},\frac{1}{b(ab+bc)},\frac{1}{a(ab+ac)}\right),$$

$$(b_1,\ b_2,\ b_3)=(ab,ac,bc),\ (x_1,\ x_2,\ x_3)=(ac,bc,ab)$$

deb olsak,

$$\frac{ab}{c(ac+bc)}+\frac{ac}{b(ab+bc)}+\frac{bc}{a(ab+ac)} \geq \frac{ac}{c(ac+bc)}+\frac{bc}{b(ab+bc)}+\frac{ab}{a(ab+ac)}$$

tengsizlikka,

$(x_1,\ x_2,\ x_3)=(bc,ab,ac)$ deb olsak,

$$\frac{ab}{c(ac+bc)}+\frac{ac}{b(ab+bc)}+\frac{bc}{a(ab+ac)} \geq \frac{bc}{c(ac+bc)}+\frac{ab}{b(ab+bc)}+\frac{ac}{a(ab+ac)}$$

tengsizlikka ega bo'lamiz.

Ularni hadma-had qo'shib

$$2\left(\frac{ab}{c(ac+bc)}+\frac{ac}{b(ab+bc)}+\frac{bc}{a(ab+ac)}\right) \geq \frac{1}{c}+\frac{1}{b}+\frac{1}{a}$$

ga ega bo'lamiz. $abc=1$ shartni hisobga olib, o'rta qiymatlar haqidagi Koshi tengsizligiga ko'ra

$$\frac{1}{c}+\frac{1}{b}+\frac{1}{a} \geq 3\sqrt[3]{\frac{1}{c}\cdot\frac{1}{b}\cdot\frac{1}{a}}=3.$$

Demak,

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}=\frac{ab}{c(ac+bc)}+\frac{ac}{b(ab+bc)}+\frac{bc}{a(ab+ac)}\geq\frac{3}{2}.$$

8-masala. (XMO –1978**)**. $\{a_1, a_2, ..., a_n\}$ – turli natural sonlardan iborat ketma-ketlik bo'lsin.

$$\sum_{k=1}^{n}\frac{a_k}{k^2}\geq\sum_{k=1}^{n}\frac{1}{k}$$

tengsizlik bajarilishini isbotlang.

Yechilishi. $(i_1, i_2, ..., i_n)$ – 1, 2, ..., n sonlarining shunday o'rin almashtirishi bo'lsinki, ular uchun $a_{i_1}<a_{i_2}<...<a_{i_n}$ bajarilsin.

$$\frac{1}{n^2}<\frac{1}{(n-1)^2}<\ldots<\frac{1}{1^2}$$

bo'lgani uchun, (1) tengsizlikka ko'ra

$$\sum_{k=1}^{n}\frac{a_k}{k^2}\geq\sum_{k=1}^{n}\frac{a_{i_k}}{k^2}.$$

Ravshanki, $a_{i_k}\geq k, k=1,2,...,n$.

Bundan $\sum_{k=1}^{n}\frac{a_k}{k^2}\geq\sum_{k=1}^{n}\frac{a_{i_k}}{k^2}\geq\sum_{k=1}^{n}\frac{k}{k^2}=\sum_{k=1}^{n}\frac{1}{k}$.

9-masala. (XMO-1964). a, b, c – biror uchburchakning tomonlari uzunliklari bo'lsin.

$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\leq 3abc$$

tengsizlikni isbotlang.

Yechilishi. $a\geq b\geq c$ deb faraz qilamiz. Dastlab quyidagini isbotlaymiz:

$$a(b+c-a)\leq b(c+a-b)\leq c(a+b-c).$$

Buning uchun

$$c(a+b-c)-b(c+a-b)=(b-c)(b+c-a)\geq 0,$$

$$b(c+a-b)-a(b+c-a)=(a-b)(a+b-c)\geq 0\,.$$

ekanligini eslatish kifoya.

Demak, (a,b,c) va $\big(a(b+c-a),b(c+a-b),c(a+b-c)\big)$ uchliklar turlicha tartiblangan bo'ladi .

(3) tengsizlikda

$(a_1,\ a_2,\ a_3)=\big(a(b+c-a),b(c+a-b),c(a+b-c)\big)$, $(b_1,\ b_2,\ b_3)=(a,b,c)$,

$(x_1,\ x_2,\ x_3)=(b,c,a)$ deb olsak,

$$\begin{aligned}&a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\leq\\&\leq ba(b+c-a)+cb(c+a-b)+ac(a+b-c)\end{aligned}$$

tengsizlikka,

$(x_1,\ x_2,\ x_3)=(c,a,b)$ deb olsak,

$$\begin{aligned}&a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\leq\\&\leq ca(b+c-a)+ab(c+a-b)+bc(a+b-c).\end{aligned}$$

tengsizlikka ega bo'lamiz.

Ohirgi ikkita tengsizliklarni qo'shib va soddalashtirib, berilgan tengsizlikni hosil qilamiz.

10-masala. (XMO-1983). a, b, c – biror uchburchakning tomonlari uzunliklari bo'lsin.

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0$$

tengsizlikni isbotlang.

Yechilishi. Umumiylikka putur etkazmagan holda $a\geq b$ deb olamiz.

Agar $a\geq b\geq c$ bo'lsa, u holda $\frac{1}{a}\leq\frac{1}{b}\leq\frac{1}{c}$ va oldingi masala yechimidan

$$c(a+b-c)\geq b(c+a-b)\geq a(b+c-a)\,.$$

ga ega bo'lamiz.

Ya’ni $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ va $\left(a(b+c-a),b(c+a-b),c(a+b-c)\right)$ uchliklar bir xil tartiblangan.

(2) tengsizlikda

$$(a_1,\ a_2,\ a_3)=\left(a(b+c-a),b(c+a-b),c(a+b-c)\right),\ (b_1,\ b_2,\ b_3)=\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right),$$

$$(x_1,\ x_2,\ x_3)=\left(\frac{1}{c},\frac{1}{a},\frac{1}{b}\right)$$

deb olsak,

$$a+b+c=\frac{1}{a}a(b+c-a)+\frac{1}{b}b(c+a-b)+\frac{1}{c}c(a+b-c)\geq$$
$$\geq\frac{1}{c}a(b+c-a)+\frac{1}{a}b(c+a-b)+\frac{1}{b}c(a+b-c)$$

tengsizlikni hosil qilamiz.

Soddalashtirishlardan so’ng bu tengsizlik berilgan tengsizlikka tengkuchli bo’lgan ushbu

$$\frac{1}{c}a(b-a)+\frac{1}{a}b(c-b)+\frac{1}{b}c(a-c)\leq 0$$

tengsizlikka keladi.

$a\geq c\geq b$ holni tahlil qilishni o’quvchilarga qoldiramiz.

11-masala. (4-Xalqaro Jautikov olimpiadasi, Almati, 2008 yil)

$abc=1$ shartni qanoatlantiruvchi ixtiyoriy $a,\ b,\ c$ musbat sonlar uchun

$$\frac{1}{(a+b)b}+\frac{1}{(b+c)c}+\frac{1}{(c+a)a}\geq\frac{3}{2}$$

tengsizlikni isbotlang.

Yechilishi. Tengsizlikning chap tomonini S orqali belgilaymiz.

$a \geq b \geq c$ deb faraz qilamiz. U holda $\frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c}$ va $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$ tengsizliklar o'rinli, ya'ni $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ va $\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$ uchliklar turlicha tartiblangan bo'ladi.

(3) tengsizlikda

$$(a_1,\ a_2,\ a_3)=\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right),\ (b_1,\ b_2,\ b_3)=\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right),\ (x_1,\ x_2,\ x_3)=\left(\frac{1}{c}, \frac{1}{a}, \frac{1}{b}\right)$$

deb olsak

$$T=\frac{1}{a(b+c)}+\frac{1}{b(c+a)}+\frac{1}{c(a+b)} \leq \frac{1}{c(b+c)}+\frac{1}{a(c+a)}+\frac{1}{b(a+b)}=S$$

tengsizlikga ega bo'lamiz.

O'rta qiymat haqidagi Koshi tengsizligini va $abc=1$ shartni hisobga olib, quyidagilarga ega bo'lamiz:

$$2S \geq S+T=\left(\frac{1}{(a+b)b}+\frac{1}{(a+b)c}\right)+\left(\frac{1}{(b+c)c}+\frac{1}{(b+c)a}\right)\left(\frac{1}{(c+a)a}+\frac{1}{(c+a)b}\right)=$$

$$=\frac{b+c}{(a+b)bc}+\frac{c+a}{(b+c)ca}+\frac{a+b}{(c+a)ab} \geq 3\sqrt[3]{\frac{b+c}{(a+b)bc} \cdot \frac{c+a}{(b+c)ca} \cdot \frac{a+b}{(c+a)ab}}=3$$

Bundan $\frac{1}{(a+b)b}+\frac{1}{(b+c)c}+\frac{1}{(c+a)a} \geq \frac{3}{2}$ kelib chiqadi.

3-§. Klassik tengsizliklarni isbotlashda trans-tengsizlikni qo'llash.

Barcha $a_1, ..., a_n$ sonlar uchun (1) tengsizlikning muhim xususiy hollarini ta'kidlab o'tamiz:

$$\frac{b_1}{a_1}+\frac{b_2}{a_2}+...+\frac{b_n}{a_n}\geq n \qquad (4)$$

$$a_1^2+a_2^2+...+a_n^2\geq a_1b_1+a_2b_2+...+a_nb_n \qquad (5)$$

bu yerda n - ixtiyoriy natural son , $(b_1, ..., b_n)$ – $a_1, a_2, …, a_n$ sonlarning ixtiyoriy o'rin almashtirishi.

1-misol (O'rta qiymatlar haqidagi Koshi tengsizligi).

$x_1, x_2, ..., x_n$ musbat sonlar uchun

$$\frac{x_1+x_2+...+x_n}{n}\geq \sqrt[n]{x_1x_2...x_n} \ ,$$

tengsizlik o'rinli, shu bilan birga tenglik $x_1= x_2 =...= x_n$ bo'lgandagina bajariladi.

Yechilishi. $G=\sqrt[n]{x_1x_2...x_n}$, $a_1=\frac{x_1}{G}$, $a_2=\frac{x_1x_2}{G^2}$, ..., $a_n=\frac{x_1x_2...x_n}{G^n}=1$ bo'lsin.

(4) tengsizlikka binoan $\frac{x_1+x_2+...+x_n}{n}\geq G$ tengsizlikka teng ekvivalent bo'lgan ushbu

$$n\leq\frac{a_1}{a_n}+\frac{a_2}{a_1}+...+\frac{a_n}{a_{n-1}}=\frac{x_1}{G}+\frac{x_2}{G}+...+\frac{x_n}{G}$$

tengsizlikka egamiz. Tenglik bajarilishi uchun $a_1= a_2 =...= a_n$ ya'ni $x_1= x_2 =...= x_n$ bo'lishi zarur va etarli.

2-misol.(O'rta geometrik va o'rta garmonik qiymatlar orasidagi tengsizlik)

$x_1, x_2, ..., x_n$ musbat sonlar uchun

$$\sqrt[n]{x_1 x_2 \ldots x_n} \geq \frac{n}{x_1^{-1} + x_2^{-1} + \ldots + x_n^{-1}}$$

tengsizlik o'rinli, shu bilan birga tenglik $x_1 = x_2 = \ldots = x_n$ bo'lgandagina bajariladi.

Yechilishi. Oldingi misoldagi $G, a_1, a_2, \ldots, a_n$ sonlarni qaraymiz. (4) tengsizlikka binoan

$$\frac{n}{x_1^{-1} + x_2^{-1} + \ldots + x_n^{-1}} \leq G$$

tengsizlikka teng ekvivalent bo'lgan ushbu

$$n \leq \frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_n}{a_1} = \frac{G}{x_1} + \frac{G}{x_2} + \ldots + \frac{G}{x_n}$$

tengsizlikka egamiz.

Tenglik bajarilishi uchun $a_1 = a_2 = \ldots = a_n$ ya'ni

$$x_1 = x_2 = \ldots = x_n$$

bo'lishi zarur va etarli.

3-misol. (O'rta kvadratik va o'rta arifmetik qiymatlar orasidagi tengsizlik)

Ixtiyoriy $x_1, x_2, \ldots, x_n$ sonlar uchun

$$\sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \ldots + x_n}{n} \quad ,$$

tengsizlik o'rinli, shu bilan birga tenglik $x_1 = x_2 = \ldots = x_n$ bo'lgandagina bajariladi.

Yechilishi.

(5) tengsizlikka ko'ra

$$x_1^2 + x_2^2 + \ldots + x_n^2 \geq x_1 x_2 + x_2 x_3 + \ldots + x_n x_1$$

$$x_1^2 + x_2^2 + \ldots + x_n^2 \geq x_1 x_3 + x_2 x_4 + \ldots + x_n x_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_1^2 + x_2^2 + \ldots + x_n^2 \geq x_1 x_n + x_2 x_1 + \ldots + x_n x_{n-1}$$

munosabatlarga ega bo'lamiz.

Bu tengsizliklarni barchasini

$x_1^2 + x_2^2 + ... + x_n^2 = x_1^2 + x_2^2 + ... + x_n^2$ tenglik bilan qo'shib, natijada

$$n(x_1^2 + x_2^2 + ... + x_n^2) \geq (x_1 + x_2 + ... + x_n)^2$$

tengsizlikni hosil qilamiz.

4-misol. (Koshi-Bunyakovskiy-Shvarts tengsizligi)

n sondan iborat ikkita a_1, a_2, …, a_n, b_1, b_2, ... , b_n ketma-ketlik berilgan bo'lsin. U holda

$$(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)\,(b_1^2 + b_2^2 + ... + b_n^2)$$

tengsizlik o'rinli. Tenglik biror o'zgarmas k son uchun $a_i = kb_i, i = 1, 2, ..., n$, bo'lgandagina bajariladi.

Yechilishi. Agar $a_1 = a_2 = … = a_n = 0$ yoki $b_1 = b_2 = … = b_n = 0$ bo'lsa, u holda tengsizlik bajariladi. Shuning uchun

$$P = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}\,,\; Q = \sqrt{b_1^2 + b_2^2 + ... + b_n^2}$$

sonlarni noldan farqli deb hisoblaymiz.

Quyidagicha aniqlangan $x_1, x_2, ..., x_{2n}$ ketma-ketlikni qaraymiz:

$$x_i = \frac{a_i}{P}, x_{n+i} = \frac{b_i}{Q}, i = 1, 2, ..., n.$$

U holda

$$2 = \frac{a_1^2 + a_2^2 + ... + a_n^2}{P^2} + \frac{b_1^2 + b_2^2 + ... + b_n^2}{Q^2} = x_1^2 + x_2^2 + ... + x_{2n}^2$$

ga egamiz.

(5) tengsizlikka ko'ra

$$x_1^2 + x_2^2 + ... + x_{2n}^2 \geq x_1x_{n+1} + x_2x_{n+2} + ... + x_nx_{2n} + x_{n+1}x_1 + x_{n+2}x_2 + ... + x_{2n}x_n =$$

$$= \frac{2(a_1b_1 + a_2b_2 + ... + a_nb_n)}{PQ}$$

ga egamiz. Natijada

$$1 \geq \frac{a_1b_1 + a_2b_2 + ... + a_nb_n}{PQ}$$

tengsizlikni hosil qilamiz.

Eslatib o'tamiz, tenglik $a_i = \frac{P}{Q} b_i, i = 1, 2, ..., n$, shart bajarilganda bo'ladi. Bu shart esa $x_i = x_{n+i}$, $i = 1, 2, ..., n$ shartiga ekvivalent.

5-misol. (Chebishev tengsizligi).

n sondan iborat ikkita a_1, a_2, …, a_n, b_1, b_2, ... , b_n ketma-ketliklar berilgan bo'lsin. Faraz qilamiz $a_1 \geq a_2 \geq ... \geq a_n$ shart bajarilsin.

U holda

a) $\frac{a_1 + a_2 + ... + a_n}{n} \cdot \frac{b_1 + b_2 + ... + b_n}{n} \leq \frac{a_1b_1 + a_2b_2 + ... + a_nb_n}{n}$, agar $b_1 \geq b_2 \geq ... \geq b_n$

b) $\frac{a_1 + a_2 + ... + a_n}{n} \cdot \frac{b_1 + b_2 + ... + b_n}{n} \geq \frac{a_1b_1 + a_2b_2 + ... + a_nb_n}{n}$, agar $b_1 \leq b_2 \leq ... \leq b_n$

Isbot.

a) (5) tengsizlikka ko'ra

$$a_1b_1 + a_2b_2 + ... + a_nb_n = a_1b_1 + a_2b_2 + ... + a_nb_n$$

$$a_1b_1 + a_2b_2 + ... + a_nb_n \geq a_1b_2 + a_2b_3 + ... + a_nb_1$$

$$a_1b_1 + a_2b_2 + ... + a_nb_n \geq a_1b_3 + a_2b_4 + ... + a_nb_2$$

……………………………………………

$$a_1b_1 + a_2b_2 + ... + a_nb_n \geq a_1b_n + a_2b_1 + ... + a_nb_{n-1}$$

munosabatlarga egamiz, ularni qo'shib

$n(a_1b_1 + a_2b_2 + ... + a_nb_n) \geq (a_1 + a_2 + ... + a_n) \cdot (b_1 + b_2 + ... + b_n)$ yoki

$$\frac{a_1 + a_2 + ... + a_n}{n} \cdot \frac{b_1 + b_2 + ... + b_n}{n} \leq \frac{a_1b_1 + a_2b_2 + ... + a_nb_n}{n}$$

ni hosil qilamiz.

b) holi shunga o’xshash isbotlanadi.

3-BOB. KARAMATA TENGSIZLIGI.

Ta'rif: $x=(x_1,x_2,...,x_n)$ va $y=(y_1,y_2,...,y_n)$ n-liklar quyidagi shartlarni qanoatlantirsin:

1. $x_1 \ge x_2 \ge ... \ge x_n$ va $y_1 \ge y_2 \ge ... \ge y_n$

2. $\sum_{i=1}^{k} x_i \ge \sum_{i-1}^{k} y_i,\quad k=1,...,n-1$, $\sum_{i=i}^{n} x_i = \sum_{i=i}^{n} y_i$, ya'ni

$$x_1 \ge y_1$$

$$x_1+x_2 \ge y_1+y_2$$

....

....

$$x_1+x_2+...+x_{n-1} \ge y_1+y_2+...+y_{n-1}$$

$$x_1+x_2+...+x_n \quad = \quad y_1+y_2+...+y_n$$

Bu holda $x=(x_1,x_2,...,x_n)$ n-lik $y=(y_1,y_2,...,y_n)$ n-likni *majorlaydi* deyiladi va bu munosabat $x \succ y$ yoki $y \prec x$ kabi yoziladi.

Misollar:

1. $$\left(\frac{1}{n},...\frac{1}{n}\right) \prec \left(\frac{1}{n-1},...\frac{1}{n-1},0\right) \prec ... \prec \left(\frac{1}{2},\frac{1}{2},0,...0\right) \prec (1,0,...0).$$

2. Agar $m \ge l$ va $c \ge 0$ bo'lsa, u holda

$$\left(\underbrace{\frac{l}{m}c,...,\frac{l}{m}c}_{m\ \text{марта}},0,...,0\right) \prec \left(\underbrace{c,...,c}_{l\ \text{марта}},0,...,0\right)$$

munosabat o'rinli.

3. Agar $a_i \ge 0$ va $\sum_{i=1}^{n} a_i = 1$ bo'lsa, u holda

$$\left(\frac{1}{n},...,\frac{1}{n}\right) \prec (a_1,...,a_n) \prec (1,0,...,0)$$

munosabat o'rinli.

4. Agar $c \geq 0$ bo'lsa, u holda

$$\frac{1}{\sum_{i=1}^{n} x_i + nc}(x_1 + c, ..., x_n + c) \prec \frac{1}{\sum_{i=1}^{n} x_i}(x_1, ..., x_n)$$

munosabat o'rinli.

5. Agar $y_1 = y_2 = ... = y_n = \frac{x_1 + x_2 + ... + x_n}{n}$ bo'lsa $(x_1, x_2, ..., x_n) \succ (y_1, y_2, ..., y_n)$ bo'ladi.

6. α, β, γ - uchburchak burchaklari bo'lsin, u holda

A) barcha uchburchaklar uchun

$$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) \prec (\alpha, \beta, \gamma) \prec (\pi, 0, 0)$$

munosabat;

B) o'tkir burchakli uchburchaklar uchun

$$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) \prec (\alpha, \beta, \gamma) \prec \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$$

munosabat;

C) o'tmas burchakli uchburchaklar uchun

$$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) \prec (\alpha, \beta, \gamma) \prec (\pi, \pi, 0)$$

munosabat o'rinli.

Lemma (uch vatar haqida). f - qavariq funksiya bo'lsin. U holda uchun har qanday $z < y < z$ uchun

$$\frac{f(y) - f(z)}{y - z} \leq \frac{f(x) - f(x)}{x - z} \leq \frac{f(x) - f(y)}{x - y}$$

qo'shtengsizlik o'rinli.

Isbot: f - qavariq funksiya bo'lganligi uchun

$$f(\lambda x + (1-\lambda)z) \leq \lambda(x) + (1-\lambda)f(z)$$

tengsizlik bajariladi, bu yerda $\lambda \in (0,1)$.

$\lambda = \frac{y-z}{x-z}$ deb olamiz va soddalashtirishlardan so'ng yuqoridagi tengsizlik

$$(x-z)f(y) \le (x-y)f(z) + (y-z)f(x)$$

tengsizlikka olib kelinadi.

Bu tengsizlik esa

$$\frac{f(y)-f(z)}{y-z} \le \frac{f(x)-f(x)}{x-z} \le \frac{f(x)-f(y)}{x-y}$$

ikkala ham tengsizlikka tengkuchli.

Natija. Qavariq f funksiya berilgan bo'lsin. U holda uchun har qanday $x_1 \ge x_2$, $y_1 \ge y_2$, $x_1 \ne y_1$, $x_2 \ne y_2$ uchun

$$\frac{f(x_1)-f(y_1)}{x_1-y_1} \ge \frac{f(x_2)-f(y_2)}{x_2-y_2}$$

tengsizlik bajariladi.

Lemma (Abel' almashtirishi). $A_k = \sum_{i=1}^{k} a_i$ bo'lsa, u holda $\sum_{k=1}^{n} a_k b_k = \sum_{k-1}^{n-1} A_k (b_k - b_{k+1}) + A_n b_n$ tenglik o'rinli.

Isbot.

$$a_1b_1 + a_2b_2 + ... + a_{n-1}b_{n-1} + a_nb_n = A_1b_1 + (A_2 - A_1)b_2 + ... + (A_{n-1} - A_{n-2})b_{n-1} + (A_n - A_{n-1})b_n =$$
$$= A_1(b_1 - b_2) + A_2(b_2 - b_3) + ... + A_{n-1}(b_{n-1} - b_n) + A_nb_n.$$

Teorema (Karamata tengsizligi**).** Qavariq (mos ravishda botiq) f funksiya berilgan bo'lsin. Agar $x \prec y$ bo'lsa

$$\sum_{i=1}^{n} f(x_i) \le \sum_{i=1}^{n} f(y_i) \ (1)$$

$$(\sum_{i=1}^{n} f(x_i) \ge \sum_{i=1}^{n} f(y_i)\ ).(1')$$

tengsizlik bajariladi.

Isbot: Qavariq f funksiya holini qarash etarli. Umumiylikka putur etkazmasdan $x_k \neq y_k$ deb hisoblashimiz mumkin.

$D_k = \dfrac{f(y_k) - f(x_k)}{y_k - x_k}$, $X_k = \sum_{i=1}^{k} x_i$, $Y_k = \sum_{i=1}^{k} y_i$ belgilashlarni kiritamiz.

U holda $Y_k \geq X_k, Y_n = X_n$.

Uch vatar haqida lemma natijasiga ko'ra $D_k \geq D_{k+1}$.

Demak, $\sum_{k=1}^{n-1} (Y_k - X_k) \cdot (D_k - D_{k+1}) + (X_n - Y_n) D \geq 0$.

Abel' almashtirishini qo'llab, $\sum_{k=1}^{n} (y_k - x_k) \cdot D_k \geq 0$ ni hosil qilamiz. Teorema isbot bo'ldi.

<u>*Eslatma*</u> *1.* Isbot qilingan tengsizlikka Karamata nomi berilishi unchalik to'g'ri emas. 1923 yilda Shur bu tengsizlikni majorlash shartini boshqacharok ifodalab isbotladi. 1920 yilda Xardi, Littlvud va Polia bu tengsizlikni ifodaladilar va uning uzluksiz analogini isbotladilar. 3 yildan keyin Karamata bu tengsizlikni umumiy holda isbotladi.

Karamata tengsizligidan foydalangan holda isbotlash mumkin bo'lgan ikkita tengsizliklarni ko'rib chiqamiz.

Misollar. 1. (Yensen tengsizligi). Agar f -qavariq funksiya bo'lsa,

$$\frac{\sum_{i=1}^{n} f(x_i)}{n} \geq f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)$$

tengsizlik o'rinli bo'ladi.

Isbot. $y_1 = y_2 = ... = y_n = \dfrac{x_1 + x_2 + ... + x_n}{n}$ deb olamiz. $(x_1, x_2, ..., x_n) \succ (y_1, y_2, ..., y_n)$ bo'lgani uchun Karamata tengsizligidan bevosita Yensen tengsizligi kelib chiqadi.

2. Ixtiyoriy musbat a,b,c lar uchun $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\le\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}$.

Isbot. $(2a,2b,2c)\succ(a+b,a+c,b+c)$ ga egamiz. Karamata tengsizligini $f(x)=\frac{1}{x}$ funksiya uchun qo'llash etarli.

4-BOB. TENGSIZLIKLARNI TRIGONOMETRIK ALMASHTIRISHLAR YORDAMIDA ISBOTLASH

1-§. Trigonometrik almashtirishlar

Ba'zida tengsizlikni isbotlashda trigonometrik almashtirish olish yaxshi foyda beradi. Almashtirish qulay olinganda tengsizlik darhol isbotlanadigan, oddiy shaklga kelib qoladi. Shuningdek trigonometrik funksiyalarning yaxshi ma'lum bo'lgan xossalari yordam berishi mumkin.

Biz dastlab bunday almashtirishlarni kiritamiz, so'ng ma'lum bo'lgan trigonometrik ayniyatlar va tengsizliklarni keltiramiz va nihoyat bir nechta olimpiada masalalarini muhokama qilamiz.

Teorema 1. Faraz qilaylik α,β,γ burchaklar $(0;\pi)$ dan olingan. U holda bu α,β,γ burchaklar biror uchburchakning ichki burchaklari bo'lishi uchun quyidagi tenglikning bajarilishi zarur va etarli

$$tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma}{2}+tg\frac{\gamma}{2}tg\frac{\alpha}{2}=1$$

Isbot. Dastlab shuni ta'kidlash joizki $\alpha=\beta=\gamma$ bo'lgan holda teoremaning tasdig'i o'rinlidir. Umumiylikka ziyon etkazmasdan $\alpha\neq\beta$ deb faraz qilaylik. $0<\alpha+\beta<2\pi$ bo'lganligi uchun $(-\pi;\pi)$ intervalda $\alpha+\beta+\gamma^1=\pi$ shartni qanoatlantiruvchi γ^1 mavjud.

Qo'shish formulalari va $tgx=ctg\left(\frac{\pi}{2}-x\right)$ formulaga ko'ra

$$tg\frac{\gamma^1}{2}=ctg\frac{\alpha+\beta}{2}=\frac{1-tg\frac{\alpha}{2}tg\frac{\beta}{2}}{tg\frac{\alpha}{2}+tg\frac{\beta}{2}},$$

ya'ni

$$tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma^1}{2}+tg\frac{\gamma^1}{2}tg\frac{\alpha}{2}=1 \qquad (1)$$

tenglik o'rinli bo'ladi. Faraz qilaylik biror $\alpha,\beta,\gamma\in(0;\pi)$ burchaklar uchun

$$tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma}{2}+tg\frac{\gamma}{2}tg\frac{\alpha}{2}=1 \qquad (2)$$

tenglik o'rinli bo'lsin.

Biz isbotlaymizki $\gamma=\gamma^1$ va bu bizga α,β,γ lar biror uchburchak burchaklari ekanligini beradi. (1) dan (2) ni ayirib $tg\frac{\gamma}{2}=tg\frac{\gamma^1}{2}$ ni hosil qilamiz. Shuning uchun $\left|\frac{\gamma-\gamma^1}{2}\right|=k\pi, k\geq 0, k\in Z$. Ammo $\left|\frac{\gamma-\gamma^1}{2}\right|\leq\left|\frac{\gamma}{2}\right|+\left|\frac{\gamma^1}{2}\right|<\pi$ tengsizlik o'rinli. Demak, $k=0$, shuning uchun $\gamma=\gamma^1$. Tasdiq isbotlandi.

Teorema 2. Faraz qilaylik α,β,γ burchaklar $(0;\pi)$ dan olingan. U holda bu α,β,γ burchaklar biror uchburchakning ichki burchaklari bo'lishi uchun quyidagi tenglikning bajarilishi zarur va etarli

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=1$$

Isbot. $0<\alpha+\beta<2\pi$ bo'lganligi uchun shunday $\gamma^1\in(-\pi;\pi)$ mavjudki $\alpha+\beta+\gamma^1=\pi$ tenglik o'rinli bo'ladi. Ko'paytmani yig'indiga keltirish va ikkilangan burchak formulalariga asosan quyidagi munosabatlar o'rinli

$$\sin^2\frac{\gamma^1}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma^1}{2}=\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha+\beta}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)=\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=$$

$$=\frac{\cos\alpha+\cos\beta}{2}=\frac{\left(1-2\sin^2\frac{\alpha}{2}\right)+\left(1-2\sin^2\frac{\beta}{2}\right)}{2}=1-\sin^2\frac{\alpha}{2}-\sin^2\frac{\beta}{2}.$$

Shunday qilib,

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma^1}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma^1}{2}=1 \qquad (1)$$

Faraz qilaylik biror $\alpha,\beta,\gamma\in(0;\pi)$ burchaklar uchun

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=1 \qquad (2)$$

tenglik o'rinli bo'lsin. (1)dan (2) ni ayirib,

$$\sin^2\frac{\gamma}{2}-\sin^2\frac{\gamma^1}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\left(\sin\frac{\gamma}{2}-\sin\frac{\gamma^1}{2}\right)=0,$$

ya'ni

$$\left(\sin\frac{\gamma}{2}-\sin\frac{\gamma^1}{2}\right)\left(\sin\frac{\gamma}{2}+\sin\frac{\gamma^1}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)=0.$$

munosabatni hosil qilamiz.

Ikkinchi qavs ichidagi ifodani quyidagicha ifodalaymiz

$$\sin\frac{\gamma}{2}+\sin\frac{\gamma^1}{2}+\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}=\sin\frac{\gamma}{2}+\cos\frac{\alpha-\beta}{2}$$

Ravshanki bu ifoda musbat qiymatlar qabul qiladi. Shuning uchun $\sin\frac{\gamma}{2}=\sin\frac{\gamma^1}{2}$, ya'ni $\gamma=\gamma^1$ bo'ladi. Demak, $\alpha+\beta+\gamma=\pi$. Tasdiq isbotlandi.

Almashtirishlar

A1. Faraz qilaylik α,β,γ lar uchburchakning ichki burchaklari bo'lsin.Quyidagicha almashtirishni qaraylik

$$A=\frac{\pi-\alpha}{2},\ B=\frac{\pi-\beta}{2},\ C=\frac{\pi-\gamma}{2}.$$

Ravshanki $A+B+C=\pi$ va $0\le A,B,C<\frac{\pi}{2}$. Bu almashtirish bizga biror masalani hal qilishda istalgan uchburchak o'rniga o'tkir burchakli uchburchakni qarash imkonini beradi. Quyidagi munosabatlar o'rinli ekanligini ta'kidlash joiz:

$$\sin\frac{\alpha}{2}=\cos A,\ \cos\frac{\alpha}{2}=\sin A,\ tg\frac{\alpha}{2}=ctgA,\ ctg\frac{\alpha}{2}=tgA$$

A2. Faraz qilaylik x, y, z lar musbat haqiqiy sonlar bo'lsin. U holda tomonlari uzunliklari $a=x+y, b=y+z, c=z+x$ lardan iborat bo'lgan uchburchak mavjud. $s=x+y+z$ bo'lsa, $(x,y,z)=(s-a,s-b,s-c)$. Shartga ko'ra x,y,z lar musbatligi uchun $s-a,\ s-b,\ s-c$ lar uchburchak tengsizligini qanoatlantiradi.

A3. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ca=1$ shartni qanoatlantirsin. Biz ushbu $f:\left(0;\frac{\pi}{2}\right)\to(0;+\infty)$, $f(x)=tgx$ funksiya yordamida quyidagicha almashtirish kiritishimiz mumkin

$$a=tg\frac{\alpha}{2}, b=tg\frac{\beta}{2}, c=tg\frac{\gamma}{2}$$

bunda α,β,γ lar biror uchburchakning burchaklari.

A4. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ca=1$ shartni qanoatlantirsin. A1 va A3 larga ko'ra quyidagilarga egamiz

$$a=ctgA,\ b=ctgB,\ c=ctgC,$$

bunda A,B,C lar o'tkir burchakli uchburchakning burchaklari.

A5. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ca=abc$ shartni qanoatlantirsin. Bu tenglikning ikkala tarafini a,b,c sonlarning ko'paytmasiga bo'lib, quyidagiga ega bo'lamiz $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=1$. A3 ga ko'ra quyidagicha almashtirish olamiz

$$\frac{1}{a}=tg\frac{\alpha}{2},\ \frac{1}{b}=tg\frac{\beta}{2},\ \frac{1}{c}=tg\frac{\gamma}{2}$$

ya'ni

$$a=ctg\frac{\alpha}{2}, b=ctg\frac{\beta}{2}, c=ctg\frac{\gamma}{2}$$

bunda α,β,γ lar biror uchburchakning burchaklari.

A6. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ca=abc$ shartni qanoatlantirsin.

A1 va A5 ga ko'ra

bunda A, B, C lar o'tkir burchakli uchburchakning burchaklari.

A7. Faraz qilaylik musbat a,b,c sonlar $a^2+b^2+c^2+2abc=1$ shartni qanoatlantirsin. Shartga ko'ra uchta son ham musbatligi uchun $a,b,c<1$ bo'ladi. Ushbu $f:(0;\pi)\to(0;1)$, $f(x)=\sin\frac{x}{2}$ funksiya hamda 2-teorema yordamida quyidagicha almashtirish olishimiz mumkin

$$a=\sin\frac{\alpha}{2},\ b=\sin\frac{\beta}{2},\ c=\sin\frac{\gamma}{2}$$

bunda α,β,γ lar biror uchburchakning burchaklari.

A8. Faraz qilaylik musbat a,b,c sonlar $a^2+b^2+c^2+2abc=1$ shartni qanoatlantirsin. A1 va A7 larga ko'ra quyidagicha almashtirish olishimiz mumkin

$$a=\cos A,\ b=\cos B,\ c=\cos C,$$

bunda A, B, C lar o'tkir burchakli uchburchakning burchaklari.

A9. Faraz qilaylik x, y, z lar musbat sonlar bo'lsin. A2 yordamida quyidagi

$$\sqrt{\frac{yz}{(x+y)(x+z)}},\ \sqrt{\frac{zx}{(y+z)(y+x)}},\ \sqrt{\frac{xy}{(z+x)(z+y)}}$$

ifodalarni ushbu

$$\sqrt{\frac{(s-b)(s-c)}{bc}},\ \sqrt{\frac{(s-c)(s-a)}{ca}},\ \sqrt{\frac{(s-a)(s-b)}{ab}}$$

ifodalarga almashtiramiz. Quyidagi ayniyatlarga ko'ra

$$\sin\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},\ \cos\frac{\alpha}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

bizning dastlabki ifodalarimiz mos ravishda quyidagi shaklga keladi

$\sin\frac{\alpha}{2},\ \sin\frac{\beta}{2},\ \sin\frac{\gamma}{2},$

bunda α,β,γ lar biror uchburchakning burchaklari.

A10. Xuddi A9 dagi kabi quyidagi

$$\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}},\ \sqrt{\frac{y(x+y+z)}{(y+z)(y+x)}},\ \sqrt{\frac{z(x+y+z)}{(z+x)(z+y)}},$$

ifodalarni mos ravishda ushbu

$$\cos\frac{\alpha}{2},\ \cos\frac{\beta}{2},\ \cos\frac{\gamma}{2},$$

bunda α,β,γ lar biror uchburchakning burchaklari.

Mashq 1. Faraz qilaylik musbat p,q,r sonlar $p^2+q^2+r^2+2pqr=1$ shartni qanoatlantirsin.U holda $p=\cos A, q=\cos B, r=\cos C$ shartni qanoatlantiruvchi o'tkir burchakli ABC uchburchak mavjudligini ko'rsating.

Mashq 2. Faraz qilaylik nomanfiy p,q,r sonlar $p^2+q^2+r^2+2pqr=1$ shartni qanoatlantirsin. U holda $p=\cos A, q=\cos B, r=\cos C$ va $A+B+C=\pi$ shartlarni qanoatlantiruvchi $A,B,C\in\left[0;\frac{\pi}{2}\right]$ burchaklar mavjudligini ko'rsating.

Quyida biz ko'plab masalalarni yechishda yordam beradigan bir qator tengsizliklar va ayniyatlar keltiramiz. Bularning deyarli barchasi yaxshi-ma'lum munosabatlar bo'lib isbotlari qiyin emas. Bu munosabatlarning ko'pchiligining isbotini adabiyotlardan topish mumkin.

Tengsizliklar

Faraz qilaylik α,β,γ lar ABC uchburchakning burchaklari bo'lsin. Quyidagi tengsizliklar o'rinli

1. $\cos\alpha+\cos\beta+\cos\gamma\le\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\le\frac{3}{2}$

2. $\sin\alpha+\sin\beta+\sin\gamma\le\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\le\frac{3\sqrt{3}}{2}$

3. $\cos\alpha\cos\beta\cos\gamma\le\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\le\frac{1}{8}$

4. $\sin\alpha\sin\beta\sin\gamma\le\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\le\frac{3\sqrt{3}}{8}$

5. $ctg\frac{\alpha}{2}+ctg\frac{\beta}{2}+ctg\frac{\gamma}{2}\ge3\sqrt{3}$

6. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma\ge\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}\ge\frac{3}{4}$

7. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma\le\cos^2\frac{\alpha}{2}+\cos^2\frac{\beta}{2}+\cos^2\frac{\gamma}{2}\le\frac{9}{4}$

8. $ctg\alpha+ctg\beta+ctg\gamma\ge tg\frac{\alpha}{2}+tg\frac{\beta}{2}+tg\frac{\gamma}{2}\ge\sqrt{3}$

Ayniyatlar

Faraz qilaylik α,β,γ lar ABC uchburchakning burchaklari bo'lsin. Quyidagi ayniyatlar o'rinli

1. $\cos\alpha+\cos\beta+\cos\gamma=1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

2. $\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

3. $\sin2\alpha+\sin2\beta+\sin2\gamma=4\sin\alpha\sin\beta\sin\gamma$

4. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2+2\cos\alpha\cos\beta\cos\gamma$

Istalgan α,β,γ burchaklar (uchburchak burchaklari bo'lishi shart emas) uchun quyidagi ayniyatlar o'rinli

$$\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$$

$$\cos\alpha+\cos\beta+\cos\gamma+\cos\left(\alpha+\beta+\gamma\right)=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$$

2-§. Trigonometrik almashtirishlarning tadbiqlari

1-masala. (Janubiy Koreya, 1998) Faraz qilaylik musbat x, y, z sonlar $x+y+z=xyz$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\le\frac{3}{2}.$$

Bu masalani yechishda o'quvchini xayoliga eng birinchi $f(t)=\frac{1}{\sqrt{1+t^2}}$ funksiya uchun Iensen tengsizligini qo'llash kelishi mumkin. Ammo bu f funksiya R^+ to'plamda yuqoriga qavariq emas. Ammo shunisi qiziqarliki $f(tg\theta)$ funksiya yuqoriga qavariq!

Isboti. Quyidagicha almashtirish olaylik

$$x=tgA,\ y=tgB,\ z=tgC,\ A,B,C\in(0;\frac{\pi}{2})$$

Ushbu $1+tg^2\alpha=\frac{1}{\cos^2\alpha}$, $\cos\alpha\ne0$ ayniyatga ko'ra berilgan tengsizlik quyidagicha ko'rinishni oladi

$$\cos A+\cos B+\cos C\le\frac{3}{2}$$

Quyidagi $tg(\pi-C)=-z=\frac{x+y}{1-xy}=tg(A+B)$ va $\pi-C, A+B\in(0;\pi)$ munosabatlardan $\pi-C=A+B$ yoki $A+B+C=\pi$ tenglikni olamiz. Demak, istalgan ABC uchburchak uchun $\cos a+\cos B+\cos C\le\frac{3}{2}$ tengsizlikni isbot qilsak etarli ekan.Bu esa quyidagi munosabatdan kelib chiqadi

$$3-2(\cos A+\cos B+\cos C)=(\sin A-\sin B)^2+(\cos A+\cos B-1)^2\ge0.$$

Isbot tugadi.

2-masala. (FML, ochiq olimpiada, Rossiya) Faraz qilaylik musbat x, y, z sonlar $x+y+z=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\sqrt{\frac{xy}{z+xy}}+\sqrt{\frac{yz}{x+yz}}+\sqrt{\frac{zx}{y+zx}}\le\frac{3}{2}$$

Isboti. Yuqoridagi tengsizlik ushbu tengsizlikka teng kuchli

$$\sqrt{\frac{yz}{(x+y)(x+z)}}+\sqrt{\frac{zx}{(y+z)(y+x)}}+\sqrt{\frac{xy}{(z+x)(z+y)}}\le\frac{3}{2}$$

A9 ga ko'ra bu tengsizlikning uchta hadini $\sin\frac{\alpha}{2}, \sin\frac{\beta}{2}, \sin\frac{\gamma}{2}$ larga almashtiramiz va demak, ushbu $\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\le\frac{3}{2}$ tengsizlikni isbotlashimiz kerak. Bu tengsizlikning o'rinli ekanligi ravshan.(Iensen tengsizligidan osongina kelib chiqadi)

Isbot tugadi.

3-masala. (Eron, 1997) Faraz qilaylik x, y, z sonlar $x, y, z>1, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$ shartlarni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\le\sqrt{x+y+z}$$

Isboti. Quyidagicha $(x, y, z)=(a+1, b+1, c+1)$ almashtirish olaylik, bunda $a, b, c>0$ va shartga ko'ra $ab+bc+ca+2abc=1$ tenglik o'rinli.U holda quyidagi tengsizlikni isbotlash etarli

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\le\sqrt{a+b+c+3}\,.$$

Ikkala tarafni kvadratga oshirib va ayrim xadlarni yo'qotib quyidagi tegsizlikka kelamiz

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\le\frac{3}{2}\,.$$

A7 dan foydalanib $(ab,bc,ca)=(\sin^2\frac{\alpha}{2},\sin^2\frac{\beta}{2},\sin^2\frac{\gamma}{2})$ ni olamiz, bunda ABC ixtiyoriy uchburchak. Demak quyidagi tengsizlikni isbotlashimiz kerak

$$\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\le\frac{3}{2}$$

Bu tengsizlikning o'rinli ekanligi ma'lum.Isbot tugadi.

4-masala. (Crux Mathematicorum and Mathematical Mayhem) Faraz qilaylik x,y,z lar musbat sonlar bo'lsin. Quyidagi tengsizlikni isbotlang

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\le 1$$

Isboti. Bu tengsizlik quyidagi tengsizlikka teng kuchli

$$\sum\frac{1}{1+\sqrt{\frac{(x+y)(x+z)}{x^2}}}\le 1$$

Berilgan tengsizlik bir jinsli bo'lganligi uchun umumiylikka ziyon etkazmasdan $xy+yz+zx=1$ deb faraz qilishimiz mumkin. A3 almashtirishdan foydalanamiz

$$\frac{(x+y)(x+z)}{x^2}=\frac{(tg\frac{\alpha}{2}+tg\frac{\beta}{2})(tg\frac{\alpha}{2}+tg\frac{\gamma}{2})}{tg^2\frac{\alpha}{2}}=\frac{1}{\sin^2\frac{\alpha}{2}},$$

qolgan xadlar ham shunga o'xshash ifodalanadi. Tengsizlik quyidagi shaklga keladi

$$\frac{\sin\frac{\alpha}{2}}{1+\sin\frac{\alpha}{2}}+\frac{\sin\frac{\beta}{2}}{1+\sin\frac{\beta}{2}}+\frac{\sin\frac{\gamma}{2}}{1+\sin\frac{\gamma}{2}}\le 1,$$

ya'ni

$$2\le\frac{1}{1+\sin\frac{\alpha}{2}}+\frac{1}{1+\sin\frac{\beta}{2}}+\frac{1}{1+\sin\frac{\gamma}{2}}.$$

Boshqa tomondan yaxshi tanish bo'lgan $\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\le\frac{3}{2}$ tengsizlik va Koshi-Bunyakovskiy-Shvarts tengsizligidan foydalanib quyidagi tengsizlikka ega bo'lamiz

$$2\le\frac{9}{\left(1+\sin\frac{\alpha}{2}\right)+\left(1+\sin\frac{\beta}{2}\right)+\left(1+\sin\frac{\gamma}{2}\right)}\le\sum\frac{1}{1+\sin\frac{\alpha}{2}}$$

Isbot tugadi.

5-masala. (Ruminiya, 2005) Faraz qilaylik musbat a,b,c sonlar $(a+b)(b+c)(c+a)=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$ab+bc+ca\le\frac{3}{4}$$

Isboti. Bu tengsizlik quyidagi tengsizlikka teng kuchli

$$(ab+bc+ca)^3\le\left(\frac{3}{4}\right)^3(a+b)^2(b+c)^2(c+a)^2$$

Bu tengsizlik bir jinsli bo'lganligi uchun umumiylikka ziyon etkazmasdan $ab+bc+ca=1$ deb faraz qilishimiz mumkin. A3 almashtirishdan foydalanamiz

$$(a+b)(b+c)(c+a)=\prod\left(\frac{\cos\frac{\gamma}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}\right)=\frac{1}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}.$$

Demak, ushbu

$$\left(\frac{4}{3}\right)^3\le\frac{1}{\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}}$$

yoki

$$4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\le\frac{3\sqrt{3}}{2}$$

tengsizlikni isbotlash etarli. Ushbu

$$\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

ayniyatga asosan quyidagi tengsizlikni isbotlashimiz kerak

$$\sin\alpha+\sin\beta+\sin\gamma\le\frac{3\sqrt{3}}{2}$$

Bu tengsizlik esa $f(x)=\sin x$ funksiya $(0;\pi)$ intervalda yuqoriga qavariqligi uchun Iensen tengsizligidan kelib chiqadi. Isbot tugadi.

6-masala. (Polsha, 1999) Faraz qilaylik musbat a,b,c sonlar $a+b+c=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$a^2+b^2+c^2+2\sqrt{3abc}\le 1.$$

Isboti. Ushbu $a=xy, b=yz, c=zx$ almashtirish bilan tengsizlik quyidagi shaklga keladi

$$x^2y^2+y^2z^2+z^2x^2+2\sqrt{3}xyz\le 1$$

bunda $x,y,z>0$ va $xy+yz+zx=1$. Yuqoridagi tengsizlik quyidagi tengsizlikka teng kuchli

$$(xy+yz+zx)^2+2\sqrt{3}xyz\le 1+2xyz(x+y+z),$$

yoki

$$\sqrt{3}\le x+y+z$$

A3 almashtirishga ko'ra

$$tg\frac{\alpha}{2}+tg\frac{\beta}{2}+tg\frac{\gamma}{2}\ge\sqrt{3}$$

tengsizlikni isbotlash etarli. Bu tengsizlik esa $f(x)=tg\frac{x}{2}$ funksiya $(0;\pi)$ intervalda qavariqligi uchun Iensen tengsizligidan kelib chiqadi. Isbot tugadi.

7-masala. Faraz qilaylik x,y,z lar musbat sonlar bo'lsin.Quyidagi tengsizlikni isbotlang

$$\sqrt{x(y+z)}+\sqrt{y\left(z+x\right)}+\sqrt{z(x+y)}\geq 2\sqrt{\frac{(x+y)(y+z)(z+x)}{x+y+z}}$$

Isboti. Tengsizlikni quyidagicha yozib olamiz

$$\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}+\sqrt{\frac{y(x+y+z)}{(y+z)(y+x)}}+\sqrt{\frac{z(x+y+z)}{(z+x)(z+y)}}\geq 2$$

A10 almashtirishga ko'ra

$$\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\geq 2$$

tengsizlikni isbot qilish etarli. A1 ga ko'ra o'tkir burchakli ABC uchburchak uchun

$$\sin A+\sin B+\sin C\geq 2$$

tengsizlikni isbotlash etarli. Bu tengsizlikni isbotlashning juda ko'p usullari mavjud. Biz Jordan tengsizligidan foydalanishni tavsiya qilamiz.

Jordan tengsizligi. Barcha $\alpha\in\left(0;\dfrac{\pi}{2}\right)$ lar uchun quyidagi tengsizlik o'rinli

$$\frac{2\alpha}{\pi}\leq\sin\alpha\leq\alpha\,.$$

U holda $\sin A+\sin B+\sin C\geq\dfrac{2A}{\pi}+\dfrac{2B}{\pi}+\dfrac{2C}{\pi}=2$

Isbot tugadi.

8-masala. Faraz qilaylik x,y,z lar musbat sonlar bo'lsin. Quyidagi tengsizlikni isbotlang

$$\sqrt{\frac{y+z}{x}}+\sqrt{\frac{z+x}{y}}+\sqrt{\frac{x+y}{z}}\geq\sqrt{\frac{16(x+y+z)^3}{3(x+y)(y+z)(z+x)}}\,.$$

Isboti. Qulaylik uchun quyidagicha belgilash olamiz:

$$\sum_{cyc}f(x,y,z)=f(x,y,z)+f(y,z,x)+f(z,x,y)$$

Berilgan tengsizlikni quyidagicha yozib olamiz

$$\sum_{cyc}(y+z)\sqrt{\frac{(x+y)(z+x)}{x(x+y+z)}}\geq\frac{4(x+y+z)}{\sqrt{3}}$$

A2 va A10 almashtirishlarga ko'ra

$$(y+z)\sqrt{\frac{(x+y)(z+x)}{x(x+y+z)}}=\frac{a}{\cos\frac{\alpha}{2}}=4R\sin\frac{\alpha}{2},$$

boshqa xadlarni ham shunday ifodalab olamiz.

Shuningdek

$$\frac{4(x+y+z)}{\sqrt{3}}=\frac{4R(\sin\alpha+\sin\beta+\sin\gamma)}{\sqrt{3}}$$

munosabat o'rinli. Bu yerda α,β,γ lar tashqi chizilgan aylana radiusi R bo'lgan uchburchakning burchaklari.

Shunday qilib quyidagi tengsizlikni isbotlashimiz kerak

$$\frac{\sqrt{3}}{2}\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)\geq\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\sin\frac{\beta}{2}\cos\frac{\beta}{2}+\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\quad.$$

Ushbu $f(x)=\cos\frac{x}{2}$ funksiya $[0;\pi]$ kesmada botiqligi uchun Iensen tengsizligiga ko'ra quyidagi tengsizlik o'rinli

$$\frac{\sqrt{3}}{2}\geq\frac{1}{3}\left(\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\right)$$

Ushbu $f(x)=\sin\frac{x}{2}$ funksiya $[0;\pi]$ kesmada o'suvchi, $f(x)=\cos\frac{x}{2}$ funksiya $[0;\pi]$ kesmada kamayuvchi bo'lganligi uchun Chebishev tengsizligiga ko'ra quyidagi tengsizlik o'rinli

$$\frac{1}{3}\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)\left(\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\right)\geq$$

$$\geq\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\sin\frac{\beta}{2}\cos\frac{\beta}{2}+\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}$$

Bu va bundan oldingi tengsizliklarga ko'ra

$$\frac{\sqrt{3}}{2}\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)\geq\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\sin\frac{\beta}{2}\cos\frac{\beta}{2}+\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}$$

tengsizlikning o'rinli ekanligi ko'rinib turibdi. Isbot tugadi.

Mashqlar

1. (Ruminiya, 2005) Faraz qilaylik musbat a,b,c sonlar $a+b+c=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\sqrt{\frac{3}{2}}$$

2. (Ukraina, 2005) Faraz qilaylik musbat a,b,c sonlar $a+b+c=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\sqrt{\frac{1}{a}-1}\sqrt{\frac{1}{b}-1}+\sqrt{\frac{1}{b}-1}\sqrt{\frac{1}{c}-1}+\sqrt{\frac{1}{c}-1}\sqrt{\frac{1}{a}-1}\geq 6$$

3. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ca=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+a}}+\frac{1}{\sqrt{a+b}}\geq 2+\frac{1}{\sqrt{2}}$$

4. (APMO, 2004) Musbat a,b,c sonlar uchun quyidagi tengsizlikni isbotlang

$$(a^2+2)(b^2+2)(c^2+2)\geq 9(ab+bc+ca)$$

5. (APMO, 2002) Faraz qilaylik musbat a,b,c sonlar $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$\sqrt{a+bc}+\sqrt{b+ca}+\sqrt{c+ab}\geq\sqrt{abc}+\sqrt{a}+\sqrt{b}+\sqrt{c}$.

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Mundarija